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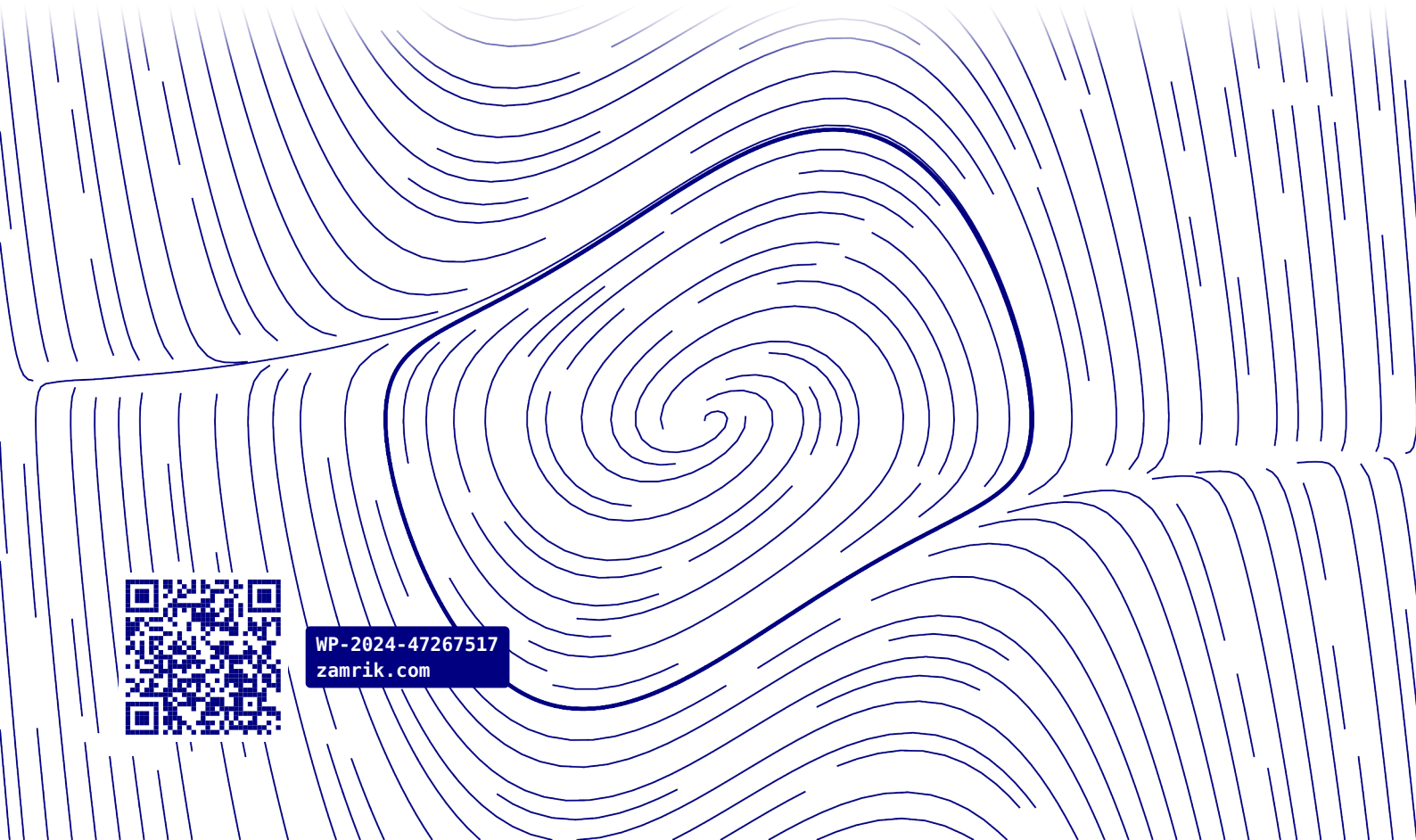
Universal Laws for the Times of the Maximum and Minimum of Random Walks

A Proof of the Mori–Majumdar–Schehr Conjecture and Its Continuum Limits

Supervised by Dr. T. Zamrik

27-AUG-2024

Phase portrait of the Van der Pol oscillator: $\dot{x} = y,\ \dot{y} = \mu\,(1 - x^2)\,y - x, \quad \mu = 0.80$



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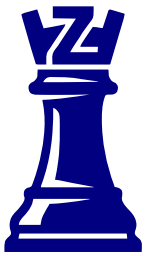
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WORKING PAPER



Universal Laws for the Times of the Maximum and Minimum of Random Walks

A Proof of the Mori–Majumdar–Schehr Conjecture and Its Continuum Limits

Supervised by Dr. T. Zamrik[1]

27-AUG-2024 WP-2024-47267517

Abstract. Let $t_{\max}$ and $t_{\min}$ be the times of the maximum and the minimum of a random walk of n steps, and $\tau = t_{\min} - t_{\max}$. Mori, Majumdar and Schehr conjectured that $P(\tau = n) = 1/(2n)$ for every symmetric continuous jump law and proved it for $n \leq 3$ and for double-exponential jumps. We prove it for every n and every i.i.d. symmetric law without atoms. The proof is the cycle lemma applied to the doubled sequence $(\xi_1, \dots, \xi_n, -\xi_1, \dots, -\xi_n)$, whose cyclic rotations are the doubled sequences of $T^r\xi$, $T(x_1, \dots, x_n) = (x_2, \dots, x_n, -x_1)$: for any law invariant under T without ties, dependent laws included, the highest point of the doubled loop is uniform on its $2n$ positions. These positions are explicit events about $t_{\max}$, $t_{\min}$ and the sign of $M + m - S_n$, each of probability $1/(2n)$. For i.i.d. steps the rest of the joint law depends on the step law through one probability, which gives the universal bound $P(\tau = j) \leq 1/(2j)$, the limit $nP(\tau = n - k) \to 1/2$ and, for every stable domain of attraction, the Lévy-flight edge value $1/2$ in averaged form. Brownian motion has a uniform maximizer of the doubled path on $[0, 2)$; on the lattice a uniform split of ties restores the law. Exact enumeration for $n \leq 7$ and Monte Carlo over ten step laws, within 4.34 standard errors, confirm the results.

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1. Introduction

A random walk of n steps reaches its highest value at some step $t_{\max}$ and its lowest at some step $t_{\min}$. Mori, Majumdar and Schehr [2], [1] studied the signed distance between the two, $\tau = t_{\min} - t_{\max}$, for Brownian motion, the Brownian bridge and discrete walks, and found its law in closed form for the continuous processes. For the discrete walk the extreme value $\tau = n$ is the event that the maximum comes first, at step 0, and the minimum last, at step n: the walk stays below its starting point throughout and ends at its lowest point. For double-exponential jumps they proved $P(\tau = n) = 1/(2n)$ for every n, and then wrote [1, after Eq. (17)]:

"our numerical simulations suggest that this result, for any n, seems to be valid

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> *for arbitrary symmetric and continuous jump distribution, and thus include Lévy flights also. [...] This remains an outstanding open problem."*

In their appendix the conjecture is stated as "for symmetric and continuous $p(\eta)$, $p_n = 1/(2n)$ for all $n \geq 1$" [1, Eq. (E5)], where p_n is the probability of the event above (written there for the reflected walk). This paper proves the conjecture and asks what else about the pair $(t_{\max}, t_{\min})$ is universal.

Universality of this kind is classical for one extreme at a time. Sparre Andersen [3], [4] showed that for steps with a symmetric continuous law the probability that the walk stays positive for a steps is $q_a = \binom{2a}{a}4^{-a}$, whatever the law, and that $t_{\max}$ has the discrete arcsine law $P(t_{\max} = k) = q_k q_{n-k}$, whose continuum limit is Lévy's arcsine law [1]. These results underlie the universal first-passage, global-maximum and record statistics of random walks and Lévy flights [5].

The joint law of the two times is another matter: the law of τ depends on the jumps, and for Lévy flights of index μ its scaling function f_μ changes with μ [1]. The conjecture singles out one corner of the joint law as universal, and the simulations of [1] single out one more feature: f_μ tends to $1/2$ at the edge $y = \pm 1$ "independently of μ" [1, Eq. (184) and Fig. 7].

Mori, Majumdar and Schehr proved the conjecture for $n \leq 3$ by direct integration [1, App. E], and for all n for the double-exponential law, where $\sum_n p_n s^n = -\frac{1}{2}\log(1-s)$ [1, Eqs. (188)–(193)]. For walks on the integers with the first time of each extreme counted they found $p_3 = 1/8$ rather than $1/6$ [1, Eq. (187)], so ties matter; and they wrote that proving the edge value for Lévy flights "appears to be a challenging task". To our knowledge, no proof of the conjecture for $n > 3$ has appeared: none of the works citing [1] that we found contains one, and we could not consult Chapter 5 of the monograph of Majumdar and Schehr [10], which treats the times of the maximum and the minimum.

The proof below belongs to a classical family. The cycle lemma of Dvoretzky and Motzkin [6] and Spitzer's combinatorial lemma [7] state that exactly one cyclic rotation of a zero-sum sequence has all its partial sums negative; Vervaat [8] rotated the Brownian bridge at its minimum to obtain the excursion; and Kabluchko, Vysotsky and Zaporozhets [9] used the Weyl chambers of the hyperoctahedral group B_n to prove distribution-free results for walks with symmetric exchangeable increments. Our argument is the cycle lemma applied to the doubled sequence $(\xi, -\xi)$.

Doubling the walk with its negated steps closes it into a loop whose rotations are the doubled walks of $T^r\xi$, $T(x_1, \dots, x_n) = (x_2, \dots, x_n, -x_1)$; when T preserves the law of the steps, the highest point of the loop is uniform on its $2n$ positions, and position 0 is the event $\tau = n$.

Main findings

- The Mori–Majumdar–Schehr conjecture holds: for every $n \geq 1$ and i.i.d. steps with any symmetric law without atoms, $P(\tau = n) = 1/(2n)$, because the highest point of the doubled walk is uniform on its $2n$ positions whenever the step law is invariant under T and has no ties (Theorem 1, Corollary 1).

- The $2n$ positions are $2n$ events about $(t_{\max}, t_{\min}, \operatorname{sign}(M + m - S_n))$ of probability exactly $1/(2n)$ each, so that given $M + m > S_n$ the time of the maximum is uniform on $\{1, \dots, n-1\}$ (Theorem 2), as exact enumeration up to $n = 7$ and Monte Carlo over ten step laws, dependent ones included, confirm.

- For i.i.d. steps the law enters the rest of the joint law of $(t_{\max}, t_{\min})$ through one probability ρ, $P(t_{\max} = a, t_{\min} = n - b) = q_a q_b \rho/(2(n - a - b))$, which gives the universal bound $P(\tau = j) \le 1/(2j)$ and the limit $nP(\tau = n - k) \to 1/2$ (Theorems 3 and 4).

- For every stable domain of attraction, Lévy flights included, $\delta^{-1} P(\tau \ge (1 - \delta)n) \to 1/2$ as $n \to \infty$ and then $\delta \to 0$, the edge value $f_\mu(\pm 1) = 1/2$ of [1] in averaged form, and every scaling density of τ/n satisfies $f(y) \le 1/(2y)$ (Theorem 7, Corollary 6).

- For Brownian motion the highest point of the doubled path is uniform on $[0, 2)$, so that given $M + m > B_1$ the time of the maximum is uniform on $(0, 1)$ (Theorem 6).

- For steps ± 1 the law fails only through ties: splitting them uniformly over the maximizers of the doubled loop, or breaking them with an independent continuous walk, gives exactly $1/(2n)$ (Theorem 5, Proposition 3).

2. Setting and the Main Theorem

Definition 1 (The walk and the times of its extremes)**.** Let $n \ge 1$ and let $\xi = (\xi_1, \dots, \xi_n)$ be a random vector in $\mathbb{R}^n$, the steps. The walk is $S_0 = 0$ and $S_k = \xi_1 + \cdots + \xi_k$ for $1 \le k \le n$. Its maximum and minimum are $M = \max_{0 \le k \le n} S_k$ and $m = \min_{0 \le k \le n} S_k$, so $m \le 0 \le M$. When each is attained at one step only, $t_{\max}$ and $t_{\min}$ denote those steps, and $\tau = t_{\min} - t_{\max} \in \{-n, \dots, n\}$.

This is the notation of [1]. The event $\tau = n$ is $\{t_{\max} = 0,\ t_{\min} = n\}$: the walk stays strictly below 0 after the start and strictly above S_n before the end. When the step law has atoms the times need a rule for ties; Section 5 treats that case.

Definition 2 (The doubled loop)**.** The doubled walk η has $2n$ steps: $\xi_1, \dots, \xi_n$ followed by $-\xi_1, \dots, -\xi_n$. Since they sum to 0, its partial sums define a function on the cycle $\mathbb{Z}/2n\mathbb{Z}$, the loop P_j:

$$P_j = S_j, \qquad P_{n+j} = S_n - S_j \qquad (0 \le j \le n),$$

both formulas giving S_n at position n and $P_{2n} = P_0 = 0$. The rotation by $r \in \mathbb{Z}$ is $(\theta_r P)_j = P_{j+r} - P_r$. The map $T : \mathbb{R}^n \to \mathbb{R}^n$ is $T(x_1, \dots, x_n) = (x_2, \dots, x_n, -x_1)$. K is the set of positions in $\mathbb{Z}/2n\mathbb{Z}$ at which the loop attains its maximum, and A is its element when K has one element.

The loop satisfies $P_{j+n} = S_n - P_j$ for every j: for $0 \le j \le n$ this is the definition, and for $j = n + i$ both sides equal S_i. Hence its minimum is S_n minus its maximum, and its first half is the walk itself, while its second half is the walk seen from its endpoint, upside down. Figure 1 draws one example.

Assumption 1 (The hypotheses)**.** The results below use some of the following. **(H1)** $T\xi$ has the same law as ξ. **(H2)** Almost surely the loop attains its maximum at one position only. **(H2′)** For every nonempty $I \subseteq \{1, \dots, n\}$ and signs $\varepsilon \in \{-1, 1\}^I$, $P(\sum_{i \in I} \varepsilon_i \xi_i = 0) = 0$.

(H1) holds for i.i.d. steps with a symmetric law, and more generally for every law invariant under signed permutations, since T is one. (H2′) holds when the steps are independent and their laws have no atoms, or when ξ has a density on $\mathbb{R}^n$. Lemma 2 shows that (H2′) implies (H2).

Lemma 1 (A rotation of the loop is the loop of T)**.** Write $P(x)$ for the loop of a step vector x. For every $x \in \mathbb{R}^n$ and every integer r, $\theta_r P(x) = P(T^r x)$. Moreover $T^n = -\mathrm{id}$ and $T^{2n} = \mathrm{id}$.

Proof. The loop $\theta_1 P(x)$ starts at 0 and has the steps of $P(x)$ read from position 1 on: $(\eta_2, \dots, \eta_{2n}, \eta_1) = (x_2, \dots, x_n, -x_1, -x_2, \dots, -x_n, x_1)$. These are the steps $(Tx, -Tx)$ of the loop $P(Tx)$, which also starts at 0; hence $\theta_1 P(x) = P(Tx)$. Since $(\theta_1 \theta_r P)_j = P_{j+r+1} - P_r - (P_{r+1} - P_r) = (\theta_{r+1} P)_j$, induction on r gives the claim for $r \geq 0$, and $\theta_{2n} = \mathrm{id}$ extends it to every integer once $T^{2n} = \mathrm{id}$ is known. Each application of T moves the first coordinate to the end and changes its sign, so after n applications every coordinate has moved once: $T^n x = -x$, and $T^{2n} x = x$. □

Lemma 2 (No ties on the loop)**.** Under (H2′) the $2n$ values $P_0, \dots, P_{2n-1}$ are almost surely distinct. Hence (H2) holds, and $t_{\max}$ and $t_{\min}$ are almost surely unique.

Proof. Let $j \neq k$ in $\mathbb{Z}/2n\mathbb{Z}$. Then $P_k - P_j$ is the sum of η over the cyclic arc of positions $j+1, \dots, k$, of some length ℓ with $1 \leq \ell \leq 2n-1$. Position i carries ξ_i and position $n+i$ carries $-\xi_i$; if the arc holds both, they cancel. What remains is $\sum_{i \in I} \varepsilon_i \xi_i$, where I is the set of indices i such that exactly one of the positions i, $n+i$ lies in the arc. The set I is not empty: otherwise the arc would be invariant under the shift by n, but an arc of length $1 \leq \ell \leq 2n-1$ has a unique first position, and the shifted arc starts $n \not\equiv 0$ later. By (H2′) and a union over the finitely many pairs (I, ε), $P(P_k = P_j) = 0$. The walk values $S_0, \dots, S_n$ are the loop values at the distinct positions $0, \dots, n$, so they are distinct as well. □

Theorem 1 (The maximizer of the doubled loop is uniform)**.** Under (H1) and (H2), A is uniform on $\mathbb{Z}/2n\mathbb{Z}$. In particular

$$P(S_j < 0 \text{ for } 1 \leq j \leq n \text{ and } S_j > S_n \text{ for } 0 \leq j \leq n-1) = \frac{1}{2n},$$

that is, the maximum is attained strictly at step 0 and the minimum strictly at step n with probability $1/(2n)$. Under (H1) and (H2′), $P(\tau = n) = P(\tau = -n) = 1/(2n)$.

Proof. The rotation θ_r subtracts the constant P_r from every loop value and relabels position $j+r$ as j, so $K(\theta_r P) = K(P) - r$, and $A(\theta_r P) = A(P) - r$ when K has one element. By

Lemma 1, $\theta_r P(\xi) = P(T^r\xi)$*, and by (H1), applied* r *times,* $T^r\xi$ *has the law of* ξ*. Hence*

$$P(A = r) = P(A(\theta_r P(\xi)) = 0) = P(A(P(T^r\xi)) = 0) = P(A = 0)$$

for every r*, and by (H2) the* $2n$ *probabilities sum to* 1*. Position* 0 *is the strict maximum exactly when* $P_j < 0$ *for* $j \neq 0$*: for* $1 \le j \le n$ *this says* $S_j < 0$*, and for* $j = n + i$ *with* $1 \le i \le n - 1$ *it says* $S_i > S_n$*; together with* $S_n < 0 = S_0$ *this is the displayed event. Under (H2′) the times are unique (Lemma 2), and the displayed event is* $\{\tau = n\}$*. The law of* $-\xi$ *satisfies (H1) and (H2′) whenever that of* ξ *does, and* $\tau(-\xi) = -\tau(\xi)$*, which gives* $P(\tau = -n) = 1/(2n)$*.* □

The proof is the cycle lemma of Dvoretzky and Motzkin [6] and Spitzer [7] applied to the zero-sum sequence $(\xi, -\xi)$, whose cyclic rotations are the doubled sequences of $T^r\xi$. Neither a density nor independence is used.

Corollary 1 (The Mori–Majumdar–Schehr conjecture)**.** If $\xi_1, \dots, \xi_n$ are i.i.d. with a symmetric law without atoms, then $P(\tau = n) = 1/(2n)$ for every $n \ge 1$.

Proof. $T\xi = (\xi_2, \dots, \xi_n, -\xi_1)$ *has independent coordinates, and* $-\xi_1$ *has the law of* ξ_1 *by symmetry, so (H1) holds. For (H2′), fix* I*,* ε *and some* $i_0 \in I$*, and condition on the coordinates other than* ξ_{i_0}*: the event* $\sum_{i\in I} \varepsilon_i \xi_i = 0$ *then pins* ξ_{i_0} *to one value, which has probability* 0 *because the law has no atoms. Integrating over the other coordinates (Fubini) gives (H2′), and Theorem 1 applies.* □

"Continuous" in [1] can thus be read as "without atoms": no density is needed, singular continuous laws are covered, and so are Lévy flights with an atomless jump law.

Corollary 2 (Dependent steps)**.** (a) If the law of ξ is invariant under signed permutations, that is, exchangeable and invariant under changing the sign of any coordinate, and ξ satisfies (H2′), for instance because it has a density on $\mathbb{R}^n$, then $P(\tau = n) = 1/(2n)$. (b) Exchangeability is not needed: if $U, V, Z_1, \dots, Z_n$ are i.i.d. standard Gaussian and

$$\xi_k = U\cos(\pi k/n) + V\sin(\pi k/n) + \tfrac{1}{2}Z_k, \qquad 1 \le k \le n,$$

then $P(\tau = n) = 1/(2n)$, although the law of ξ is not exchangeable for $n \ge 3$.

Proof. T *is a signed permutation, so (H1) holds; a density gives (H2′) because each hyperplane* $\sum_{i\in I} \varepsilon_i x_i = 0$ *is Lebesgue-null. (b) Put* $\varphi = \pi/n$ *and* $U' = U\cos\varphi + V\sin\varphi$*,* $V' = -U\sin\varphi + V\cos\varphi$*. The addition formulas give* $\xi_{k+1} = U'\cos(\pi k/n) + V'\sin(\pi k/n) + \frac{1}{2}Z_{k+1}$ *for* $1 \le k \le n - 1$*, and at* $k = n$*, where* $\cos\pi = -1$ *and* $\sin\pi = 0$*,* $-\xi_1 = U'\cos(\pi n/n) + V'\sin(\pi n/n) - \frac{1}{2}Z_1$*. So* $T\xi$ *is built from* $(U', V', Z_2, \dots, Z_n, -Z_1)$ *exactly as* ξ *is built from* $(U, V, Z_1, \dots, Z_n)$*, and these two vectors have the same law, because a standard Gaussian vector in the plane is invariant under rotation and* $-Z_1$ *has the law of* Z_1*. This is (H1). The vector* ξ *is Gaussian with covariance* $\mathrm{Cov}(\xi_j, \xi_k) = \cos(\pi(j - k)/n) + \frac{1}{4}\mathbf{1}\{j = k\}$*, which is positive definite, so* ξ *has a density and (H2′) holds. For* $n \ge 3$*,* $\mathrm{Cov}(\xi_1, \xi_2) = \cos(\pi/n)$ *differs from* $\mathrm{Cov}(\xi_1, \xi_3) = \cos(2\pi/n)$*, so the law is not exchangeable.* □

Examples of (a) are the multivariate Student law and random signs on the magnitudes of an equicorrelated Gaussian vector; the law in (b), the rotating phase, is invariant under the powers of T only.

Remark 1 (Equicorrelated Gaussian steps are not T-invariant). Gaussian steps with unit variances and a common correlation $c \neq 0$ are exchangeable but do not satisfy (H1): $\mathrm{Cov}(\xi_1, \xi_n) = c$, while the corresponding coordinates of $T\xi$ have $\mathrm{Cov}(\xi_2, -\xi_1) = -c$. Theorem 1 says nothing about them.

As a control, with $c = 1/2$, they give $2nP(\tau = n) = 1.7222$ at $n = 3$ and 8.9926 at $n = 20$ (Table 2): exchangeability alone does not give the $1/(2n)$ law.

Corollary 3 (The walk conditioned on $\tau = n$ is a rotated loop)**.** Under (H1) and (H2), the loop rotated to its maximum, $V = \theta_A P$, is independent of A and has the law of P given $A = 0$. Consequently the loop range $\Lambda = \max_j P_j - \min_j P_j = 2\max(M, S_n - m) - S_n$ satisfies: under (H1) and (H2′), the law of $-S_n$ given $\tau = n$ is the unconditional law of Λ.

Proof. For a measurable set B of loops and $r \in \mathbb{Z}/2n\mathbb{Z}$, using $A(\theta_r P) = A - r$ and $\theta_r P(\xi) = P(T^r \xi)$, which has the law of $P(\xi)$,

$$P(V \in B, A = r) = P(\theta_r P \in B,\ A(\theta_r P) = 0) = P(P \in B,\ A = 0).$$

Summing over r gives $P(V \in B) = 2n\, P(P \in B, A = 0) = P(P \in B \mid A = 0)$, by Theorem 1; and then $P(V \in B, A = r) = P(V \in B)/(2n) = P(V \in B)\, P(A = r)$. A rotation shifts all loop values by one constant, so $\Lambda(V) = \Lambda(P)$. On $\{A = 0\}$ the maximum of the loop is $P_0 = 0$ and, by $P_{j+n} = S_n - P_j$, its minimum is S_n, so $\Lambda = -S_n$ there. Hence the law of $\Lambda(P)$ is the law of $\Lambda(V)$, which is the law of Λ given $A = 0$, which is the law of $-S_n$ given $A = 0$; under (H2′), $\{A = 0\} = \{\tau = n\}$. The formula for Λ follows from $\min_j P_j = S_n - \max_j P_j$ and $\max_j P_j = \max(M, S_n - m)$. □

This is the doubled-loop form of Vervaat's rotation [8], and the input of Theorem 3.

Proposition 1 (A cycle lemma for the doubled sequence)**.** Let $x \in \mathbb{R}^n$ have no vanishing nonempty signed subset sum. Then the $2n$ vectors $x, Tx, \dots, T^{2n-1}x$ are distinct, and exactly one of them has a walk with its maximum strictly at step 0 and its minimum strictly at step n. Hence, for every $a \in (0, \infty)^n$ with no vanishing nonempty signed subset sum, exactly $2^{n-1}(n-1)! = 2^n n!/(2n)$ of the $2^n n!$ signed permutations of a have $\tau = n$.

Proof. The argument of Lemma 2 is deterministic: the loop values of x, and of every signed permutation of it, are distinct, so $A(x)$ is defined. By Lemma 1 and the proof of Theorem 1, $A(T^r x) = A(x) - r$. As r runs over $0, \dots, 2n-1$, $A(T^r x)$ runs over $\mathbb{Z}/2n\mathbb{Z}$ once, so exactly one r gives $A(T^r x) = 0$, which is the stated event; and $T^r x = T^s x$ with $0 \leq s < r < 2n$ would give $A(x) - r = A(x) - s$, which is impossible. The coordinates of a are nonzero and pairwise distinct (their differences and the coordinates themselves are signed subset sums), so its $2^n n!$ signed permutations are distinct vectors. T maps this set to itself, and by the first part it acts freely with orbits of size $2n$; each of the $2^n n!/(2n)$ orbits contains exactly one vector with $\tau = n$. □

This is Spitzer's lemma [7] for the zero-sum sequence $(x, -x)$, restricted to rotations that keep the doubled form; for laws invariant under signed permutations Theorem 1 follows from it by conditioning on the magnitudes.

Remark 2 (The order of T). As a signed permutation, T is one n-cycle carrying one sign change. This is the signed cycle type of the Coxeter element $s_0 s_1 \cdots s_{n-1}$ of the hyperoctahedral group B_n, so the two are conjugate, and their common order $2n$ is the Coxeter number $h(B_n)$ [11]. Theorem 1 therefore reads $P(\tau = n) = 1/h(B_n)$.

This is a coincidence of form, which the proof does not use; the signed cycle types and orders were compared by computer for $n \leq 8$.

Algorithm 1 — Exact enumeration of the loop cells over T-orbits

Inputs: n and a vector a of n positive integers. *Output:* for each position $r \in \mathbb{Z}/2n\mathbb{Z}$, the number of signed permutations of a whose loop has its maximum at r; and a verdict on each identity of Theorem 1, Proposition 1 and Theorem 2.

1. Test genericity: for every nonzero $e \in \{-1, 0, 1\}^n$ require $\sum_i e_i a_i \neq 0$, the deterministic form of (H2′). A set that fails, such as $a = (1, 2, \dots, n)$ for $n \geq 3$, is kept as the tied control.
2. List the $2^n n!$ signed permutations x of a as rows of an integer array, so that every comparison below is exact.
3. For each row form the loop $P_0, \dots, P_{2n-1}$ of Definition 2, its set of maximizers K and, when $|K| = 1$, its maximizer A; on a generic set require $|K| = 1$ on every row.
4. Compute the walk's $t_{\max}$, $t_{\min}$ and the sign of $M + m - S_n$, translate them into a position by the dictionary of Theorem 2, and require it to equal A.
5. Apply T to every row, recompute the maximizers, and require $K(Tx) = K(x) - 1$; iterate T and require $T^r x \neq x$ for $0 < r < 2n$ and $T^{2n} x = x$, so that the rows split into free orbits of size $2n$.
6. Count the rows with $A = r$ for each r and require the count $2^{n-1}(n-1)!$ at every position; on the tied set, count instead the rows with $K = \{0\}$, with $0 \in K$, and the sum of $\mathbf{1}\{0 \in K\}/|K|$.

Cost. Steps 3 and 4 take $O(n)$ operations per row, step 5 takes $O(n^2)$ and step 1 takes 3^n sums, so the work grows as $2^n n! \, n^2$; the largest case, $n = 7$, has 645120 rows for each magnitude set.

Method. Algorithm 1 on every signed permutation of each magnitude set, in integer arithmetic, for $n = 1, \dots, 7$.

Parameters. Five generic sets at each n: the magnitudes $1 + \lfloor 10^6 |X_i| \rfloor$ of Gaussian, Cauchy and Laplace draws, integers drawn uniformly from $1, \dots, 10^6$, and the powers $1, 2, \dots, 2^{n-1}$. The tied control is $a = (1, 2, \dots, n)$, not generic for $n \geq 3$ since $1 + 2 - 3 = 0$.

Result. On every row of every generic set the loop maximum is unique, $K(Tx) = K(x) - 1$, T acts freely with $T^{2n} = \mathrm{id}$, and the dictionary of Theorem 2 returns A. Each of the $2n$ positions holds exactly $2^{n-1}(n-1)!$ rows, for instance 46080 of 645120 at $n = 7$ (Table 1). The values 1/2, 1/4, 1/6 proved in [1] for $n \leq 3$ are reproduced exactly; their integral

(E14) evaluates to 0.166666666667, and their double-exponential generating function, by quadrature of their Eqs. (124), (126) and (189), gives 0.346573590280 at $s = 1/2$ against $-\frac{1}{2}\log\frac{1}{2} = 0.346573590280$, and agrees at $s = 0.1$, 0.9, 0.99. On the tied set the strict count falls below $2^{n-1}(n-1)!$ and the weak count exceeds it from $n = 4$ on, while the count weighted by $1/|K|$ equals it at every n (Theorem 5).

Error. None for the counts, which are integer comparisons and agree exactly. The quadratures agree with their closed forms within the absolute tolerance 10^{-9} fixed before the run.

Source. Algorithm 1, integer arithmetic, all signed permutations of each magnitude set; `derive.py`.

Columns. n; the number $2^n n!$ of signed permutations; the number of them in each of the $2n$ loop cells, the same for every cell and every generic set; the number of generic sets enumerated. The last three columns use the tied set $a = (1, \dots, n)$: the number of rows with $K = \{0\}$ (strict), the sum of $\mathbf{1}\{0 \in K\}/|K|$ (weighted) and the number with $0 \in K$ (weak).

Table 1: Exact counts per loop cell for every signed permutation of generic and tied magnitudes, $n \le 7$.

n	$2^n n!$	**count per cell**	**generic sets**	**tied: strict**	**tied: weighted**	**tied: weak**
1	2	1	5	1	1	1
2	8	2	5	2	2	2
3	48	8	5	8	8	8
4	384	48	5	44	48	52
5	3840	384	5	356	384	420
6	46080	3840	5	3544	3840	4152
7	645120	46080	5	42816	46080	49360

Method. For each step law and each n, simulate independent walks, form the loop, and record its maximizer A together with the position read off from $(t_{\max}, t_{\min}, \operatorname{sign}(M + m - S_n))$. Each of the $2n$ pooled cell frequencies is scored by its z-value against $1/(2n)$, and each seed by a chi-square test of uniformity over the $2n$ cells. A control, whose hypotheses fail, must be detected with $|z| \ge 10$ in some cell. Corollary 3 is tested at $n = 5$ by a two-sample Kolmogorov–Smirnov test of $-S_n$ on $\{\tau = n\}$ against Λ on all paths.

Parameters. $n = 3, 4, 5, 6, 10, 20$; three seeds of 10^6 paths. Ten laws satisfy (H1) and (H2′): seven i.i.d. (Gaussian; uniform on $(-1, 1)$; Laplace; Cauchy; symmetric stable with $\alpha = 1/2$, drawn as in [12]; the singular Cantor law on $[-\frac{1}{2}, \frac{1}{2}]$, with 40 ternary digits; magnitudes uniform on $[1, 2]$ with random signs), two dependent laws invariant under signed permutations (multivariate Student with three degrees of freedom; random signs on the magnitudes of a Gaussian vector with common correlation 0.8), and the rotating

phase of Corollary 2(b). Controls: equicorrelated Gaussian steps (Remark 1), the centered exponential $E-1$, and steps ± 1 under the first-time rule. Tolerances, fixed in advance: $|z| \le 5$ per cell, chi-square $p \ge 10^{-5}$.

Result. Table 2 lists $2nP(\tau = n)$. For the ten laws every one of the $2n$ cells at every n agrees with $1/(2n)$; the largest $|z|$ over all of them is 4.34, for the stable law, and the smallest of the 180 per-seed chi-square p-values is 0.0053. The standard error of $2nP(\tau = n)$ runs from 0.0013 at $n = 3$ to 0.0036 at $n = 20$. No path of a continuous law had a tied loop maximum. Every control is detected at every n, with a largest cell $|z|$ of at least 134. For Corollary 3 at $n = 5$ the medians of $-S_n$ given $\tau = n$ and of Λ are 3.064 and 3.061 (Gaussian), 8.165 and 8.164 (Cauchy), 3.837 and 3.853 (Laplace), against 1.511, 5.007 and 2.001 for $|S_n|$; the smallest Kolmogorov–Smirnov p-value over nine seeds is 0.188, and the control comparison with $|S_n|$ gives a p-value that underflows to 0.

Error. The ten laws deviate from $1/(2n)$ by at most 4.34 standard errors over all pooled cells, the size of sampling noise over that many tests; the controls deviate by more than a hundred.

Source. Monte Carlo, 3×10^6 paths per law and n (three seeds of 10^6); `derive.py`.

Columns. The step law; $2n$ times the frequency of $\tau = n$ at each n, which Theorem 1 puts at 1 for the first ten rows; the largest $|z|$ over all $2n$ loop cells at all six n, or, for a control, the smallest over n of the largest cell $|z|$.

Table 2: Twice n times the frequency of $\tau = n$ for ten step laws that satisfy the hypotheses and three controls.

step law	$n = 3$	4	5	6	10	20	**max** $\lvert z \rvert$
Gaussian	1.0012	1.0024	1.0012	0.9975	0.9977	1.0003	2.63
uniform on $(-1, 1)$	1.0002	1.0002	0.9956	1.0011	1.0007	0.9992	3.08
Laplace	0.9964	1.0000	1.0010	0.9986	0.9991	1.0020	2.80
Cauchy	1.0007	0.9996	0.9993	0.9978	0.9962	0.9945	3.27
symmetric stable, $\alpha = 1/2$	1.0002	1.0021	1.0010	1.0007	0.9955	0.9980	4.34
Cantor (singular)	1.0004	1.0015	1.0004	0.9993	1.0026	0.9998	3.30
$\lvert\xi\rvert \sim U[1, 2]$, random sign	1.0002	1.0034	1.0024	0.9987	1.0008	0.9973	2.28
Student t_3 vector (dependent)	1.0008	0.9990	0.9978	0.9969	0.9974	0.9962	2.67
random signs on equicorrelated $\lvert Y\rvert$ (dependent)	1.0004	1.0025	1.0011	0.9978	1.0001	1.0011	2.77
rotating phase (not exchangeable)	1.0008	1.0004	0.9979	0.9991	0.9999	1.0036	2.84
control: equicorrelated Gaussian	1.7222	2.1283	2.5415	2.9575	4.6613	8.9926	≥ 559
control: $E - 1$, E exponential	1.7983	1.9153	1.9762	2.0236	2.1040	2.1343	≥ 315
control: ± 1 steps, first-time rule	0.7504	0.9993	0.9377	0.9392	0.9578	0.9903	≥ 134

Script. `fig_loop.py`: one walk of $n = 12$ Gaussian steps stored by the research run, drawn with its doubled loop.

Shows. Definition 2 and the dictionary of Theorem 2 on one path.

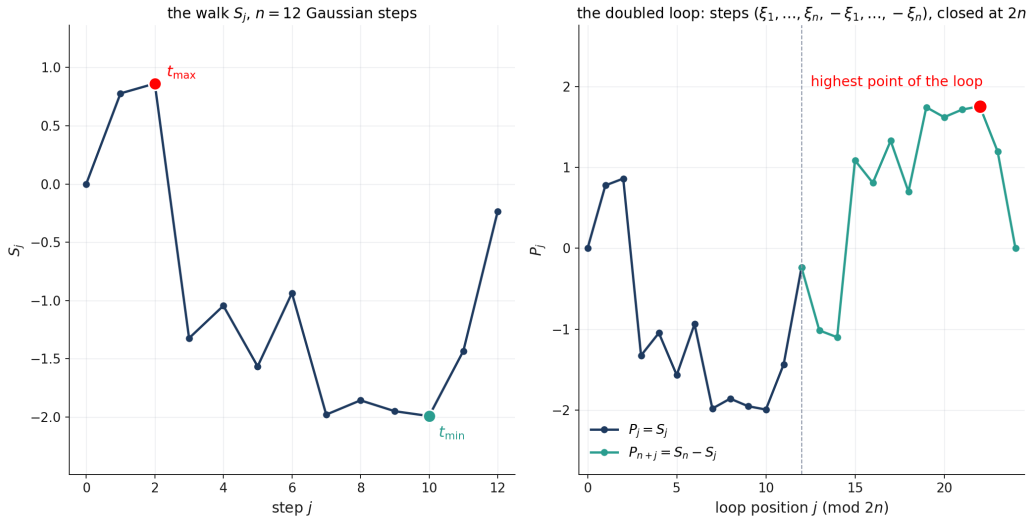


Figure 1: Left, a walk of twelve Gaussian steps with the time of its maximum in red and of its minimum in teal; right, its doubled loop of twenty-four positions, the walk followed by the walk seen from its endpoint upside down, with the loop's highest point marked.

Here $t_{\max} = 2$ with $M = 0.86$ and $t_{\min} = 10$ with $S_n - m = 1.76$. Since the rise from the minimum to the end exceeds the rise from the start to the maximum, $M + m < S_n$, and the loop peaks in its second half, at position $n + t_{\min} = 22$. Rotating the loop so that this point comes first gives the doubled loop of $T^{22}\xi$, whose walk has $\tau = n$.

3. The 2n Events

Theorem 2 (The $2n$ events of probability one over $2n$)**.** Let $n \geq 2$ and assume (H1) and (H2′). Define

$$C_0 = \{t_{\max} = 0,\ t_{\min} = n\}, \qquad C_n = \{t_{\max} = n,\ t_{\min} = 0\},$$

$$C_k = \{t_{\max} = k,\ M + m > S_n\}, \qquad C_{n+k} = \{t_{\min} = k,\ M + m < S_n\} \qquad (1 \leq k \leq n-1).$$

Then $C_r = \{A = r\}$ up to a null set for every $r \in \mathbb{Z}/2n\mathbb{Z}$. Hence the $2n$ events partition the sample space up to a null set, and each has probability exactly $1/(2n)$.

Proof. Positions $0, \ldots, n$ of the loop carry $S_0, \ldots, S_n$, whose maximum M sits at $t_{\max}$; positions $n, \ldots, 2n$ carry $S_n - S_j$, $0 \leq j \leq n$, whose maximum $S_n - m$ sits at $n + t_{\min}$ (read modulo $2n$). By Lemma 2 the loop values are almost surely distinct. If $M > S_n - m$ the loop maximum is at $t_{\max}$, and if $S_n - m > M$ it is at $n + t_{\min}$. Equality $M = S_n - m$ is possible only when the two positions coincide, $t_{\max} \equiv n + t_{\min} \pmod{2n}$, which for $t_{\max}, t_{\min} \in \{0, \ldots, n\}$ means $(t_{\max}, t_{\min}) = (0, n)$ or $(n, 0)$; conversely on C_0, $M = 0 = S_n - m$, and on C_n, $M = S_n = S_n - m$. So $\{A = 0\} = C_0$ and $\{A = n\} = C_n$. For $1 \leq k \leq n-1$, position k lies in the first half only, so $A = k$ exactly when $M > S_n - m$, that is $M + m > S_n$, and $t_{\max} = k$: this is C_k. Likewise $A = n + k$ exactly when $M + m < S_n$ and $t_{\min} = k$, which is C_{n+k}. Theorem 1 gives each probability $1/(2n)$. □

The sign of $M+m-S_n = M-(S_n-m)$ compares the rise from the start to the maximum with the rise from the minimum to the end.

Corollary 4 (The sign of $M+m-S_n$ and a conditional uniform law)**.** Under the hypotheses of Theorem 2: (a) $P(M+m > S_n) = P(M+m < S_n) = (n-1)/(2n)$, and $\{M+m = S_n\} = C_0 \cup C_n$ up to a null set, with probability $1/n$; (b) given $M+m > S_n$, $t_{\max}$ is uniform on $\{1,\dots,n-1\}$, and given $M+m < S_n$, $t_{\min}$ is uniform on $\{1,\dots,n-1\}$.

Proof. By the proof of Theorem 2, $\{M+m > S_n\}$ *is the union of* $C_1,\dots,C_{n-1}$, $\{M+m < S_n\}$ *the union of* $C_{n+1},\dots,C_{2n-1}$, *and* $\{M+m = S_n\}$ *the union of* C_0 *and* C_n, *up to null sets. Each* C_r *has probability* $1/(2n)$*; dividing gives the conditional laws.* □

Corollary 5 (The joint law of $t_{\max}$ and the sign for exchangeable laws)**.** Let the law of ξ be invariant under signed permutations and satisfy (H2′), and put $u_k = q_k q_{n-k}$ with $q_a = \binom{2a}{a}4^{-a}$. Then for $0 \le k \le n$

$$P(t_{\max} = k,\ M+m \ge S_n) = \frac{1}{2n}, \qquad P(t_{\max} = k,\ M+m < S_n) = u_k - \frac{1}{2n},$$

and the joint law of $(t_{\max}, \operatorname{sign}(M+m-S_n))$ is the same for every such law. The same holds for $t_{\min}$ with the roles of the two signs exchanged.

Proof. On $\{t_{\max} = 0\}$, $M = 0 \le S_n - m$, *with equality exactly on* C_0*; on* $\{t_{\max} = n\}$, $M+m-S_n = m \le 0$, *with equality exactly on* C_n. *Hence* $\{t_{\max} = k, M+m \ge S_n\}$ *is* C_k *for every* $0 \le k \le n$, *of probability* $1/(2n)$ *by Theorem 2. For laws invariant under signed permutations without ties the time of the maximum has Sparre Andersen's law* $P(t_{\max} = k) = u_k$ *[3], [4]; subtracting gives the second formula. The statement for* $t_{\min}$ *follows by applying this to* $-\xi$. □

The exact enumeration confirms both ingredients for each magnitude set separately: on every generic set with $n \le 7$ the counts give $P(t_{\max} = k) = u_k$ and the full joint law of Corollary 5.

Remark 3 (Exchangeability is needed for the joint law). The rotating phase of Corollary 2(b) satisfies (H1) and (H2′), so Theorem 2 holds for it, and its $2n$ cells pass every test. Its time of the maximum, however, does not follow Sparre Andersen's law, and its cells $\{t_{\max} = k, M+m < S_n\}$ miss $u_k - 1/(2n)$ by up to $|z| = 613$ over the six values of n simulated; Table 3 shows them at $n = 6$ in its last column.

Only the $2n$ events of Theorem 2 follow from invariance under T.

Method. For every simulated path of Table 2 and every row of Table 1, the position given by the dictionary of Theorem 2 is compared with the maximizer of the loop. For the nine laws invariant under signed permutations, the pooled frequencies of $\{t_{\max} = k, M+m \ge S_n\}$ and $\{t_{\max} = k, M+m < S_n\}$ are scored against Corollary 5, and the combinations that Theorem 2 excludes are counted.

Parameters. As for Table 2. Excluded combinations are allowed in a fraction 10^{-5} of the paths, the floating-point allowance for ties stated before the run.

Result. The dictionary agrees with the loop maximizer on every enumerated row and on all simulated paths but one: a path of the stable law with $\alpha = 1/2$ at $n = 6$, out of 3×10^6, recorded with $t_{\max} = 1$ and $M + m = S_n$, which Theorem 2 excludes. A huge step had made several partial sums equal in double precision; re-examined in exact arithmetic, the path lies in cell 1. For the nine exchangeable laws every cell of Corollary 5 agrees with its prediction at every n, with a largest $|z|$ of 4.34; Table 3 shows $n = 6$, where a cell near $1/12$ has standard error 0.00016.

Error. Within 4.34 standard errors for every cell, and exact for the enumeration. The check of excluded combinations first demanded none and was relaxed to the allowance after that one path had been examined; it is the only check changed after a run.

Source. Corollary 5 for the exact columns; Monte Carlo, 3×10^6 paths per law, for the others; `derive.py`.

Columns. k; then $P(t_{\max} = k, M + m \geq S_n)$, exact and for Gaussian steps; then $P(t_{\max} = k, M + m < S_n)$, exact $(u_k - 1/12)$ and for Gaussian, Cauchy and multivariate Student steps, all invariant under signed permutations, and for the rotating phase, which is invariant under T only.

Table 3: The joint law of the time of the maximum and the sign of $M + m - S_n$ at $n = 6$.

k	$\geq$: exact	$\geq$: Gaussian	$<$: exact	$<$: Gaussian	$<$: Cauchy	$<$: Student t_3	$<$: rotating phase
0	0.08333	0.08313	0.14225	0.14238	0.14223	0.14190	0.19407
1	0.08333	0.08307	0.03971	0.03967	0.03967	0.03983	0.01093
2	0.08333	0.08326	0.01921	0.01920	0.01919	0.01931	0.00254
3	0.08333	0.08375	0.01432	0.01427	0.01429	0.01440	0.00141
4	0.08333	0.08314	0.01921	0.01926	0.01914	0.01922	0.00259
5	0.08333	0.08358	0.03971	0.03958	0.03991	0.03966	0.01097
6	0.08333	0.08351	0.14225	0.14220	0.14210	0.14225	0.19419

Script. `fig_cells.py`: the pooled loop-cell frequencies at $n = 6$ for the ten laws of Table 2, and the joint law of $(t_{\max}, t_{\min})$ at $n = 4$ for five i.i.d. laws, both read from `results.npz`.

Shows. Theorem 2 against its non-universal complement.

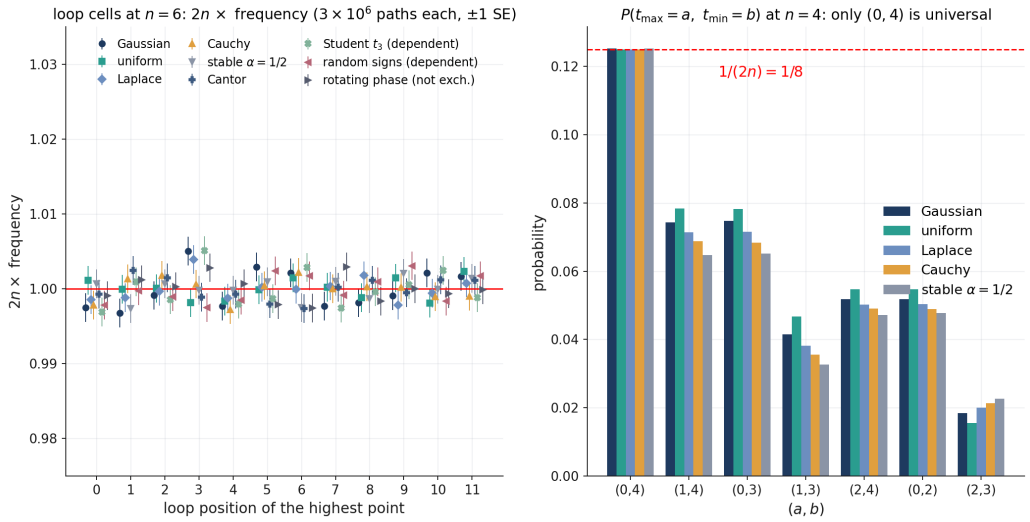


Figure 2: Left, twice n times the frequency of each of the twelve loop cells at n equal to six, for ten step laws, two of them dependent and one not exchangeable, all scattered about one with bars of one standard error; right, the probabilities of seven pairs of times of the maximum and minimum at n equal to four for five laws, equal only at the corner pair.

Every law sits on the red line at 1, cell by cell; on the right only the corner $(0, 4)$, the cell C_0, is common to the five laws (Tables 4 and 5).

4. What Is Not Universal

Theorem 3 (The joint law through one law-dependent probability)**.** Let $\xi_1, \dots, \xi_n$ be i.i.d. with a symmetric law without atoms, and let $a, b \geq 0$ with $L = n - a - b \geq 1$. Then

$$P(t_{\max} = a,\ t_{\min} = n - b) = \frac{q_a q_b\, \rho_{a,b,L}}{2L}, \qquad \rho_{a,b,L} = P(\max(R_a, R'_b) < \Lambda_L),$$

where R_a is the maximum of an a-step walk conditioned to stay positive ($R_0 = 0$), R'_b is an independent copy for b steps, Λ_L is the loop range of Corollary 3 for an independent, unconditioned walk of L steps, and all three walks have the step law of ξ_1. By $x \mapsto -x$ the same holds for $(t_{\max}, t_{\min}) = (n - b, a)$.

Proof. Split the steps into the blocks $I = (1, \dots, a)$, $II = (a + 1, \dots, n - b)$ and $III = (n - b + 1, \dots, n)$. Comparing S_a and S_{n-b} with every other partial sum, the event is $E_1 \cap E_2 \cap E_3 \cap C$, where $E_1 = \{S_a > S_j,\ j < a\}$ depends on block I; $E_2 = \{S_a > S_j > S_{n-b},\ a < j < n - b\} \cap \{S_a > S_{n-b}\}$, which says that block II, as a walk of L steps, has its maximum strictly at its start and its minimum strictly at its end, depends on block II; $E_3 = \{S_j > S_{n-b},\ j > n - b\}$ depends on block III; and $C = \{S_j > S_{n-b},\ j < a\} \cap \{S_j < S_a,\ j > n - b\}$ links them. Put $R_{\mathrm{I}} = S_a - \min_{j \leq a} S_j$, $R_{\mathrm{II}} = S_a - S_{n-b}$ and $R_{\mathrm{III}} = \max_{j \geq n-b} S_j - S_{n-b}$; then $C = \{R_{\mathrm{I}} < R_{\mathrm{II}},\ R_{\mathrm{III}} < R_{\mathrm{II}}\}$. The blocks are independent. Read backwards from step a, block I is an a-step walk with the same step law, and E_1 says that it stays positive; so $P(E_1) = q_a$ by Sparre Andersen's theorem [3], and given E_1, R_{I}

is the maximum of that walk, with the law of R_a. Likewise $P(E_3) = q_b$, and given E_3, R_{III} has the law of R'_b. By Theorem 1 for the L-step walk of block II, $P(E_2) = 1/(2L)$, and given E_2, R_{II} is minus the endpoint of that walk, which by Corollary 3 has the law of Λ_L. Multiplying the three independent factors and the conditional probability of C gives the formula. □

Since $\rho_{0,0,L} = 1$, Theorem 1 is the case $a = b = 0$: the step law enters the joint law of $(t_{\max}, t_{\min})$ only through ρ.

Theorem 4 (The bound and the edge limit)**.** Let $\xi_1, \dots, \xi_n$ be i.i.d. with a symmetric law without atoms. Then (i) $P(\tau = j) \le 1/(2j)$ for every $1 \le j \le n$, with equality at $j = n$; and (ii) for every fixed $k \ge 0$, $nP(\tau = n - k) \to 1/2$, and for fixed $a, b \ge 0$, $nP(t_{\max} = a,\ t_{\min} = n - b) \to q_a q_b/2$, as $n \to \infty$.

Proof. The event $\tau = j$ is the union over $a + b = n - j$ of $\{t_{\max} = a, t_{\min} = n - b\}$, with $L = j$. By Theorem 3 and $\rho \le 1$,

$$P(\tau = j) \le \frac{1}{2j} \sum_{a+b=n-j} q_a q_b = \frac{1}{2j},$$

because $\sum_{a+b=k} q_a q_b$ is the coefficient of x^k in $(1-x)^{-1/2}(1-x)^{-1/2} = (1-x)^{-1}$, the generating function of q_a being $(1-x)^{-1/2}$. At $j = n$ the sum has the single term $a = b = 0$ with $\rho = 1$. (ii) The loop range dominates the endpoint: $\Lambda_L = 2\max(M, S_L - m) - S_L$ is at least $S_L - 2m \ge S_L$ when $S_L \ge 0$, and at least $2M - S_L \ge -S_L$ when $S_L < 0$; so $\Lambda_L \ge |S_L|$. Hence $1 - \rho_{a,b,L} \le P(R_a \ge |S_L|) + P(R'_b \ge |S_L|)$, with S_L independent of R_a and R'_b. For a non-degenerate step law, $P(|S_L| \le c) \to 0$ for every c as $L \to \infty$ (Rogozin's bound on the concentration function [13]), while R_a and R'_b have fixed laws; so $\rho_{a,b,L} \to 1$ for fixed a, b. With $L = n - a - b$, $nq_a q_b \rho_{a,b,L}/(2L) \to q_a q_b/2$, and summing over the finitely many pairs $a + b = k$ gives $nP(\tau = n - k) \to \frac{1}{2}\sum_{a+b=k} q_a q_b = 1/2$. □

No i.i.d. symmetric atomless law puts more than $1/(2j)$ on $\tau = j$; light-tailed laws come close to the bound near the edge and heavy-tailed laws stay below it (Table 6).

Example 1 (Already $n = 3$ depends on the law)**.** For $n = 3$ and i.i.d. symmetric atomless steps, the event $\{t_{\max} = 1, t_{\min} = 3\}$ is $\{\xi_1 > 0,\ \xi_2 < 0,\ \xi_3 < 0,\ |\xi_1| < |\xi_2| + |\xi_3|\}$, and the signs are independent of the magnitudes, so

$$P(t_{\max} = 1,\ t_{\min} = 3) = \tfrac{1}{8}\, P(|\xi_1| < |\xi_2| + |\xi_3|).$$

This is Theorem 3 with $a = 1$, $b = 0$, $L = 2$: $R_1 = |\xi|$ and $\Lambda_2 = |\xi_1| + |\xi_2|$. The value is $5/48 = 0.104167$ for steps uniform on $(-1, 1)$, $3/32 = 0.093750$ for Laplace steps and $1/8 = 0.125000$ for magnitudes uniform on $[1, 2]$, where the inequality always holds; by quadrature it is 0.097957 for Gaussian and 0.090792 for Cauchy steps.

Monte Carlo with 3×10^6 paths gives 0.104376, 0.093638, 0.125011, 0.098065 and 0.090820, at $z = +1.19$, -0.67, $+0.06$, $+0.63$ and $+0.17$. The cells C_0 and C_3 have probability $1/6$ for all five laws, while this neighboring cell takes five different values. The case $n = 2$ is

still universal: the six pairs $(t_{\max}, t_{\min})$ have probability 1/4 at $(0,2)$ and $(2,0)$ and 1/8 at the other four, since each of those four is fixed by the signs of the two steps and the order of their magnitudes.

Method. For Theorem 3, $\rho_{a,b,L}$ is estimated from samples that never see the joint law: maxima of walks conditioned to stay positive, drawn by rejection, and loop ranges of unconditioned walks. The prediction $q_a q_b \rho_{a,b,L}/(2L)$ is compared with the direct frequency of $\{t_{\max} = a, t_{\min} = n-b\}$ from the simulations of Table 2. For the non-universal cells the joint law is compared between laws, and between magnitude sets in the enumeration.

Parameters. $n = 4, 5$, every pair with $a + b \le n - 1$, Gaussian and Cauchy steps; 3×10^6 samples of each conditioned maximum and each loop range. Joint laws at $n = 3, 4, 5$ for five i.i.d. laws, 3×10^6 paths each.

Result. Over all 50 pairs the direct frequency agrees with the formula within 2.44 standard errors. At $n = 4$, ρ falls from 1 at $(a,b) = (0,0)$ to 0.1393 for Gaussian and 0.1977 for Cauchy steps at $(0,3)$, where the formula gives 0.02177 and 0.03089 against direct frequencies 0.02170 and 0.03080; ρ carries the whole difference between the two laws. Tables 4 and 5 give the joint laws at $n = 4$ and $n = 5$: only $(0,n)$ is common to the five laws, and the largest difference between two laws in some cell is $|z| = 103$ at $n = 3$, 160 at $n = 4$ and 185 at $n = 5$. In the enumeration, the number of signed orderings with $(t_{\max}, t_{\min}) = (1,n)$ is 4 or 6 of 48 at $n = 3$ and takes four values from 23040 to 26880 of 645120 at $n = 7$, depending on the magnitude set, while every loop cell holds the same count for every set.

Error. The representation agrees within 2.44 standard errors over 50 comparisons. The non-universal cells differ between laws by over a hundred standard errors; two cells at $n = 5$, $(0,2)$ and $(3,5)$, differ by about four and are left unresolved.

Source. Monte Carlo, 3×10^6 paths per law; `derive.py`.

Columns. The pair $(t_{\max}, t_{\min})$ with $t_{\max} < t_{\min}$ (the others are mirror images under $x \mapsto -x$); its frequency for five i.i.d. laws; the largest two-law difference in units of its standard error, and, at $(0,n)$, the value of Theorem 1.

Table 4: The joint law of the times of the maximum and minimum at $n = 4$ for five step laws.

$(t_{\max}, t_{\min})$	**Gaussian**	**uniform**	**Laplace**	**Cauchy**	**stable** $\alpha = 1/2$	**spread,** $\lvert z \rvert$
$(0, 1)$	0.02165	0.01549	0.02635	0.03067	0.03598	159
$(0, 2)$	0.05183	0.05471	0.05035	0.04897	0.04777	39
$(0, 3)$	0.07477	0.07827	0.07158	0.06832	0.06511	62
$(0, 4)$	0.12530	0.12502	0.12500	0.12495	0.12527	1.3: $1/(2n)$
$(1, 2)$	0.01835	0.01575	0.01991	0.02124	0.02276	63
$(1, 3)$	0.04154	0.04675	0.03822	0.03563	0.03266	88
$(1, 4)$	0.07429	0.07838	0.07146	0.06876	0.06469	65
$(2, 3)$	0.01837	0.01556	0.02002	0.02127	0.02261	63
$(2, 4)$	0.05185	0.05468	0.05025	0.04909	0.04720	42
$(3, 4)$	0.02187	0.01559	0.02648	0.03112	0.03638	160

Source. As Table 4.

Columns. As Table 4, at $n = 5$. A spread of a few standard errors, as at $(0, 2)$ and $(3, 5)$, leaves the cell unresolved; the theory makes no claim there.

Table 5: The joint law of the times of the maximum and minimum at $n = 5$ for five step laws.

$(t_{\max}, t_{\min})$	**Gaussian**	**uniform**	**Laplace**	**Cauchy**	**stable** $\alpha = 1/2$	**spread,** $\lvert z\rvert$
$(0,1)$	0.01101	0.00624	0.01506	0.02002	0.02484	185
$(0,2)$	0.03048	0.03072	0.03042	0.03063	0.03092	4
$(0,3)$	0.04532	0.04828	0.04327	0.04093	0.03878	57
$(0,4)$	0.05888	0.06107	0.05735	0.05411	0.05145	51
$(0,5)$	0.10012	0.09956	0.10010	0.09993	0.10010	2.3: $1/(2n)$
$(1,2)$	0.00904	0.00619	0.01087	0.01291	0.01460	102
$(1,3)$	0.02336	0.02579	0.02175	0.02044	0.01927	54
$(1,4)$	0.03435	0.03763	0.03186	0.02899	0.02643	78
$(1,5)$	0.05904	0.06122	0.05711	0.05440	0.05166	51
$(2,3)$	0.00887	0.00622	0.01060	0.01211	0.01336	89
$(2,4)$	0.02320	0.02584	0.02172	0.02052	0.01927	54
$(2,5)$	0.04557	0.04820	0.04348	0.04117	0.03872	57
$(3,4)$	0.00903	0.00617	0.01093	0.01285	0.01475	103
$(3,5)$	0.03041	0.03066	0.03051	0.03075	0.03098	4
$(4,5)$	0.01093	0.00633	0.01507	0.01981	0.02465	182

Method. Simulate i.i.d. walks at large n, estimate $nP(\tau = n-k)$ from $\pm\tau$ together, and compare with the bound $n/(2(n-k))$ of Theorem 4(i), checked at every $1 \le j \le n$, and with the limit $1/2$. For Theorem 7 the mean of $nP(\tau = n-k)$ over $0 \le k \le K$ stands in for $\delta^{-1}P(\tau \ge (1-\delta)n)$, with $K \approx \delta n$.

Parameters. $n = 50, 200$; Gaussian, uniform, Cauchy and stable ($\alpha = 1/2$) steps; three seeds of 4×10^6 paths; $k \le 8$; $\delta = 0.02$, 0.05, 0.1. Tolerance: an excess over $1/(2j)$ of at most five standard errors.

Result. Table 6. At $k = 0$ all eight estimates agree with $1/2$ within 2.18 standard errors. The bound is respected at every j for all four laws and both n, the largest excess of an estimate over it being 2.42 standard errors. Gaussian and uniform steps lie on the bound (ρ close to 1); Cauchy and stable steps fall below it. With $K = 4$ at $n = 200$ ($\delta = 0.02$) the edge means are 0.5065, 0.5067, 0.5032 and 0.5010 (Gaussian, uniform, Cauchy, stable) against the bound 0.5051; at $\delta = 0.1$ they range from 0.5054 to 0.5275 against 0.5269. Every mean decreases towards $1/2$ as δ decreases.

Error. Every excess over the bound is within sampling error, at most 2.42 standard errors against a tolerance of five, and the values at $k = 0$ agree with $1/2$ within 2.18 standard errors.

Source. Monte Carlo, 1.2×10^7 paths per law and n, estimates symmetrized over $\pm\tau$; `derive.py`.

Columns. The step law and n; $nP(\tau = n-k)$ at $k = 0, 1, 4, 8$; the bound $n/(2(n-8))$ of Theorem 4(i) at $k = 8$; the mean of $nP(\tau = n-k)$ over $0 \le k \le K$ with $K = n/10$, and the mean of the bound over the same range.

Table 6: The edge of the law of τ: $nP(\tau = n-k)$ against the universal bound and the limit one half.

step law	n	$k=0$	$k=1$	$k=4$	$k=8$	**bound,** $k=8$	**mean,** $k \le K$	**bound, mean**
Gaussian	50	0.4998	0.5088	0.5422	0.5904	0.5952	0.5266	0.5270
Gaussian	200	0.5014	0.5058	0.5100	0.5164	0.5208	0.5272	0.5269
uniform	50	0.4993	0.5124	0.5432	0.5919	0.5952	0.5276	0.5270
uniform	200	0.4982	0.5040	0.5132	0.5235	0.5208	0.5275	0.5269
Cauchy	50	0.5004	0.5070	0.5224	0.5435	0.5952	0.5137	0.5270
Cauchy	200	0.5045	0.5022	0.5033	0.5092	0.5208	0.5138	0.5269
stable $\alpha = 1/2$	50	0.5005	0.5042	0.5085	0.5148	0.5952	0.5059	0.5270
stable $\alpha = 1/2$	200	0.4970	0.5026	0.5034	0.5050	0.5208	0.5054	0.5269

5. Lattice Walks and Ties

Theorem 5 (The tie identity)**.** Under (H1) alone,

$$\mathbb{E}\Big[\frac{\mathbf{1}\{0 \in K\}}{|K|}\Big] = \frac{1}{2n}, \qquad \text{hence} \qquad P(K = \{0\}) \le \frac{1}{2n} \le P(0 \in K).$$

Proof. The weights $w_r = \mathbf{1}\{r \in K\}/|K|$, $r \in \mathbb{Z}/2n\mathbb{Z}$, *sum to* 1. *Since* $K(\theta_r P) = K(P) - r$, $w_r(P) = w_0(\theta_r P)$, *and by Lemma 1 and (H1)* $\mathbb{E}[w_r(P(\xi))] = \mathbb{E}[w_0(P(T^r\xi))] = \mathbb{E}[w_0(P(\xi))]$. *Summing over the* $2n$ *positions,* $2n\,\mathbb{E}[w_0] = 1$. *The inequalities follow from* $\mathbf{1}\{K = \{0\}\} \le w_0 \le \mathbf{1}\{0 \in K\}$. □

For a walk with ties, $\{K = \{0\}\}$ is the strict event of Theorem 1 and $\{0 \in K\}$ the weak one. Picking one maximizer of the loop uniformly at random, and declaring $\tau = n$ when it is position 0, gives exactly $1/(2n)$ for every law satisfying (H1), such as steps ± 1.

Proposition 2 (The lattice conventions linked)**.** Let the steps be i.i.d. with $P(\xi_i = 1) = P(\xi_i = -1) = 1/2$. Write $\text{strict}(n) = P(S_j < 0,\ 1 \le j \le n;\ S_j > S_n,\ 0 \le j < n)$, $\text{weak}(n) = P(S_j \le 0 \text{ and } S_j \ge S_n,\ 0 \le j \le n)$, and $\text{first}(n)$ for the probability of $\tau = n$ when $t_{\max}$ and $t_{\min}$ are the first times the extremes are reached, the rule of [1]. Then $\text{strict}(n+1) = \text{first}(n)/2$ and $\text{weak}(n) = 2\,\text{first}(n+1)$. In particular, if $n\,\text{first}(n) \to 1/2$, then $n\,\text{strict}(n) \to 1/4$ and $n\,\text{weak}(n) \to 1$.

Proof. Under the first-time rule, $t_{\max} = 0$ exactly when $M = 0$, that is $S_j \le 0$ for all j, and $t_{\min} = n$ exactly when $S_j > S_n$ for all $j < n$; so $\mathrm{first}(n) = P(S_j \le 0\ \forall j,\ S_j > S_n\ \forall j < n)$. *The strict event at $n+1$ forces $\xi_1 = -1$. Given that, $S'_j = S_{j+1} + 1$, $0 \le j \le n$, is a walk of n fair steps, and on the integers $S_{j+1} < 0$ means $S'_j \le 0$, while $S_{j+1} > S_{n+1}$ means $S'_j > S'_n$ and $S_0 > S_{n+1}$ means $S'_n \le 0$; so the strict event at $n+1$ is $\{\xi_1 = -1\}$ intersected with the first-time event of S', and* $\mathrm{strict}(n+1) = \mathrm{first}(n)/2$. *The first-time event at $n+1$ forces $\xi_{n+1} = -1$, since $S_{n+1} < S_n$. Given that, $S_{n+1} = S_n - 1$, and on the integers $S_j > S_n - 1$ means $S_j \ge S_n$, while $S_{n+1} \le 0$ follows from $S_n \le 0$; so the first-time event at $n+1$ is $\{\xi_{n+1} = -1\}$ intersected with the weak event at n, and* $\mathrm{first}(n+1) = \mathrm{weak}(n)/2$. *The limits follow by substitution.* □

Since $n\,\mathrm{first}(n) \to 1/2$ for the lattice walk [1, App. C], the strict and weak values straddle $1/(2n)$ by factors of two at large n.

Proposition 3 (Ties broken by an independent continuous walk)**.** Theorem 1 holds verbatim for walks with values in an abelian group carrying a translation-invariant total order. Consequently, if $\xi_1, \dots, \xi_n$ are i.i.d. fair ± 1 steps and $\zeta_1, \dots, \zeta_n$ are i.i.d. standard Gaussian and independent of them, and if ties in the time of the maximum of S are broken in favor of the highest value of $Z_j = \zeta_1 + \cdots + \zeta_j$ and ties in the time of the minimum in favor of the lowest, then $P(\tau = n) = 1/(2n)$ for every n.

Proof. Lemmas 1 and 2 and the proof of Theorem 1 use only addition and the fact that a rotation subtracts one constant from every loop value, which preserves a translation-invariant order; so they hold in such a group. Take $\mathbb{R}^2$ with the lexicographic order and the steps (ξ_i, ζ_i). (H1) holds because both coordinates are i.i.d. and symmetric. (H2′) holds because $\sum_{i \in I} \varepsilon_i (\xi_i, \zeta_i) = 0$ requires $\sum_{i \in I} \varepsilon_i \zeta_i = 0$, an event of probability 0. The lexicographic maximum of the walk (S_j, Z_j) is attained among the maximizers of S, at the one with the largest Z_j, and the lexicographic minimum among the minimizers of S, at the one with the smallest Z_j; these are the tie-broken times, and Theorem 1 gives $P(\tau = n) = 1/(2n)$. □

Example 2 (No deterministic rule is rotation compatible)**.** Take $n = 3$ and $\xi = (1, -1, 1)$. The loop is $(P_0, \dots, P_5) = (0, 1, 0, 1, 0, 1)$, with $K = \{1, 3, 5\}$, and $T\xi = (-1, 1, -1)$, $T^2\xi = (1, -1, 1) = \xi$. A rule $a(\xi) \in K$ that commutes with rotation, $a(T\xi) = a(\xi) - 1$, would give $a(\xi) = a(T^2\xi) = a(\xi) - 2 \pmod 6$, which is impossible.

So no deterministic choice, such as the first-time or last-time rule, can carry the proof of Theorem 1 to walks with ties; Theorem 5 and Proposition 3 use a random choice instead.

Method. Exact enumeration of all 2^n paths of the fair ± 1 walk in rational arithmetic, under the strict, first-time, last-time and weak rules and the uniform split of Theorem 5; a dynamic program over the strip between the running extremes for large n; and Monte Carlo with a Gaussian tie-breaking walk for Proposition 3.

Parameters. Enumeration for $n \le 22$; dynamic program to $n = 400$; tie-breaking walk at $n = 3, 4, 5, 6, 8, 10$, three seeds of 10^6 paths.

Result. Table 7. The uniform split gives exactly $1/(2n)$ at every $n \le 22$, strict $\le 1/(2n) \le$ weak throughout, and the identities of Proposition 2 hold exactly on every row. The first-time value is $1/8$ at $n = 3$, [1, Eq. (187)], and equals $1/(2n)$ only at $n = 1$, 2 and 4. The dynamic program agrees exactly with the enumeration up to $n = 22$, and at $n = 400$ gives $n\,\mathrm{strict}(n) = 0.25042$, $n\,\mathrm{first}(n) = 0.49958$ and $n\,\mathrm{weak}(n) = 0.99668$, approaching $1/4$, $1/2$ and 1. With ties broken by the Gaussian walk, $2nP(\tau = n)$ is 0.9993 ± 0.0013, 0.9966 ± 0.0015, 0.9954 ± 0.0017, 1.0000 ± 0.0019, 0.9994 ± 0.0022 and 1.0030 ± 0.0025 at the six values of n.

Error. None for the enumeration. The tie-broken frequencies lie within 2.63 standard errors of $1/(2n)$. As the first-time value equals $1/(2n)$ at $n = 4$, $P(\tau = n)$ alone misses the lattice control there (0.9993 in Table 2); its $2n$ cells detect it at every n.

Source. Exact enumeration of all 2^n paths in rational arithmetic; `derive.py`.

Columns. n; $P(\tau = n)$ under the strict rule, the first-time rule of [1], the weak rule, and the uniform split of Theorem 5; and $2n$ times the first-time value.

Table 7: The probability of $\tau = n$ for the fair ± 1 walk under four rules for ties, exactly.

n	**strict**	**first time**	**weak**	**split** $1/\lvert K\rvert$	$2n\times$ **first time**
1	1/2	1/2	1/2	1/2	1.0000
2	1/4	1/4	1/4	1/4	1.0000
3	1/8	1/8	1/4	1/6	0.7500
4	1/16	1/8	3/16	1/8	1.0000
5	1/16	3/32	5/32	1/10	0.9375
6	3/64	5/64	9/64	1/12	0.9375
7	5/128	9/128	15/128	1/14	0.9844
8	9/256	15/256	7/64	1/16	0.9375
10	7/256	49/1024	91/1024	1/20	0.9570
12	91/4096	83/2048	307/4096	1/24	0.9727
16	1065/65536	63/2048	3769/65536	1/32	0.9844
20	3383/262144	12921/524288	49113/1048576	1/40	0.9858
22	49113/4194304	93995/4194304	179775/4194304	1/44	0.9860

6. Continuum Limits

Theorem 6 (The Brownian loop)**.** Let B be a standard Brownian motion on $[0, 1]$, with maximum M at time $t_{\max}$ and minimum m at time $t_{\min}$, and define the loop on the circle $\mathbb{R}/2\mathbb{Z}$ by $L(t) = B(t)$ and $L(1 + t) = B(1) - B(t)$ for $t \in [0, 1]$. Then L has almost surely

a unique maximizer U, and U is uniform on $[0, 2)$. Equivalently, for $0 < u < 1$,

$$P(t_{\max} \in \mathrm{d}u,\ M + m > B_1) = \tfrac{1}{2}\,\mathrm{d}u, \qquad P(t_{\min} \in \mathrm{d}u,\ M + m < B_1) = \tfrac{1}{2}\,\mathrm{d}u,$$

$P(M + m > B_1) = 1/2$, and $P(t_{\max} \in \mathrm{d}u,\ M + m < B_1) = (1/(\pi\sqrt{u(1-u)}) - \tfrac{1}{2})\,\mathrm{d}u$.

Proof. For $s \in [0, 1]$ the rotated loop $\theta_s L(t) = L(t + s) - L(s)$ is the loop of

$$W(u) = \begin{cases} B(s + u) - B(s), & 0 \le u \le 1 - s, \\ B(1) - B(s) - B(u - 1 + s), & 1 - s \le u \le 1, \end{cases}$$

as a direct check of the two halves of $\theta_s L$ shows. W is a Brownian motion: it runs through the increments of B on $[s, 1]$ and then through those on $[0, s]$ with their sign changed, and these are independent and Brownian. Likewise $\theta_1 L$ is the loop of $-B$, and $\theta_{1+s} = \theta_s\theta_1$. So every rotation of L has the law of L. The maximizer and the minimizer of B are almost surely unique and interior, and (M, m, B_1) has a joint density (the reflection series, as in Feller [14]), so $P(M + m = B_1) = 0$. The loop's maximum is $\max(M, B_1 - m)$, attained at $t_{\max}$ in the first half and at $1 + t_{\min}$ in the second, so it is attained once. Since $U(\theta_s L) = U(L) - s$, the law of U is invariant under every rotation of the circle, and the only such probability law is the uniform one. Finally $U \in \mathrm{d}u$ with $0 < u < 1$ means $t_{\max} \in \mathrm{d}u$ and $M > B_1 - m$, and $U \in 1 + \mathrm{d}u$ means $t_{\min} \in \mathrm{d}u$ and $B_1 - m > M$; the last formula subtracts the first from Lévy's arcsine density of $t_{\max}$. □

The proof uses only that B is symmetric with stationary independent increments and that its loop has a unique maximizer; we do not prove that uniqueness for other Lévy processes.

Method. Brownian paths on a grid, with the extremes inside each grid interval drawn exactly from the Brownian bridge between the grid values, $\frac{1}{2}(x+y\pm\sqrt{(y-x)^2 - 2h\log V})$ with V uniform, and their times placed at the interval midpoints. Histograms of U, of $t_{\max}$ on each side of $M + m = B_1$ and of τ are tested by chi-square against Theorem 6 and the densities f_{BM}, f_{BB} of [1]; the moments of τ are compared with [1] by batch means. The Brownian bridge is the control.

Parameters. 4000 steps, $h = 2.5 \times 10^{-4}$; three seeds of 10^6 paths for each process; 40 bins for U and τ, 20 for $t_{\max}$; chi-square $p \ge 10^{-5}$, moments within five standard errors.

Result. Pooled, U is uniform on $[0, 2)$ ($p = 0.194$), $t_{\max}$ is uniform on $\{M + m > B_1\}$ ($p = 0.438$), and its density on the complement is the arcsine density minus $1/2$ ($p = 0.233$); the smallest per-seed p-value is 0.0496. The frequency of $M + m > B_1$ is 0.50066, 2.28 standard errors from $1/2$. The histograms of τ match f_{BM} ($p = 0.44$) and f_{BB} ($p = 0.128$), and the moments in Table 8 agree with [1] within 1.09 standard errors. The doubled bridge loop's maximizer is far from uniform: its chi-square p-value underflows to 0 (Figure 3).

Error. Every test passed and every moment lies within 1.09 standard errors. Placing an extreme at the midpoint of its interval errs by at most $h/2$, far below the bin width of 0.05.

Source. Monte Carlo, 3×10^6 paths of 4000 steps with exact extremes between grid points, batch-means standard errors; exact values from the closed forms of [1]; `derive.py`.

Columns. The quantity; for Brownian motion and then the Brownian bridge on $[0, 1]$, its simulated value, with the standard error in units of the last digit in parentheses, and its exact value (the frequency of $M + m > X_1$ is simulated only).

Table 8: Moments of τ for Brownian motion and the Brownian bridge against the exact values of Mori, Majumdar and Schehr.

quantity	**Brownian motion**	**exact**	**Brownian bridge**	**exact**
$\langle \lvert\tau\rvert \rangle$	0.59086(13)	0.59086	0.43005(10)	0.42996
$\langle \tau^2 \rangle$	0.40087(16)	0.40090	0.21508(9)	0.21498
$\mathrm{cov}(t_{\min}, t_{\max})$	-0.07544(5)	-0.07545	-0.02417(4)	-0.02416
$P(M + m > X_1)$	0.50066		0.49970	

Script. `fig_brownian.py`: the pooled histograms of the loop maximizer and of τ, with f_{BM} and f_{BB} evaluated from the formulas of [1], all read from `results.npz`.

Shows. Theorem 6, its failure for the bridge, and the bound of Corollary 6.

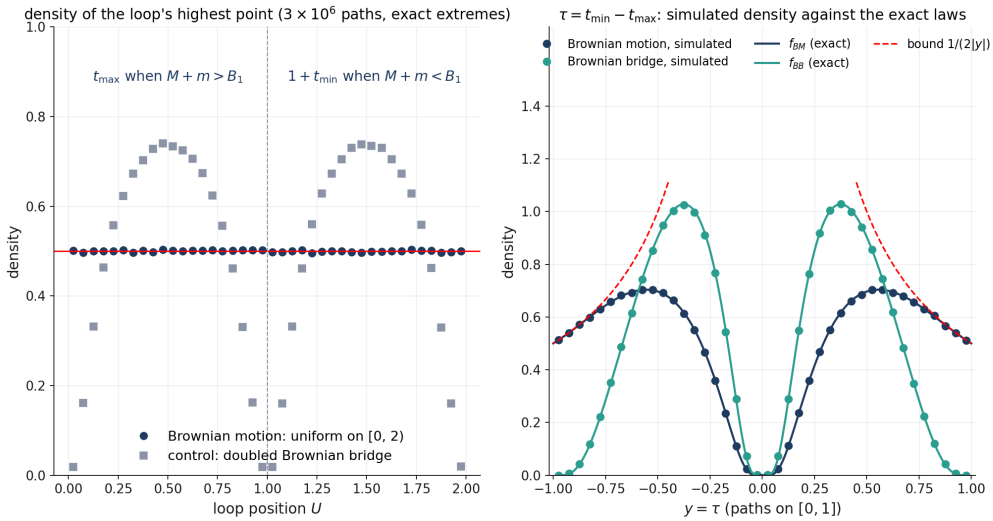


Figure 3: Left, the density of the highest point of the doubled Brownian loop over its two halves, flat at one half for Brownian motion and humped for the doubled Brownian bridge; right, the simulated density of tau for Brownian motion and the Brownian bridge against the exact laws, with the bound one over twice the absolute value of y.

Both halves of the circle are flat on the left. On the right f_{BM} meets the bound $1/(2|y|)$ at the edge and stays below it (Remark 4), while f_{BB} vanishes at $y = \pm 1$.

Theorem 7 (The averaged edge value for stable domains of attraction)**.** Let $\xi_1, \xi_2, \ldots$ be

i.i.d. with a symmetric law without atoms in the domain of attraction of a symmetric α-stable law, $0 < \alpha \le 2$. Then

$$\lim_{\delta\downarrow 0} \liminf_{n\to\infty} \frac{1}{\delta}\, P(\tau \ge (1-\delta)n) = \lim_{\delta\downarrow 0} \limsup_{n\to\infty} \frac{1}{\delta}\, P(\tau \ge (1-\delta)n) = \frac{1}{2}.$$

Proof. Put $K = \lfloor \delta n \rfloor$, so that $\{\tau \ge (1-\delta)n\} = \{\tau \ge n - K\}$, and let $H_j = \sum_{i\le j} 1/i$. Upper bound. By Theorem 4(i), $P(\tau \ge n-K) \le \frac{1}{2}\sum_{j=n-K}^{n} 1/j = \frac{1}{2}(H_n - H_{n-K-1})$, which tends to $\frac{1}{2}\log(1/(1-\delta))$ as $n\to\infty$; divided by δ, this tends to $1/2$ as $\delta \to 0$. Lower bound. By Theorem 3 and the proof of Theorem 4(ii), $1-\rho_{a,b,L} \le \pi(a,L) + \pi(b,L)$ with $\pi(a,L) = P(R_a \ge |S'_L|)$, S'_L an independent walk. Let c_n be the norming sequence, $S_n/c_n \to Y$ in law with Y symmetric α-stable. By the invariance principle for walks conditioned to stay positive (Iglehart [15] and Bolthausen [16] for finite variance, Doney [17] for asymptotically stable walks, in the form recalled as Eq. (3.3) of Caravenna and Chaumont [18]) and the continuity of the supremum in the Skorokhod topology, R_a/c_a converges in law as $a\to\infty$; hence the family $\{R_a/c_a : a \ge 1\}$ is tight. The sequence c_n is regularly varying of index $1/\alpha$, so Potter's bounds give a constant C with $c_a/c_L \le C(a/L)^{1/(2\alpha)}$ for $a_0 \le a \le L$, and the finitely many $a < a_0$ have R_a of fixed law. For $a \le \delta n$ and $L \ge (1-\delta)n$, and any $x > 0$,

$$\pi(a,L) \le P(R_a > x\, c_a) + P\Big(\frac{|S'_L|}{c_L} \le Cx\Big(\frac{\delta}{1-\delta}\Big)^{1/(2\alpha)}\Big) + o(1),$$

where the $o(1)$, as $n\to\infty$, covers $a < a_0$. As $n\to\infty$ the second term tends to the probability that $|Y|$ lies below the same level, and $|Y|$ has a continuous law with no atom at 0. Choosing x large and then δ small makes $s(\delta) = \limsup_n \sup\{\pi(a,L) : a \le \delta n,\ L \ge (1-\delta)n\}$ tend to 0. Then

$$P(\tau \ge n-K) = \sum_{k=0}^{K} \sum_{a+b=k} \frac{q_a q_b\, \rho_{a,b,n-k}}{2(n-k)} \ge (1 - 2s(\delta) - o(1))\, \tfrac{1}{2}(H_n - H_{n-K-1}),$$

and the lower limit, divided by δ, tends to $1/2$ as $\delta \to 0$. □

The only outside input is the invariance principle for the conditioned walk, used as recalled in [18]. The theorem averages over a vanishing edge window and does not give the existence of a scaling limit.

Corollary 6 (Consequences for the scaling densities)**.** Let the steps be i.i.d. with a symmetric law without atoms, and suppose that τ/n converges in law to a limit with density f on $[-1,1]$, as assumed in [1, Eqs. (16) and (183)]. Then (a) $f(y) \le 1/(2y)$ for almost every $y \in (0,1]$; (b) if the step law is in the domain of attraction of a symmetric α-stable law, $0 < \alpha \le 2$, then $\delta^{-1}\int_{1-\delta}^{1} f(y)\,\mathrm{d}y \to 1/2$ as $\delta \downarrow 0$, so $f(1) = 1/2$ whenever f is continuous at 1; (c) if $f(1) = 1/2$ and f has a left derivative at 1, then $f'(1-) \ge -1/2$.

Proof. For $0 < y_1 < y_2 \le 1$, convergence in law to a law with a density gives $P(y_1 n < \tau \le y_2 n) \to \int_{y_1}^{y_2} f$, while Theorem 4(i) bounds the left side by $\sum_{y_1 n < j \le y_2 n} 1/(2j) \to \frac{1}{2}\log(y_2/y_1) = \int_{y_1}^{y_2} \mathrm{d}y/(2y)$. An inequality between the integrals over every interval gives

the pointwise bound almost everywhere, by Lebesgue's differentiation theorem. (b) $\int_{1-\delta}^{1} f = \lim_n P(\tau > (1-\delta)n)$*, which differs from* $P(\tau \geq (1-\delta)n)$ *by at most one point mass, of size at most* $1/(2(1-\delta)n)$ *by Theorem 4(i); Theorem 7 gives the limit of the average, and a continuous* f *equals the limit of its averages. (c) By (a), for almost every small* $h > 0$*,* $f(1-h) \leq 1/(2(1-h)) = \frac{1}{2} + \frac{1}{2}h + O(h^2)$*, while* $f(1-h) = \frac{1}{2} - hf'(1-) + o(h)$*; comparing gives* $-f'(1-) \leq \frac{1}{2}$*.* □

Part (b) proves in averaged form the edge value $f_\mu(\pm 1) = 1/2$ that [1, Fig. 7] observed for every index μ; part (a) needs no stable domain of attraction.

Remark 4 (The Brownian density against the bound). For Brownian motion on $[0, 1]$, the exact density of $y = \tau$ [1, Eq. (3)], with the Abel value $1/2$ of $\sum_{k\geq 1}(-1)^{k+1}$ [1, Eq. (58)], reads, for $0 < y < 1$,

$$f_{\mathrm{BM}}(y) = \frac{1}{2y} - \frac{1}{y}\sum_{k\geq 1}(-1)^{k+1} \times \operatorname{sech}^2\Big(\frac{k\pi}{2}\sqrt{\frac{y}{1-y}}\Big).$$

The series alternates and its terms decrease in k, since sech^2 decreases on $[0, \infty)$, so its sum lies strictly between 0 and its first term, $\operatorname{sech}^2(\frac{\pi}{2}\sqrt{y/(1-y)}) \leq 4e^{-\pi\sqrt{y/(1-y)}}$. Hence $f_{\mathrm{BM}}(y) < 1/(2y)$, and the gap is $O(e^{-\pi/\sqrt{1-y}})$ as $y \to 1$.

So f_{BM} is the bound of Corollary 6(a) minus a correction that vanishes with its derivative faster than any power of $1 - y$: its edge value $1/2$ [1, Eq. (4)] and edge slope $-1/2$ are those of $1/(2y)$.

Method. Evaluate f_{BM} from the formulas of [1]: the Poisson-resummed series of their Eq. (64) for $y \leq 1/2$ and the Abel-summed form of Remark 4 for $y > 1/2$. Compare the two forms, the normalization and moments with [1], and f_{BM} with $1/(2y)$ on a grid and integrated over the edge window $[1 - \delta, 1]$.

Parameters. 400 odd terms for Eq. (64) and 200 terms for the Abel form; 2000 grid points on $(0, 1)$; $\delta = 0.01$, 0.05, 0.1 and 0.5; tolerances 10^{-12} between the two forms and 10^{-6} for integrals.

Result. The two series agree to 7.77×10^{-16} on $[0.3, 0.7]$; $2\int_0^1 f_{\mathrm{BM}} = 1$ and the mean of $|\tau|$ and of τ^2 reproduce $(4\log 2 - 1)/3$ and $(7\zeta(3) - 2)/16$ of [1]. On all 2000 points $f_{\mathrm{BM}}(y) \leq 1/(2y)$, the largest value of $f_{\mathrm{BM}}(y) - 1/(2y)$ being 0, and $f_{\mathrm{BM}}(1 - 10^{-6}) = 0.500000500001$. The integrated gap $\frac{1}{2}\log(1/(1-\delta)) - \int_{1-\delta}^{1} f_{\mathrm{BM}}$ is 9.1×10^{-17} at $\delta = 0.01$, 2.7×10^{-8} at $\delta = 0.05$, 5.4×10^{-6} at $\delta = 0.1$ and 0.032 at $\delta = 0.5$, so the edge average of Corollary 6(b) is $1/2$ to within rounding for the smallest window.

Error. Rounding error between the two forms, within 10^{-6} for the integrals; the gap to the bound is positive and collapses as Remark 4 predicts.

Script. `fig_pn.py`: $2nP(\tau = n)$ from Tables 2 and 7 and from the tie-breaking runs, and the edge at $n = 200$ from Table 6, all read from `results.npz` and `numbers.json`.

Shows. Theorem 1 across laws and tie rules, and Theorem 4 at the edge.

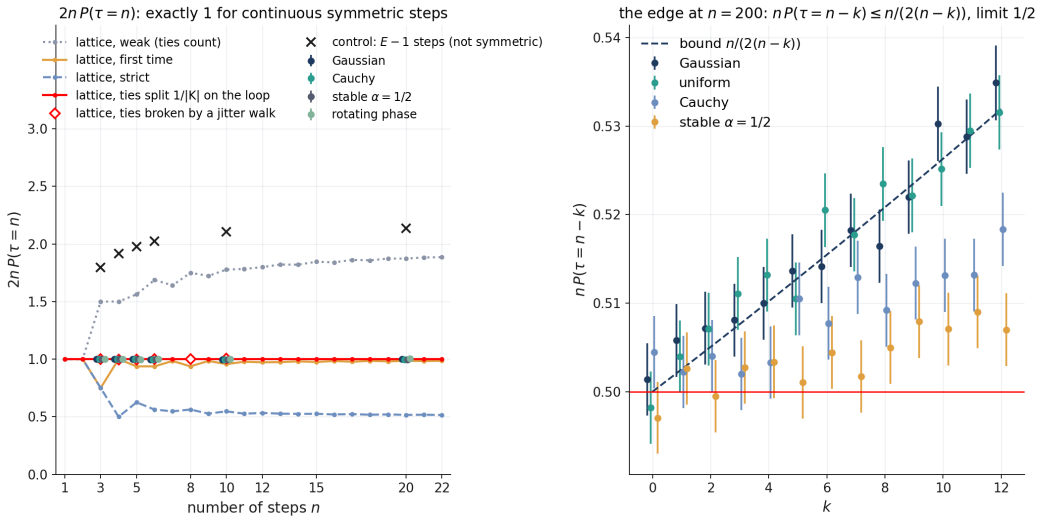


Figure 4: Left, twice n times the probability that tau equals n against n, at one for continuous symmetric steps, for the uniform split of ties and for ties broken by a jitter walk, but not for the strict, first-time and weak lattice rules or for asymmetric steps; right, n times the probability that tau equals n minus k at n equal to two hundred for four laws, under the bound and approaching one half.

On the left the strict and weak lattice rules settle near 1/2 and 2 (Proposition 2), and the first-time rule touches 1 only at $n = 1$, 2 and 4. On the right, with bars of two standard errors, the light-tailed laws ride the dashed bound within Monte Carlo error and the heavy-tailed laws stay under it.

Remark 5 (What is proved for Lévy flights). For Lévy flights with an atomless symmetric jump law, the finite-n statement $P(\tau = n) = 1/(2n)$ of [1, Eq. (184)] is Corollary 1, for every n. The edge value $f_\mu(\pm 1) = 1/2$ of their scaling function is proved here in the averaged form of Theorem 7 for every index $0 < \mu \leq 2$, and as a pointwise value under the assumption that the scaling limit exists and its density is continuous at the edge (Corollary 6). Neither of those two assumptions is proved here.

What is missing is the convergence of $(t_{\max}, t_{\min})/n$ in a stable domain of attraction, which needs the continuity of the argmax map at stable paths, and the regularity of the limit density at the edge.

7. Bridges and Interfaces

Remark 6 (Why the device does not reach bridges). For a bridge, $S_n = 0$, but T does not preserve that constraint: the coordinates of $T\xi$ sum to $\sum_i \xi_i - 2\xi_1$. The doubled loop of a bridge is therefore not rotation invariant, and it carries a built-in tie, $P_0 = P_n = 0$. Its maximum, $\max(M, -m)$, is positive unless the path is identically zero, so it is never at position 0 or n, and $\tau = \pm n$ is impossible.

This agrees with $f_{\mathrm{BB}}(\pm 1) = 0$ in [1]; the doubled Brownian bridge serves as the control of Theorem 6 (Figure 3).

Proposition 4 (The cyclic laws of a bridge)**.** Let ξ be exchangeable with $\sum_i \xi_i = 0$ almost surely, and suppose the values $S_0, \dots, S_{n-1}$ are almost surely distinct. Read $t_{\max}$ and $t_{\min}$ in $\{0, \dots, n-1\}$, identifying step n with step 0, and put $D = (t_{\min} - t_{\max}) \bmod n$. Then $t_{\max}$ is uniform on $\{0, \dots, n-1\}$ and independent of D, and for $1 \le d \le n-1$

$$P(\tau = d) = \Big(1 - \frac{d}{n}\Big) P(D = d), \qquad P(\tau = -(n-d)) = \frac{d}{n}\, P(D = d).$$

Proof. The walk of a bridge is already a loop on $\mathbb{Z}/n\mathbb{Z}$, and its rotation by r, $S_{j+r} - S_r$, is the walk of the cyclically shifted steps $(\xi_{r+1}, \dots, \xi_n, \xi_1, \dots, \xi_r)$, which have the law of ξ by exchangeability. The proofs of Theorem 1 and Corollary 3, with $\mathbb{Z}/n\mathbb{Z}$ in place of $\mathbb{Z}/2n\mathbb{Z}$, show that $t_{\max}$ is uniform and that the walk rotated to its maximum is independent of $t_{\max}$; D is the position of the minimum of the rotated walk, so it is independent of $t_{\max}$ too. Given $D = d$, $\tau = d$ when $t_{\max} \le n-1-d$, which has probability $(n-d)/n$, and $\tau = d-n$ otherwise. □

This is the cycle lemma itself [6], [7], and Vervaat's rotation [8] for the Brownian bridge. The first formula is the discrete form of $f_{\mathrm{BB}}(y) = (1-y) f_{\mathrm{BE}}(y)$ [1, Eq. (109)]. The law of D depends on the step law.

Method. Exact cyclic enumeration of the $n!$ orderings of generic zero-sum vectors, and Monte Carlo for exchangeable zero-sum loops. For Corollary 7, the interface $h = B - \int_0^1 B$ is built from every Brownian path of Theorem 6, and $h_{\max} + h_{\min} - h(0) - h(1)$ is compared with $M + m - B_1$ path by path.

Parameters. Enumeration for $n = 3, \dots, 7$, three vectors each. Monte Carlo at $n = 6$ for Gaussian steps minus their mean and for centered uniform and Cauchy steps, 3×10^6 loops each. Interface tolerance 10^{-12}.

Result. In the enumeration each of the n positions holds the maximum in exactly $(n-1)!$ orderings, for every vector, while the counts of D differ between vectors for $n = 4, \dots, 7$. In the simulation $P(t_{\max} = k)$ deviates from $1/6$ by at most 0.00040, independence from D passes with $p \ge 0.178$, and $P(\tau = d) = (1 - d/n) P(D = d)$ holds within 1.20 standard errors. The law of D is not universal: $P(D = 1)$, $P(D = 2)$, $P(D = 3)$ are 0.1087, 0.2452, 0.2916 for the Gaussian loop and 0.2506, 0.1648, 0.1689 for the Cauchy loop, apart by $|z| = 462$. No doubled bridge loop peaks at position 0 or n. The interface identity holds on every path, with a largest discrepancy of 8.9×10^{-16}.

Error. Exact for the enumeration; within 1.20 standard errors for Proposition 4; rounding error only for the interface identity, which is algebraic.

Corollary 7 (A free-boundary Edwards–Wilkinson interface)**.** Let $h(x) = B(x) - \int_0^1 B(u)\,\mathrm{d}u$, $x \in [0,1]$, be the stationary Edwards–Wilkinson interface with free ends in the form used by [1, Eq. (200)], with highest point $h_{\max}$ at $x_{\max}$ and lowest point $h_{\min}$ at $x_{\min}$. Then $P(h_{\max} + h_{\min} > h(0) + h(1)) = 1/2$; given this event $x_{\max}$ is uniform on $(0,1)$, and given the reverse inequality $x_{\min}$ is uniform on $(0,1)$.

Proof. Subtracting the constant $c = \int_0^1 B$ moves every value by $-c$ and no position, so $x_{\max} = t_{\max}$, $x_{\min} = t_{\min}$ and $h_{\max} + h_{\min} - h(0) - h(1) = (M-c) + (m-c) - (-c) - (B_1 - c) =$

$M + m - B_1$. *Theorem 6 gives the three statements.* □

By Brownian scaling the same holds on an interval of any length. For periodic boundaries, where the interface is a Brownian bridge, the device adds nothing to the uniform position of the maximum (Proposition 4).

8. Limits

Remark 7 (What is not proved). (a) For Lévy flights, the pointwise edge value $f_\mu(\pm 1) = 1/2$ and the existence of the scaling limit of $(t_{\max}, t_{\min})/n$ (Remark 5); Theorem 7 is an average over a vanishing edge window. (b) A loop theorem for symmetric Lévy processes other than Brownian motion, which needs the uniqueness of the loop's maximizer. (c) The bound $P(\tau = j) \le 1/(2j)$ of Theorem 4(i) for exchangeable laws that are not i.i.d.: the exact enumeration finds it on every generic magnitude set with $n \le 7$, but the proof of Theorem 4 uses independence through Theorem 3. (d) Theorem 7 imports the invariance principle for walks conditioned to stay positive, which we use as recalled in [18] and have not re-derived.

Two further limits concern the record. The checks of the combinations that Theorem 2 excludes allow a fraction 10^{-5} of floating-point ties; one stable path used it, and that check was relaxed only after the path had been examined. And we claim novelty only to our knowledge: the works citing [1] that we found contain no proof, but we could not consult Chapter 5 of [10].

9. Conclusion

Doubling a walk with its negated steps closes it into a loop whose rotations are the doubled walks of $T^r\xi$, so for any step law invariant under T and free of ties the highest point of the loop is uniform on its $2n$ positions. Position 0 is the event that the maximum comes first and the minimum last, which proves the conjecture of Mori, Majumdar and Schehr, $P(\tau = n) = 1/(2n)$, for every n and every i.i.d. symmetric atomless law, and for dependent laws as well; the other positions are $2n$ explicit events of the same probability. The rest of the joint law of the two times depends on the step law, through one probability for i.i.d. steps, which bounds $P(\tau = j)$ by $1/(2j)$ and fixes the edge, in averaged form for every stable domain of attraction. The Brownian loop has a uniform maximizer, and on the lattice a uniform split of ties restores the law. The pointwise edge value for Lévy flights, a loop theorem for other Lévy processes and the bound for exchangeable laws that are not i.i.d. remain open.

References

[1] Mori, F., Majumdar, S. N., & Schehr, G. (2020). Distribution of the time between maximum and minimum of random walks. *Physical Review E*. https://doi.org/10.1103/PhysRevE.101.052111

[2] Mori, F., Majumdar, S. N., & Schehr, G. (2019). Time between the maximum and the minimum of a stochastic process. *Physical Review Letters*. https://doi.org/10.1103/PhysRevLett.123.200201

[3] Sparre Andersen, E. (1953). On the fluctuations of sums of random variables. *Mathematica Scandinavica*. https://doi.org/10.7146/math.scand.a-10385

[4] Sparre Andersen, E. (1954). On the fluctuations of sums of random variables II. *Mathematica Scandinavica*. https://doi.org/10.7146/math.scand.a-10407

[5] Majumdar, S. N. (2010). Universal first-passage properties of discrete-time random walks and Lévy flights on a line: statistics of the global maximum and records. *Physica A*. https://doi.org/10.1016/j.physa.2010.01.021

[6] Dvoretzky, A., & Motzkin, Th. (1947). A problem of arrangements. *Duke Mathematical Journal*. https://doi.org/10.1215/S0012-7094-47-01423-3

[7] Spitzer, F. (1956). A combinatorial lemma and its application to probability theory. *Transactions of the American Mathematical Society*. https://doi.org/10.1090/S0002-9947-1956-0079851-X

[8] Vervaat, W. (1979). A relation between Brownian bridge and Brownian excursion. *The Annals of Probability*. https://doi.org/10.1214/aop/1176995155

[9] Kabluchko, Z., Vysotsky, V., & Zaporozhets, D. (2017). Convex hulls of random walks, hyperplane arrangements, and Weyl chambers. *Geometric and Functional Analysis*. https://doi.org/10.1007/s00039-017-0415-x

[10] Majumdar, S. N., & Schehr, G. (2024). *Statistics of Extremes and Records in Random Sequences*. Oxford University Press. https://doi.org/10.1093/9780191838781.001.0001

[11] Humphreys, J. E. (1990). *Reflection Groups and Coxeter Groups*. Cambridge University Press. https://doi.org/10.1017/CBO9780511623646

[12] Chambers, J. M., Mallows, C. L., & Stuck, B. W. (1976). A method for simulating stable random variables. *Journal of the American Statistical Association*. https://doi.org/10.1080/01621459.1976.10480344

[13] Rogozin, B. A. (1961). On the increase of dispersion of sums of independent random variables. *Theory of Probability and Its Applications*. https://doi.org/10.1137/1106010

[14] Feller, W. (1951). The asymptotic distribution of the range of sums of independent random variables. *The Annals of Mathematical Statistics*. https://doi.org/10.1214/aoms/1177729589

[15] Iglehart, D. L. (1974). Functional central limit theorems for random walks conditioned to stay positive. *The Annals of Probability*. https://doi.org/10.1214/aop/1176996607

[16] Bolthausen, E. (1976). On a functional central limit theorem for random walks conditioned to stay positive. *The Annals of Probability*. https://doi.org/10.1214/aop/1176996098

[17] Doney, R. A. (1985). Conditional limit theorems for asymptotically stable random walks. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*. https://doi.org/10.1007/BF00534868

[18] Caravenna, F., & Chaumont, L. (2008). Invariance principles for random walks conditioned to stay positive. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*. https://doi.org/10.1214/07-AIHP119

Zamrik Quant Research
The research arm of Zamrik Quant Lab

About the Desk

Stochastic Analysis Desk

Zamrik Quant Research

How random processes behave — Itô calculus, diffusions, local time, Fokker–Planck — in theory and in application.

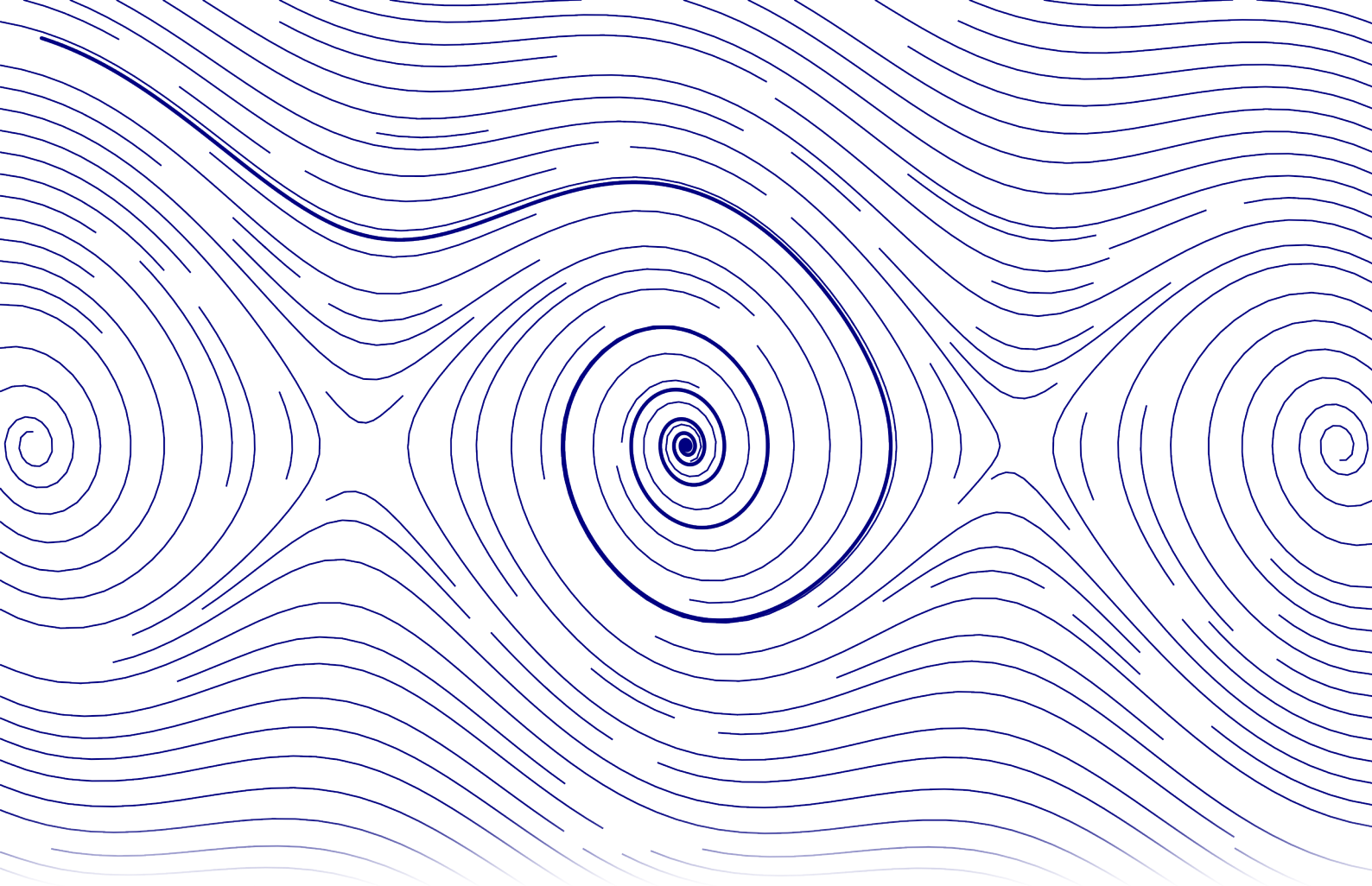
Stochastic analysis is the grammar of every other desk: a price, an order queue, a population or a value function is a functional of a random path, and what can be said about it depends on what can be proved about the path. The Stochastic Analysis desk works on those proofs and on the exact formulas they produce.

Its core questions are the classical ones, asked with care: when a diffusion first reaches a boundary, fixed or moving; how long it spends at each level before it does; and what happens at the boundary itself — reflection, killing, a sticky or skewed point, a reset to the start. The answers come through Itô calculus and martingale methods, local time and Tanaka's formula, Ray–Knight theorems, time changes, and the Feynman–Kac and Fokker–Planck equations that turn questions about paths into equations for expectations and densities. Wherever the problem allows, the desk derives the law in closed form and then checks it against an independently built simulation.

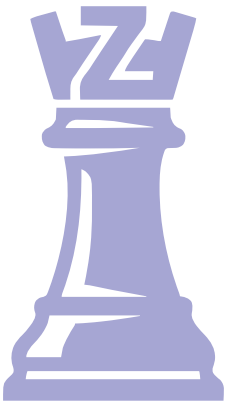
The results travel. The same calculus corrects a barrier that is checked once a day, explains why most stocks lose to the average stock, separates real correlation from the noise of random matrices, sets the acceptance rate of a Metropolis sampler, and models a neuron's membrane and a day's rainfall. The desk also studies the numerics that decide whether a simulated path tells the truth: the strong and weak convergence of Euler–Maruyama and Milstein schemes, Brownian-bridge corrections for barriers monitored in discrete time, and the error left when a continuous process is sampled on a grid.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).

Notes



Phase portrait of the damped pendulum: $\dot{\theta} = \omega,\ \dot{\omega} = -\sin\theta - \delta\,\omega, \quad \delta = 0.23$



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