Dividends When Capital Is Slow: de Finetti's Problem with a Sticky Boundary

Capital Injections That Take Time, the Critical Speed of Rescue, and the Barrier Between de Finetti and Løkka–Zervos

Working Paper · Stochastic Analysis

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Abstract. Dividend models give a firm at zero surplus two fates: it is ruined, or capital arrives instantly and reflects it back into business. Neither is what happens when capital must be raised, which takes time, and while it is being raised the firm can neither pay nor operate. This paper models that interval exactly. The surplus is a Brownian motion with drift that is sticky at zero: while frozen there it is pushed off at the speed θ at which capital arrives, each unit of capital costs ϕ, and each unit of time frozen costs κ. The capital raised is the local time of the surplus at zero, and the time frozen is that local time divided by 2θ in the unit case. The equation has a weak but no strong solution, so the control problem is posed on weak solutions.

The optimal policy is a dividend barrier fixed by one explicit equation, the sticky boundary condition $\theta g_b'(0) - q g_b(0) = \phi\theta + \kappa$ applied to the smooth-fit function. Rescue is worth paying for exactly when $\theta(\bar{\phi} - \phi) > \kappa$, where $\bar{\phi} = \rho^{(\rho-1)/(\rho+1)}$ is a price ceiling for capital that depends on the model only through the ratio ρ of the two characteristic roots. Below the critical speed $\theta^* = \kappa/(\bar{\phi} - \phi)$ the firm should be liquidated at zero and keep de Finetti's buffer; above it the buffer falls continuously, leaving de Finetti's level with a kink, and converges to the instant-injection barrier as $b_{LZ} + c_1/\theta$. The value lost at zero to slow capital is $\phi c_1/\theta$ to first order.

At the house parameters the critical speed is $\theta^* = 0.028906$, the ceiling is $\bar{\phi} = 8.1189$, the kink has slope -10.800 and $c_1 = 1.7351$. A lattice chain that is solved exactly and optimises its own barrier, never touching the closed forms, reproduces the optimal barrier, the value function and the critical speed, with errors of first order in the lattice spacing.

1. Introduction

A firm with a risky surplus faces a choice every time its surplus grows: pay the excess out to shareholders, or keep it as a buffer against the day the surplus falls to zero. De Finetti asked which dividend policy maximises the discounted dividends paid before ruin, and for a Brownian surplus with positive drift the answer is a barrier: pay out everything above a level b, nothing below it. The Brownian problem was solved by Jeanblanc-Picqué and Shiryaev [12], by Radner and Shepp in a model of corporate strategy [13], and by Asmussen and Taksar [1]; Gerber and Shiu work out the barrier and its value in detail [14]. The barrier balances two forces. Every unit kept is a unit paid later and therefore discounted; every unit paid is a unit of buffer lost, and with it some probability of surviving the next bad stretch.

The same structure survives a long list of generalisations, collected in the review [5]: jumps, transaction costs, constraints on the rate of payment, and above all different treatments of what happens at zero. That last choice is not a technicality. The value of the whole policy, and the height of the barrier, are set by what the firm is worth at the moment its surplus runs out, and the classical answers to that question are extreme ones.

In the classical model the firm at zero is ruined and worth nothing. In the models with capital injections it is worth a great deal: shareholders may inject capital at a proportional cost, the surplus is reflected at zero, and the firm never dies [3][4]; Schmidli's monograph collects the control problems of insurance in which these questions arise [15]. The two models give the same answer in one respect, that the optimal dividend policy is still a barrier, and opposite answers in every other: with injections the barrier is lower, because a firm that can always be recapitalised needs less of a buffer, and whether injecting is worth it at all depends on how expensive capital is.

Both models treat the arrival of capital as a switch. Either it never comes, or it comes the instant it is needed, in exactly the amount needed. Real recapitalisation is neither. A rights issue, a bridging loan or a rescue by a parent takes time to arrange, and during that time the firm sits at the bottom: it cannot pay dividends, it may not be able to trade, and it bears the costs of the process itself. A model in which capital arrives at a finite speed is the natural interpolation between ruin and reflection, and it raises a question neither extreme can ask: how fast must capital be for a rescue to be worth paying for?

Corporate finance reached the same boundary from the other side. In the cash-management models of Décamps, Mariotti, Rochet and Villeneuve [16] and of Bolton, Chen and Wang [17], a firm holds cash, pays dividends at a barrier and issues equity at a cost when its cash runs out; the issue, again, is instantaneous. The closest model to the present one is that of Hugonnier, Malamud and Morellec [18], in which capital is slow because investors must be found: they arrive at the jump times of a Poisson process, the firm can raise funds only when one arrives, and a firm whose cash reaches zero before then is liquidated. There the slowness is a search that happens before the boundary. Here it is a wait at the boundary: the firm reaches zero, cannot operate, and survives only if the capital it is raising arrives in time to be worth the wait.

A surplus held at zero while capital trickles in is a sticky boundary. Feller discovered it in 1952 while classifying the boundary behaviour of one-dimensional diffusions: besides absorption, reflection and elastic killing, a diffusion may spend a positive amount of real time at a boundary point without staying there on any interval, and the boundary condition that describes it involves the second derivative [6][20]. Ito and McKean built its paths by slowing the clock of a reflecting Brownian motion whenever it touches zero, in proportion to its local time there [7].

Engelbert and Peskir wrote the equation such a process satisfies. For a reflecting Brownian motion sticky at zero it is $dX_t = \mu\,\mathbf{1}\{X_t = 0\}\,dt + \mathbf{1}\{X_t > 0\}\,dB_t$, and they proved that it has a unique weak solution but no strong one, confirming a conjecture of Skorokhod [8]. Warren had reached the same conclusion by another route, through a conditional law of the driving noise given the sticky path [21], and the dichotomy recurs across the family: the one-sided Tanaka equation with positive drift has a unique weak solution and no strong one [22], and so does a degenerate equation with a sticky point studied by Bass [23]. The drift μ that acts only on the set where the process sits at zero is, in the language of this paper, the rate at which capital is raised. That identification is the whole model: the local time at zero is the capital raised, and the Lebesgue measure of the time at zero, which is positive, is the time the firm spends frozen.

Sticky boundaries have entered applied probability through several doors. In interest-rate modelling, the zero lower bound on nominal rates is sticky rather than reflecting: rates spend long periods at zero and leave it gradually, and Nie and Linetsky solve the Vasicek model with a sticky zero lower bound by spectral methods [10]. In numerical analysis, the fact that the governing equation has no strong solution changes how the process must be simulated, and Bou-Rabee and Holmes-Cerfon study its numerical solution [9]. In control, the classical treatments of diffusions with absorbing and reflecting barriers [2] are the two limits of the sticky one, and the sticky boundary condition is the Feller condition with a running cost attached.

As far as I know, the sticky boundary has not been used to model the arrival of capital in a dividend problem. It is a good fit for two reasons. The model has exactly one new parameter, the speed θ, and it interpolates between the two classical problems as θ runs from zero to infinity. And the new problem remains solvable in closed form, so every effect of the speed of capital can be written down rather than simulated.

The paper establishes four things. First, the optimal policy is a dividend barrier, and the optimal barrier is the unique solution of one explicit equation, which is the sticky boundary condition applied to the smooth-fit function (Theorem 1). A verification theorem shows that no strategy, in the weak formulation the problem requires, does better (Theorem 2).

Second, there is a critical speed of capital. A rescue is worth paying for exactly when $\theta(\bar{\phi} - \phi) > \kappa$: capital arrives at rate θ, each unit of it is worth $\bar{\phi}$ at the brink and costs ϕ, and the difference must beat the cost κ of each unit of time spent frozen (Theorem 3). Below the critical speed the firm should be liquidated at zero and behaves exactly as de Finetti's.

Third, the shape of the transition. The ceiling $\bar{\phi} = \rho^{(\rho-1)/(\rho+1)}$ depends only on the ratio of the model's two characteristic roots (Theorem 4). The barrier is continuous in the speed, but leaves de Finetti's level with a kink of explicit slope (Theorem 5), falls as the speed rises, and approaches the instant-injection barrier with a correction of order $1/\theta$; the value lost at zero is the price of capital times that extra buffer (Theorem 6). The discounted time frozen and capital raised under the optimal policy are explicit too (Theorem 7).

Fourth, every closed form is checked against a lattice chain that is solved exactly as a linear system and optimises its own barrier, and never uses the ordinary differential equation.

2. The Model

Definition 1 (The surplus, sticky at zero)**.** Fix a drift m, a volatility $\sigma > 0$ and a speed $\theta > 0$. On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ carrying a Brownian motion W, let D be a nondecreasing right-continuous adapted process with $D_{0-} = 0$, the cumulative dividends. The surplus sticky at zero is a process $X_t \geq 0$ satisfying

$$X_t = x + \int_0^t \mathbf{1}\{X_s > 0\}\,(m\,ds + \sigma\,dW_s) + \theta \int_0^t \mathbf{1}\{X_s = 0\}\,ds - D_t.$$

Away from zero, X moves as a Brownian motion with drift m and volatility σ; at zero it has no diffusion and is pushed upward at the rate θ.

With $D \equiv 0$, $m = 0$ and $\sigma = 1$ this is the reflecting Brownian motion sticky at zero of Engelbert and Peskir, with their parameter μ equal to θ [8]. They show it has a unique weak solution and no strong one. The drift and the volatility are absorbed by Girsanov's theorem, the drift $m\mathbf{1}\{X > 0\}$ being bounded, and by scaling, so the same holds for Definition 1 without dividends; the solution is the Ito and McKean time change of a reflecting Brownian motion with drift, whose clock is slowed at zero by its local time divided by θ [7].

Definition 2 (Capital injected and time frozen)**.** For the surplus of Definition 1, the cumulative capital injected and the cumulative time frozen at zero are

$$C_t = \theta \int_0^t \mathbf{1}\{X_s = 0\}\,ds, \qquad F_t = \int_0^t \mathbf{1}\{X_s = 0\}\,ds,$$

so that $C_t = \theta F_t$. C_t is the total push the surplus has received at zero, and F_t is the Lebesgue measure of the set of times at which the firm has been frozen.

The two processes increase only on the set $\{t : X_t = 0\}$, which has no interior but has positive measure. That is the defining feature of a sticky point: the surplus leaves zero immediately after every visit and returns immediately, so there is no interval on which it rests, and yet the total time it spends there grows at a positive rate while capital is being raised. In the model this set is the firm's spell of administration, broken into infinitely many pieces too short to see one at a time.

Definition 3 (Admissible strategies)**.** An admissible strategy is a filtered probability space with a Brownian motion W, a dividend process D_t as in Definition 1 with jumps $\Delta D_t \leq X_{t-}$, and a stopping time τ, the liquidation time, with $X_\tau = 0$ on $\{\tau < \infty\}$, such that the surplus of Definition 1 exists on $[0, \tau)$. The set of admissible strategies from x is written $\Pi(x)$.

A strategy therefore decides two things: when to pay dividends, and whether to rescue the firm each time it reaches zero or to liquidate it there. Liquidation is allowed only at zero because liquidating a firm with positive surplus is dominated by first paying that surplus out as a dividend and then liquidating. The strategy carries its own probability space because the equation of Definition 1 has no strong solution, a point taken up in Remark 1.

The barrier strategy π_b at level $b > 0$ pays out exactly what is needed to keep $X \leq b$, rescues at every visit to zero, and never liquidates. The liquidating barrier strategy $\pi_b^\dagger$ pays the same dividends and sets τ equal to the first time X reaches zero.

Definition 4 (The shareholders' value)**.** Fix a discount rate $q > 0$, a price ϕ per unit of capital and a cost $\kappa \geq 0$ per unit of time frozen, and write $c = \phi\theta + \kappa$. The value of an admissible strategy π from x is

$$\begin{aligned} J(x;\pi) &= \mathbb{E}_x\Big[\int_{[0,\tau)} e^{-qt}\,dD_t - \phi\int_0^\tau e^{-qt}\,dC_t - \kappa\int_0^\tau e^{-qt}\,dF_t\Big] \\ &= \mathbb{E}_x\Big[\int_{[0,\tau)} e^{-qt}\,dD_t - c\int_0^\tau e^{-qt}\,\mathbf{1}\{X_t = 0\}\,dt\Big], \end{aligned}$$

and the value function is $V(x) = \sup_{\pi\in\Pi(x)} J(x;\pi)$.

The two costs enter only through the combination c, because both capital and frozen time accrue at a constant rate while the firm is frozen: $\phi\theta$ per unit of time for capital, κ for administration. The cost ϕ is at least one when capital is raised at a discount to its value to the firm, as a rights issue is, and κ collects everything else: fees, lost business, management time. A liquidated firm pays nothing further and is worth nothing, which is why the integrals stop at τ.

Assumption 1 (Standing assumptions)**.** Throughout, $m > 0$, $\sigma > 0$, $q > 0$, $\theta > 0$, $\phi \geq 1$ and $\kappa \geq 0$. The roots of $\frac{1}{2}\sigma^2 r^2 + mr - q = 0$ are written $r_1 > 0 > r_2$, their ratio $\rho = -r_2/r_1$, and the operator $\mathcal{L} = \frac{1}{2}\sigma^2\,d^2/dx^2 + m\,d/dx$ is the generator away from zero.

The positive drift is what makes the problem interesting: with $m \leq 0$ the ratio ρ is at most one, de Finetti's barrier collapses to zero, and it is optimal to pay everything out at once. With $m > 0$ the ratio exceeds one, and it is the single number that the price ceiling of Theorem 4 depends on. The restriction $\phi \geq 1$ says that capital is never cheaper than its face value; without it, a firm could profit by raising capital and paying it straight back out, and the barrier equation of Theorem 1 can lose its positive root. Two identities are used repeatedly: $r_1 + r_2 = -2m/\sigma^2$ and $r_1 r_2 = -2q/\sigma^2$.

Remark 1 (Why the problem is posed on weak solutions). The equation of Definition 1 has no strong solution [8]: there is no way to write the sticky surplus as a function of the Brownian motion that drives it. A control problem in which the controller chooses dividends adapted to a fixed Brownian motion, and the surplus then follows, is therefore not available. Each admissible strategy carries its own probability space, as in Definition 3.

This is the standard weak formulation of stochastic control [24], and nothing in the results depends on it beyond the Ito formula of Lemma 1, which holds on any space carrying a solution. The reason the strong formulation fails is instructive. During the time the surplus spends at zero the driving Brownian motion keeps moving, but the surplus ignores it; the time at zero is additional randomness that the driving noise does not contain. Engelbert and Peskir make this precise by showing that smooth approximations of the equation, which do have strong solutions, fail to converge in probability as the smoothing is removed [8]. Remark 4 draws the practical consequence for simulation.

Remark 2 (Reading the stickiness). Since $C_t = \theta F_t$, each unit of capital raised costs $1/\theta$ units of frozen time. In the unit case $m = 0$, $\sigma = 1$, the push at zero is half the right local time at zero, $C_t = \frac{1}{2}\ell_t^0(X)$, and the time frozen is $F_t = \ell_t^0(X)/(2\theta)$ [8].

The first identity is the economic content of the model. A firm that needs one unit of capital spends $1/\theta$ units of time frozen while it arrives, so θ is literally the speed of the capital market as the firm experiences it. The two limits recover the classical boundaries: as $\theta \to \infty$ the frozen time per unit of capital vanishes and the surplus is instantaneously reflected, which is the model with instant injections; as $\theta \to 0$ the surplus is stuck at zero for ever, which with liquidation available is ruin. The second identity says that the capital raised is the local time, the quantity that measures how much a reflecting process has been pushed. Stickiness adds nothing to the push; it only charges time for it.

Lemma 1 (Ito's formula for the sticky surplus)**.** Let $f \in C^2([0,\infty))$, with one-sided derivatives at zero, and let X be the surplus of Definition 1 under some admissible strategy. Then, for $t < \tau$,

$$
\begin{aligned}
e^{-qt} f(X_t) = f(x) &+ \int_0^t e^{-qs} (\mathcal{L} - q) f(X_s)\, \mathbf{1}\{X_s > 0\}\, ds \\
&+ \int_0^t e^{-qs} (\theta f'(0) - q f(0))\, \mathbf{1}\{X_s = 0\}\, ds + \int_0^t e^{-qs} \sigma f'(X_s)\, \mathbf{1}\{X_s > 0\}\, dW_s \\
&- \int_0^t e^{-qs} f'(X_{s-})\, dD^c_s + \sum_{s \le t} e^{-qs} [f(X_s) - f(X_{s-})],
\end{aligned}
$$

where D^c is the continuous part of D.

Proof. The surplus is a semimartingale whose continuous martingale part is $\int \sigma \mathbf{1}\{X_s > 0\}\, dW_s$, so its quadratic variation is $\sigma^2 \int \mathbf{1}\{X_s > 0\}\, ds$. The ordinary Ito formula for semimartingales with jumps [11] applied to $e^{-qt} f(X_t)$ gives the formula with the drift term $f'(X_s)(m\mathbf{1}\{X_s > 0\} + \theta \mathbf{1}\{X_s = 0\})$ and the second-order term $\frac{1}{2} f''(X_s) \sigma^2 \mathbf{1}\{X_s > 0\}$. On the set $\{X_s = 0\}$ the first reduces to $\theta f'(0)$ and the second vanishes, and grouping the terms by whether X_s is positive or zero gives the statement. □

The point of the lemma is the boundary term. Where a reflecting process would contribute an integral against its local time, and an absorbed one would stop, the sticky surplus contributes an ordinary time integral over the set where it sits at zero, with integrand $\theta f'(0) - q f(0)$. That integrand is the sticky boundary operator, and it is where the speed of capital enters every equation that follows.

3. Barrier Strategies

Lemma 1 turns the control problem into equations by the usual argument. Away from zero the surplus moves by $\mathcal{L}$, discounting costs q, and paying a unit of dividend gains one unit while removing a unit of surplus, worth V'. So on $(0, \infty)$ the value function should satisfy

$$
\max\{(\mathcal{L} - q) V(x),\ 1 - V'(x)\} = 0, \tag{3.1}
$$

the familiar variational inequality of the dividend problem [1][2]: either it is optimal to wait, and the value is a discounted harmonic function of $\mathcal{L}$, or it is optimal to pay, and the value rises with slope one. Where the two regions meet, the principle of smooth fit asks the value to be twice differentiable [26], and that condition is what fixes the barrier in Theorem 1.

At zero there are two choices. Rescuing the firm keeps the surplus in the sticky regime, where by Lemma 1 the value earns the drift $\theta V'(0)$, is discounted at q, and pays the running cost c for as long as it is frozen. Liquidating it takes the value to zero. The dynamic programming principle therefore gives the boundary condition

$$\max\{\theta V'(0) - qV(0) - c,\ -V(0)\} = 0. \tag{3.2}$$

The first term is Feller's sticky boundary operator [6] with a cost attached. Substituting $qV(0) = \frac{1}{2}\sigma^2 V''(0+) + mV'(0)$ from the equation, rescue reads $\theta V'(0) = \frac{1}{2}\sigma^2 V''(0+) + mV'(0) + c$: the push at zero balances the generator's action there and the running cost. With $m = 0$, $\sigma = 1$ and no cost this is Feller's condition $f'(0+) = \frac{1}{2\mu} f''(0+)$ with $\mu = \theta$ [6][8].

The two classical problems are the two limits of this condition. Dividing by θ and letting $\theta \to \infty$ gives $V'(0) = \phi$, the condition of instant injection at price ϕ. Letting $\theta \to 0$ with $\kappa > 0$ makes rescue worth $-\kappa/q$ and forces $V(0) = 0$, which is ruin.

Proposition 1 (The value of a barrier strategy)**.** Under Assumption 1, for every $b > 0$ the barrier strategy π_b of Definition 3 is admissible, and its value is

$$J(x;\pi_b) = A\,e^{r_1 x} + B\,e^{r_2 x} \quad (0 \le x \le b), \qquad J(x;\pi_b) = x - b + J(b;\pi_b) \quad (x > b),$$

where (A, B) is the unique solution of the linear system

$$Ar_1 e^{r_1 b} + Br_2 e^{r_2 b} = 1, \qquad (\theta r_1 - q)A + (\theta r_2 - q)B = c.$$

Proof. Admissibility: let Y be a Brownian motion with drift m and volatility σ reflected into $[0, b]$ by its regulators L at zero and U at b [25], time-change it by the inverse T of $t \mapsto t + L_t/\theta$, and set $X_t = Y_{T_t}$, $D_t = U_{T_t}$. The time X spends at zero up to t is $t - T_t = L_{T_t}/\theta$, so the push L_{T_t} equals θ times the frozen time, which is Definition 1 [7][8]. Value: the determinant of the system is $\theta r_1 r_2(e^{r_1 b} - e^{r_2 b}) - q(r_1 e^{r_1 b} - r_2 e^{r_2 b})$, which is strictly negative because $r_1 r_2 < 0$, so the solution exists and is unique. Put $g(x) = Ae^{r_1 x} + Be^{r_2 x}$. Then $(\mathcal{L} - q)g = 0$ on $(0, b)$ because r_1, r_2 solve the characteristic equation; $\theta g'(0) - qg(0) = c$ by the second equation; and D is continuous and grows only when $X = b$, where $g'(b) = 1$ by the first. Lemma 1 therefore gives $e^{-qt} g(X_t) = g(x) + c\int_0^t e^{-qs}\mathbf{1}\{X_s = 0\}\,ds - \int_0^t e^{-qs}\,dD_s + M_t$ with M a martingale, g' being bounded on $[0, b]$. Taking expectations and letting $t \to \infty$, with g bounded on $[0, b]$, gives $g(x) = J(x;\pi_b)$. Above b the strategy pays $x - b$ at once and continues from b. □

The formula is the sticky counterpart of the classical barrier value [1]. The only change is the second equation, which replaces $g(0) = 0$ in the ruin model and $g'(0) = \phi$ in the instant-injection model by the sticky condition, and it is linear in (A, B), so the whole family of barrier values stays in closed form.

Theorem 1 (The optimal barrier solves one explicit equation)**.** Under Assumption 1, for $b > 0$ let $g_b(x) = A_b e^{r_1 x} + B_b e^{r_2 x}$ be the smooth-fit function, with

$$A_b = \frac{-r_2\, e^{-r_1 b}}{r_1(r_1 - r_2)}, \qquad B_b = \frac{r_1\, e^{-r_2 b}}{r_2(r_1 - r_2)},$$

so that $g_b'(b) = 1$ and $g_b''(b) = 0$. Then $\Psi_\theta(b) = \theta g_b'(0) - q g_b(0)$ is strictly increasing in b, equals $\theta - m$ at $b = 0$, and tends to infinity. Hence the equation $\Psi_\theta(b) = c$ has exactly one root $b_\theta > 0$, and in closed form it reads

$$r_1(\theta + \tfrac{1}{2}\sigma^2 r_1)e^{-r_2 b} - r_2(\theta + \tfrac{1}{2}\sigma^2 r_2)e^{-r_1 b} = c\,(r_1 - r_2).$$

The barrier value of Proposition 1 with $b = b_\theta$ is g_{b_θ}.

Proof. The coefficients solve $g_b'(b) = 1$ and $g_b''(b) = 0$, as substitution shows. Then $g_b'(0) = (r_1 e^{-r_2 b} - r_2 e^{-r_1 b})/(r_1 - r_2)$, and a direct computation gives $\frac{d}{db} g_b(0) = -g_b'(0)$ and $\frac{d}{db} g_b'(0) = -r_1 r_2 (e^{-r_2 b} - e^{-r_1 b})/(r_1 - r_2) \geq 0$. So $\Psi_\theta'(b) = \theta \frac{d}{db} g_b'(0) + q\, g_b'(0) > 0$, since both terms of $g_b'(0)$ are positive. At $b = 0$, $g_0'(0) = 1$ and $g_0(0) = (r_1 + r_2)/(r_1 r_2) = m/q$, so $\Psi_\theta(0) = \theta - m$; and both $g_b'(0)$ and $-g_b(0)$ grow like $e^{-r_2 b}$, so $\Psi_\theta(b) \to \infty$. Under Assumption 1, $c - (\theta - m) = \theta(\phi - 1) + \kappa + m > 0$, so the root is positive and unique. Multiplying $\Psi_\theta(b) = c$ by $r_1 - r_2$ and using $q = -\frac{1}{2}\sigma^2 r_1 r_2$ gives the closed form. Finally g_{b_θ} satisfies both equations of Proposition 1, whose solution is unique. ☐

The theorem says that the optimal barrier is where the smooth-fit function first satisfies the sticky boundary condition. Read as a function of the barrier, $\Psi_\theta(b)$ is the net rate at which a frozen firm gains value: the push θ times the marginal value $g_b'(0)$, less the discounting of what the firm is worth. Raising the barrier raises the marginal value at zero and lowers the value itself, and the optimum is where that net rate just pays the running cost c.

Lemma 2 (The candidate is concave below the barrier)**.** For every $b > 0$ the smooth-fit function satisfies $g_b''' > 0$, $g_b'' < 0$ and $g_b' \geq 1$ on $[0, b)$, and $q\, g_b(b) = m$. Extended linearly above b, it satisfies $(\mathcal{L} - q)g_b \leq 0$ on (b, ∞).

Proof. From Theorem 1, $A_b > 0$ and $B_b < 0$, since $r_2 < 0 < r_1$. Then $g_b''' = A_b r_1^3 e^{r_1 x} + B_b r_2^3 e^{r_2 x}$ is a sum of two positive terms. So g_b'' is strictly increasing, and it vanishes at b, hence is negative on $[0, b)$; so g_b' is strictly decreasing there, and it equals one at b, hence exceeds one on $[0, b)$. At b the equation $\frac{1}{2}\sigma^2 g_b'' + m g_b' - q g_b = 0$ with $g_b'' = 0$ and $g_b' = 1$ gives $q g_b(b) = m$. Above b the extension has slope one and no curvature, so $(\mathcal{L} - q)g_b(x) = m - q g_b(x) \leq m - q g_b(b) = 0$, because g_b is increasing. ☐

These are the properties a verification argument needs from a candidate: it is a supersolution of the variational inequality everywhere, it pays dividends only where its slope is one, and nowhere would paying a dividend beat waiting, because the slope is at least one. None of them involve the speed θ: they hold for every smooth-fit function, and the speed enters only through which barrier is chosen and through the boundary condition at zero.

4. Optimality

Theorem 2 (Verification)**.** Under Assumption 1, let b_{DF} be de Finetti's barrier, the b with $g_b(0) = 0$, and $V_{DF} = g_{b_{DF}}$ extended linearly. (i) If $g_{b_\theta}(0) \geq 0$, the barrier strategy π_{b_θ}, which rescues at every visit to zero, is optimal and $V = g_{b_\theta}$. (ii) If $\theta V'_{DF}(0) \leq c$, the liquidating barrier strategy $\pi^\dagger_{b_{DF}}$ is optimal and $V = V_{DF}$.

Proof. The argument is set out in the derivation that follows. It applies Lemma 1 to the candidate along an arbitrary admissible strategy, uses the supersolution properties of Lemma 2 and the boundary inequality at zero to bound the value of that strategy by the candidate, and then checks that each inequality is an equality along the strategy named in the theorem. □

The two conditions are what each candidate needs at zero, and nothing else. A rescuing candidate already satisfies the sticky boundary condition with equality; what it must also show is that liquidating is never better, which is $g_{b_\theta}(0) \geq 0$. A liquidating candidate already has $V_{DF}(0) = 0$; what it must show is that a rescue would not pay, which is the inequality $\theta V'_{DF}(0) - qV_{DF}(0) \leq c$. The theorem leaves open whether exactly one of the two conditions holds, and Theorem 3 shows that it does, with a single threshold in θ between them. The structure follows the classical proofs for the ruin and injection problems [1][3].

Fix $x \geq 0$, an admissible strategy (D, τ) and one of the two candidates V^*, which by Lemma 2 is C^2 on $(0, \infty)$, has $V^{*\prime} \geq 1$, and satisfies $(\mathcal{L} - q)V^* \leq 0$. Apply Lemma 1 up to $t \wedge \tau$. The jump sum is bounded by $-\sum e^{-qs}\Delta D_s$, because $V^*(X_{s-}) - V^*(X_s) \geq \Delta D_s$ by the mean value theorem and $V^{*\prime} \geq 1$; the continuous dividend term is bounded by $-\int e^{-qs} dD^c_s$ for the same reason. Writing $\Theta = \theta V^{*\prime}(0) - qV^*(0)$ for the boundary operator,

$$e^{-q(t\wedge\tau)}V^*(X_{t\wedge\tau}) \leq V^*(x) + \int_0^{t\wedge\tau} e^{-qs}\, \Theta\, \mathbf{1}\{X_s = 0\}\, ds - \int_{[0,t\wedge\tau]} e^{-qs}\, dD_s + M_{t\wedge\tau}, \quad (4.1)$$

with M a martingale because $V^{*\prime}$ is bounded by $V^{*\prime}(0)$.

In case (i), $\Theta = c$ exactly. In case (ii), $\Theta = \theta V'_{DF}(0) \leq c$ by hypothesis. Either way the boundary integral is at most $c \int_0^{t\wedge\tau} e^{-qs}\mathbf{1}\{X_s = 0\} ds$. Taking expectations and rearranging,

$$\mathbb{E}_x\Big[\int_{[0,t\wedge\tau]} e^{-qs}\, dD_s - c\int_0^{t\wedge\tau} e^{-qs}\, \mathbf{1}\{X_s = 0\}\, ds\Big] \leq V^*(x) - \mathbb{E}_x\big[e^{-q(t\wedge\tau)}V^*(X_{t\wedge\tau})\big]. \quad (4.2)$$

The candidate is nonnegative, since it is increasing and $V^*(0) \geq 0$ in both cases, so the last expectation can be dropped; this is where $g_{b_\theta}(0) \geq 0$ is used, because at the liquidation time the strategy receives nothing while the candidate is worth $V^*(0)$. Letting $t \to \infty$ by monotone convergence gives $J(x; \pi) \leq V^*(x)$ for every admissible π.

For the named strategy every inequality is an equality: dividends are paid only at the barrier, where $V^{*\prime} = 1$; $(\mathcal{L} - q)V^* = 0$ below it; the boundary term is exactly c in case (i), and in case (ii) the strategy stops at zero, so the boundary term never accrues and the terminal term is $V_{DF}(0) = 0$. And $e^{-qt}V^*(X_t) \to 0$ because the surplus stays in $[0, b]$. So $J(x; \pi) = V^*(x)$, and V^* is the value function.

5. The Speed of Capital

Theorem 3 (The critical speed of rescue)**.** Under Assumption 1 with $\kappa > 0$, let $\bar{\phi} = V'_{DF}(0)$ and define the critical speed

$$\theta^* = \frac{\kappa}{\bar{\phi} - \phi} \quad \text{if } \phi < \bar{\phi}, \qquad \theta^* = \infty \quad \text{otherwise.}$$

Then $g_{b_\theta}(0) \geq 0$ if and only if $\theta \geq \theta^*$, and $\theta V'_{DF}(0) \leq c$ if and only if $\theta \leq \theta^*$. Consequently, for $\theta < \theta^*$ the liquidating strategy $\pi^\dagger_{b_{DF}}$ is optimal, for $\theta > \theta^*$ the rescuing strategy π_{b_θ} is optimal, and at θ^* both are, with $b_{\theta^*} = b_{DF}$. The optimal barrier $b^*(\theta)$, equal to b_{DF} for $\theta \leq \theta^*$ and to b_θ above, is continuous in θ.

Proof. By the proof of Theorem 1, $\frac{d}{db} g_b(0) = -g'_b(0) < 0$, so $g_b(0)$ is strictly decreasing in b and vanishes only at b_{DF}; hence $g_{b_\theta}(0) \geq 0$ if and only if $b_\theta \leq b_{DF}$. Since Ψ_θ is strictly increasing, $b_\theta \leq b_{DF}$ if and only if $\Psi_\theta(b_{DF}) \geq c$. But $\Psi_\theta(b_{DF}) = \theta g'_{b_{DF}}(0) - q g_{b_{DF}}(0) = \theta\bar{\phi}$, so the condition is $\theta\bar{\phi} \geq \phi\theta + \kappa$, that is $\theta(\bar{\phi} - \phi) \geq \kappa$, which holds if and only if $\theta \geq \theta^*$ when $\phi < \bar{\phi}$ and never otherwise. The second condition, $\theta V'_{DF}(0) = \theta\bar{\phi} \leq \phi\theta + \kappa$, is the complementary inequality. The optimality statements follow from Theorem 2. At θ^* the equality $\Psi_{\theta^*}(b_{DF}) = c$ identifies b_{DF} as the unique root, so the two branches of b^* meet and b^* is continuous. □

The whole decision reduces to one comparison of flows. While the firm is frozen, capital arrives at rate θ; each unit is worth $\bar{\phi}$ to a firm at the brink that would otherwise be liquidated, and costs ϕ; and administration costs κ per unit of time. A rescue pays for itself exactly when $\theta(\bar{\phi} - \phi) > \kappa$. Nothing else in the model enters: not the barrier, not the value away from zero, not the frozen time the rescue will eventually accumulate.

Proposition 2 (Faster capital, a lower barrier and a higher value at zero)**.** Under the assumptions of Theorem 3 with $\phi < \bar{\phi}$, the optimal barrier $b^*(\theta)$ is strictly decreasing and the value at zero $V^*(0) = g_{b^*(\theta)}(0)$ is strictly increasing on (θ^*, ∞). On $(0, \theta^*]$ both are constant, at b_{DF} and 0.

Proof. Let $F(b, \theta) = \Psi_\theta(b) - \phi\theta - \kappa$, so that b_θ solves $F(b_\theta, \theta) = 0$. By Theorem 1, $\partial_b F = \Psi'_\theta(b) > 0$, and $\partial_\theta F = g'_b(0) - \phi$. At the root, $\theta(g'_{b_\theta}(0) - \phi) = q g_{b_\theta}(0) + \kappa$, and for $\theta > \theta^*$ the right side is positive by Theorem 3, so $\partial_\theta F > 0$. The implicit function theorem gives $\frac{db_\theta}{d\theta} = -\partial_\theta F / \partial_b F < 0$. Then $\frac{d}{d\theta} g_{b_\theta}(0) = -g'_{b_\theta}(0) \frac{db_\theta}{d\theta} > 0$. The statement on $(0, \theta^*]$ is Theorem 3. □

The proof contains the economics. The sign of $\partial_\theta F$ is the sign of $g'_b(0) - \phi$, the marginal value of surplus at the brink net of the price of capital, and at the optimum this equals

$(qV^*(0)+\kappa)/\theta$. Whenever rescue is worth anything, a unit of surplus at zero is worth more than the capital that replaces it, so faster capital makes being at zero less costly, the firm needs less buffer against arriving there, and it pays dividends sooner.

Corollary 1 (The two limits: de Finetti and Lokka-Zervos)**.** Under the assumptions of Theorem 3, (i) for every $\theta \le \theta^*$ the optimal policy is de Finetti's, with barrier $b_{DF} = 2\ln\rho/(r_1 - r_2)$ and liquidation at zero [1]; (ii) if $\phi < \bar{\phi}$ then, as $\theta \to \infty$, $b^*(\theta)$ decreases to b_{LZ}, the unique b with $g_b'(0) = \phi$, and $g_{b^*(\theta)} \to g_{b_{LZ}}$ uniformly on compacts. The limit $g_{b_{LZ}}$ is the value of paying dividends at b_{LZ} while injecting capital instantly at price ϕ whenever the surplus reaches zero, the problem of Lokka and Zervos [3].

Proof. Part (i) is Theorem 3, and $g_b(0) = 0$ *reads* $(r_1/r_2)e^{-r_2 b} = (r_2/r_1)e^{-r_1 b}$*, that is* $e^{(r_1-r_2)b} = \rho^2$*. (ii) The function* $b \mapsto g_b'(0)$ *increases from 1 at* $b = 0$ *to infinity, so* b_{LZ} *exists and is unique for* $\phi \ge 1$*. For* $\theta > \theta^*$ *the proof of Proposition 2 gives* $g_{b_\theta}'(0) - \phi = (qg_{b_\theta}(0)+\kappa)/\theta > 0$*, so* $b_\theta > b_{LZ}$*; as* b_θ *decreases in* θ *it has a limit* $\ge b_{LZ}$*, and letting* $\theta \to \infty$ *in the same identity, with* $g_{b_\theta}(0)$ *bounded, gives* $g'(0) = \phi$ *at the limit, which is therefore* b_{LZ}*. The coefficients* A_b*,* B_b *are continuous in* b*, so the functions converge uniformly on compacts. Finally* $g_{b_{LZ}}$ *has* $g'(b_{LZ}) = 1$*,* $g''(b_{LZ}) = 0$ *and* $g'(0) = \phi$*, which are the barrier, smooth-fit and injection conditions of the instant-injection problem.* □

So the sticky model is exactly the bridge its construction promised. At one end the firm cannot be saved and behaves as in de Finetti's problem; at the other, capital is instant and the policy is the instant-injection one. The new content lies between, and it is not a gradual interpolation: the policy sits at de Finetti's for a whole range of positive speeds before it starts to move.

Theorem 4 (A price ceiling for capital, fixed by one ratio)**.** Under Assumption 1, the marginal value at zero in de Finetti's problem is

$$\bar{\phi} = V_{DF}'(0) = \rho^{(\rho-1)/(\rho+1)}, \qquad \rho = -r_2/r_1.$$

(i) $\bar{\phi} > 1$, $\bar{\phi}$ is strictly increasing in ρ, and $\bar{\phi} \to 1$ as $\rho \downarrow 1$. (ii) If $\kappa > 0$, rescue is optimal at some speed if and only if $\phi < \bar{\phi}$; for $\phi \ge \bar{\phi}$ liquidation at zero is optimal at every speed. (iii) In the instant-injection limit, $g_{b_{LZ}}(0) > 0$ if and only if $\phi < \bar{\phi}$.

Proof. With $e^{(r_1-r_2)b_{DF}} = \rho^2$ *and* $-r_2/(r_1 - r_2) = \rho/(1+\rho)$*,* $e^{-r_2 b_{DF}} = \rho^{2\rho/(1+\rho)}$ *and* $e^{-r_1 b_{DF}} = \rho^{-2/(1+\rho)}$*. Substituting into* $g_b'(0) = (r_1 e^{-r_2 b} - r_2 e^{-r_1 b})/(r_1 - r_2)$ *and dividing through by* r_1*,* $\bar{\phi} = (\rho^{2\rho/(1+\rho)} + \rho^{(\rho-1)/(1+\rho)})/(1+\rho) = \rho^{2\rho/(1+\rho)}\rho^{-1} = \rho^{(\rho-1)/(\rho+1)}$*. (i)* $\ln\bar{\phi} = \frac{\rho-1}{\rho+1}\ln\rho$ *is a product of two positive increasing functions on* $\rho > 1$*, and tends to zero as* $\rho \downarrow 1$*. (ii) is Theorem 3, since* $\theta^* < \infty$ *exactly when* $\phi < \bar{\phi}$*. (iii)* $b \mapsto g_b(0)$ *is decreasing with its zero at* b_{DF}*, and* $b \mapsto g_b'(0)$ *is increasing, so* $g_{b_{LZ}}(0) > 0$ *if and only if* $b_{LZ} < b_{DF}$*, if and only if* $\phi = g_{b_{LZ}}'(0) < g_{b_{DF}}'(0) = \bar{\phi}$*.* □

The ceiling is the value, at the moment of ruin, of one more unit of surplus to a firm that cannot be rescued. It is the most a unit of capital can ever be worth to the firm, and the theorem says that no speed of arrival changes that. What the speed changes, through

Theorem 3, is how far below the ceiling the price must be before waiting for the capital pays. The ratio ρ grows as the drift grows relative to the discount rate, so the ceiling is high for profitable, patient firms and falls to one as the drift vanishes, where no capital is worth more than its face value.

Theorem 5 (The barrier leaves de Finetti's with a kink)**.** Under the assumptions of Theorem 3 with $\phi < \bar{\phi}$, the optimal barrier b^* is continuous on $(0, \infty)$, constant on $(0, \theta^*]$, and continuously differentiable on (θ^*, ∞). At θ^* its left derivative is zero and its right derivative is

$$s = -\frac{\bar{\phi} - \phi}{\theta^* \, \partial_b g_b'(0)\big|_{b_{DF}} + q\bar{\phi}} < 0, \qquad \partial_b g_b'(0) = \frac{-r_1 r_2 \, (e^{-r_2 b} - e^{-r_1 b})}{r_1 - r_2},$$

so b^* is not differentiable at θ^*. The value at zero has a kink of the same kind, with right derivative $-\bar{\phi} s > 0$.

Proof. Continuity and constancy are Theorem 3. On (θ^*, ∞) the root b_θ of $F(b, \theta) = \Psi_\theta(b) - \phi\theta - \kappa$ is continuously differentiable by the implicit function theorem, F being smooth with $\partial_b F > 0$, and $\frac{db_\theta}{d\theta} = -(g_b'(0) - \phi)/\Psi_\theta'(b)$. As $\theta \downarrow \theta^*$, $b_\theta \to b_{DF}$, $g_b'(0) \to \bar{\phi}$ and $\Psi_\theta'(b) \to \theta^* \partial_b g_b'(0)|_{b_{DF}} + q\bar{\phi}$, which gives s; its numerator is positive because $\phi < \bar{\phi}$, so $s < 0 \neq 0$. For the value, $\frac{d}{d\theta} g_{b_\theta}(0) = -g_{b_\theta}'(0) \, \frac{db_\theta}{d\theta} \to -\bar{\phi} s$. □

The kink is what a threshold looks like when the objects on either side of it agree. At θ^* the rescuing firm and the liquidating firm hold the same buffer and have the same value, so nothing jumps. But the liquidating firm's policy does not respond to the speed at all, while the rescuing firm's responds at once and at a finite rate. A firm just above the critical speed therefore cuts its buffer sharply as capital speeds up, and a firm just below it does not move.

Theorem 6 (The cost of slow capital falls as one over its speed)**.** Under the assumptions of Theorem 3 with $\phi < \bar{\phi}$, let $V_{LZ}(0) = g_{b_{LZ}}(0)$ and

$$c_1 = \frac{q \, V_{LZ}(0) + \kappa}{\partial_b g_b'(0)\big|_{b_{LZ}}} > 0.$$

Then, as $\theta \to \infty$,

$$b^*(\theta) = b_{LZ} + \frac{c_1}{\theta} + O(\theta^{-2}), \qquad V^*(0) = V_{LZ}(0) - \frac{\phi \, c_1}{\theta} + O(\theta^{-2}).$$

Proof. The expansion is derived in the derivation that follows, by writing the barrier equation as $\theta P(b) + R(b) = 0$ with P vanishing at b_{LZ} and expanding about it. The constant is positive because $V_{LZ}(0) > 0$ by Theorem 4 and $\partial_b g_b'(0) > 0$. □

The value lost at zero is, to first order, the price of capital times the extra buffer, and the statement is more than a coincidence of constants. Raising the barrier by δ lowers the value at zero by $g_b'(0) \, \delta$, and at b_{LZ} that marginal value is exactly ϕ by the injection condition. So slow capital costs a firm what it would cost to buy, at the market price, the extra buffer the slowness forces it to hold. The rate $1/\theta$ says that halving the time it takes to raise a unit of capital halves that cost, all the way down to the instant-injection limit, and Lokka and Zervos's value [3] is approached no faster than that.

Rearrange the barrier equation of Theorem 1 by powers of θ: $\Psi_\theta(b) = \phi\theta + \kappa$ becomes $\theta P(b) + R(b) = 0$ with

$$P(b) = g_b'(0) - \phi, \qquad R(b) = -(q\, g_b(0) + \kappa). \tag{5.1}$$

By Corollary 1, $P(b_{LZ}) = 0$ and $b_\theta \to b_{LZ}$. Put $b_\theta = b_{LZ} + \delta$ and expand, using $P'(b) = \partial_b g_b'(0) > 0$ and the smoothness of P and R:

$$\theta\big(P'(b_{LZ})\,\delta + O(\delta^2)\big) + R(b_{LZ}) + O(\delta) = 0. \tag{5.2}$$

The leading balance is between $\theta\delta$ and a constant, so $\delta = O(1/\theta)$, the remainders are $O(1/\theta)$, and

$$\delta = -\frac{R(b_{LZ})}{\theta\, P'(b_{LZ})} + O(\theta^{-2}) = \frac{qV_{LZ}(0) + \kappa}{\theta\, \partial_b g_b'(0)|_{b_{LZ}}} + O(\theta^{-2}) = \frac{c_1}{\theta} + O(\theta^{-2}). \tag{5.3}$$

For the value, expand $b \mapsto g_b(0)$ about b_{LZ} with $\frac{d}{db} g_b(0) = -g_b'(0)$ from the proof of Theorem 1:

$$V^*(0) = g_{b_{LZ}}(0) - g_{b_{LZ}}'(0)\,\delta + O(\delta^2) = V_{LZ}(0) - \phi\,\frac{c_1}{\theta} + O(\theta^{-2}), \tag{5.4}$$

because $g_{b_{LZ}}'(0) = \phi$. Both statements of Theorem 6 follow.

Two features of the computation are worth keeping. The frozen-time cost κ survives in the constant c_1 even though its weight in the equation, κ/θ, vanishes: it is of the same order as the correction it produces. And the whole expansion is local to b_{LZ}, so it needs nothing of the de Finetti branch; it is the fast-capital end of the curve, where the rescue is always worth it.

Theorem 7 (Time frozen and capital raised under the optimal policy)**.** Under the assumptions of Theorem 3, let $\theta > \theta^*$ and $b = b_\theta$, and for $0 \le x \le b$ let $u(x) = \mathbb{E}_x\big[\int_0^\infty e^{-qt}\,\mathbf{1}\{X_t = 0\}\,dt\big]$ under the optimal strategy. Then $u(x) = Ce^{r_1 x} + De^{r_2 x}$, where (C, D) is the unique solution of

$$Cr_1 e^{r_1 b} + Dr_2 e^{r_2 b} = 0, \qquad (\theta r_1 - q)C + (\theta r_2 - q)D = -1.$$

The discounted capital raised is $\theta u(x)$, and the discounted dividends paid are $V^*(x) + c\,u(x)$.

Proof. The system has the same determinant as in Proposition 1, which is nonzero. The function u it defines satisfies $(\mathcal{L} - q)u = 0$ on $(0, b)$, $u'(b) = 0$ and $\theta u'(0) - qu(0) = -1$. Lemma 1 along the optimal strategy gives $e^{-qt}u(X_t) = u(x) - \int_0^t e^{-qs}\mathbf{1}\{X_s = 0\}\,ds - \int_0^t e^{-qs}u'(X_s)\,dD_s + M_t$, and the dividend integral vanishes because D grows only when $X = b$, where $u' = 0$. Taking expectations and letting $t \to \infty$, u being bounded on $[0, b]$, gives the first claim. The second is $C = \theta F$ from Definition 2. The third is Definition

4 rearranged: the value is the discounted dividends less c times the discounted frozen time. □

The theorem splits the value of the rescued firm into its three flows: what it pays out, what it spends on capital, and what it loses to administration. The split matters because the three respond differently to the speed. As θ grows, the frozen time vanishes, like $1/\theta$, but the capital raised does not, because the firm reaches zero no less often when it is rescued quickly; it converges to the discounted injections of the instant-injection policy.

Remark 3 (When a frozen day costs nothing). If $\kappa = 0$, then $\theta^* = 0$ whenever $\phi < \bar{\phi}$: rescue is optimal at every positive speed, and the plateau of Theorem 3 disappears. If $\phi \geq \bar{\phi}$, liquidation is optimal at every speed.

With free administration the speed of capital never decides whether to rescue the firm; only its price does, through the ceiling of Theorem 4. The speed still decides how much buffer to hold: the barrier falls from b_{DF} at $\theta \downarrow 0$ to b_{LZ} at $\theta \to \infty$, now without a flat stretch at the slow end, since at every positive speed the rescuing firm's barrier responds. The critical speed is therefore entirely an effect of the cost of being frozen. That is a useful separation for applications: the price of capital decides whether a rescue can ever pay, and the cost of the wait decides how fast the rescue has to be.

Example 1 (The house parameters)**.** Take drift $m = 0.5$, volatility $\sigma = 1$, discount rate $q = 0.05$, price of capital $\phi = 1.2$ and cost of frozen time $\kappa = 0.2$. The roots are $r_1 = 0.091608$ and $r_2 = -1.091608$, so $\rho = 11.916$.

The two classical barriers are $b_{DF} = 2 \ln \rho/(r_1 - r_2) = 4.1884$ and $b_{LZ} = 1.4938$, the latter from $g_b'(0) = 1.2$. The ceiling is $\bar{\phi} = 11.916^{0.84516} = 8.1189$, well above the price of capital, and the critical speed is $\theta^* = 0.2/(8.1189 - 1.2) = 0.028906$. The policy across speeds, from Theorems 1 and 3:

speed θ	**regime**	**barrier** b^*	$V^*(0)$	$V^*(1)$
0.02	liquidate	4.1884	0.0000	5.2167
0.05	rescue	3.9892	1.4670	5.8052
0.1	rescue	3.6610	3.3430	6.5971
0.3	rescue	3.0305	5.6910	7.7108
1	rescue	2.3582	7.1837	8.5803
2	rescue	2.0523	7.6784	8.9217
10	rescue	1.6479	8.2312	9.3467
∞	instant	1.4938	8.4207	n/a

The barrier falls monotonically and the value at zero rises, as Proposition 2 requires, from de Finetti's policy at $\theta = 0.02$ to within 0.19 of the instant-injection value at $\theta = 10$. One number is the same in every row: by Lemma 2, $qV^*(b^*) = m$, so the value at the barrier is $m/q = 10$ whatever the speed. The speed decides where that level is reached, not what the firm is worth there.

Example 2 (The ceiling and the kink)**.** At the parameters of Example 1, vary the price of capital ϕ and read off the critical speed $\theta^* = \kappa/(\bar{\phi} - \phi)$ with $\bar{\phi} = 8.1189$ and $\kappa = 0.2$; then compute the slope of the barrier just above θ^* from Theorem 5.

The critical speed is 0.028094 at $\phi = 1$, 0.028906 at 1.2, 0.032685 at 2, 0.048556 at 4 and 0.094387 at 6, and it reaches 1.6817 at $\phi = 8$, just below the ceiling. Most of the range of prices barely matters: raising the price from 1 to 6 multiplies the critical speed by less than three and a half. The last two units before the ceiling multiply it by eighteen, because θ^* has a pole at $\bar{\phi}$. Capital priced near the ceiling is worth raising only if it is fast, and capital at or above it is not worth raising at all.

The kink of Theorem 5 has slope $s = -10.800$ at the house price. Just above the critical speed, a gain of 0.01 in the speed of capital lowers the barrier by about 0.108, some two and a half per cent of b_{DF}; just below it, the same gain changes nothing. A finite difference of the barrier equation over $\theta^* + 10^{-6}$ returns -10.80005, agreeing with the formula to five figures.

6. Computation

Algorithm 1 — Computing the barrier, the value and the critical speed

Inputs: the drift m, volatility σ, discount rate q, price of capital ϕ, cost of frozen time κ, and the speed θ. *Output:* the regime, the optimal barrier b^* and the value function V^*.

1. Compute the roots $r_{1,2} = (-m \pm \sqrt{m^2 + 2q\sigma^2})/\sigma^2$ and the ratio $\rho = -r_2/r_1$.
2. Set $b_{DF} = 2\ln\rho/(r_1 - r_2)$ and $\bar{\phi} = \rho^{(\rho-1)/(\rho+1)}$ (Theorem 4).
3. If $\phi \geq \bar{\phi}$, or $\theta(\bar{\phi} - \phi) \leq \kappa$, the regime is liquidation: return $b^* = b_{DF}$ and $V^* = g_{b_{DF}}$ (Theorem 3).
4. Otherwise the regime is rescue, and b^* is the root of $r_1(\theta + \frac{1}{2}\sigma^2 r_1)e^{-r_2 b} - r_2(\theta + \frac{1}{2}\sigma^2 r_2)e^{-r_1 b} = (\phi\theta + \kappa)(r_1 - r_2)$ (Theorem 1). The left side is increasing and the root lies in $(0, b_{DF})$, so bisect on that interval.
5. Return b^* and $V^* = g_{b^*}$, with A_{b^*} and B_{b^*} from Theorem 1, extended linearly above b^*.

Cost. Two square roots, a logarithm, a power, and one bracketed root-finding problem with a monotone function: about fifty bisection steps reach machine precision, and the whole computation takes microseconds. The bracket in step 4 is the useful fact: Theorem 3 shows that in the rescue regime the barrier is strictly below de Finetti's, so the search interval is known before the search starts.

What makes it work. The only equation solved numerically is the boundary condition at zero. Every other condition is built into the smooth-fit family, so the problem that looked two-dimensional, a barrier and a value function, collapses to one scalar root.

Algorithm 2 — An independent check: the sticky lattice chain

Inputs: the model parameters, the speed θ, a lattice spacing h with $h < \sigma^2/m$, and a barrier Kh. *Output:* the chain's value V_k^h at every lattice point kh, $0 \le k \le K$.

1. On the points kh with $1 \le k \le K$, let the chain jump up at rate $\sigma^2/(2h^2) + m/(2h)$ and down at rate $\sigma^2/(2h^2) - m/(2h)$; these rates reproduce the drift and the variance of $\mathcal{L}$ to first order.
2. At $k = 0$, let the chain jump up only, at rate θ/h, and charge the running cost c per unit of time spent there: each jump raises h of capital after a mean wait of h/θ, so the frozen time per unit of capital is $1/\theta$, exactly as in Remark 2.
3. At $k = K$, replace every up-jump by a dividend of h, the chain staying at K.
4. Solve the tridiagonal linear system $(Q - q)V^h = -r$ exactly, where Q is the generator of the chain and r collects the dividend and cost rates.
5. To find the chain's own optimal barrier, repeat for every K up to a cap and keep the one that maximises V_0^h.

Cost. One tridiagonal solve is linear in K, so the search in step 5 is quadratic in $1/h$: at $h = 0.005$ it solves about sixteen hundred systems and takes under a second. The method is the Markov-chain approximation of Kushner and Dupuis [27], applied to a sticky boundary as in [9]; its accuracy is first order in h.

Why it is independent. The chain never evaluates g_b, the barrier equation or the smooth-fit conditions. It knows the model only through its rates, and it chooses its barrier by brute force, so agreement between the chain and Theorem 1 is evidence rather than arithmetic.

Remark 4 (Why the check is a chain and not an Euler scheme). A scheme that drives the surplus with a given Brownian path, such as an Euler scheme for Definition 1 or for a smoothed version of it, cannot converge in probability to the sticky surplus. If it did, the limit would be adapted to the driving Brownian motion and would be a strong solution, which does not exist [8].

The failure is concrete. Engelbert and Peskir smooth the equation by giving the surplus a small volatility, of order the square root of the smoothing width, in a thin band around zero; the smoothed equations have strong solutions, and those solutions, driven by one and the same Brownian path, fail to converge as the band shrinks [8]. A simulation of the dividend problem driven by a fixed noise path would inherit the same instability, and its errors would not shrink with the step. The sticky surplus has to be approximated in law, not path by path, and a Markov chain that matches the generator, boundary condition included, does exactly that [9]. This is the point at which the absence of a strong solution, a fact about the equation, becomes a fact about how the problem can be computed.

The chain's own barrier. *Method.* For three speeds, the lattice chain of Algorithm 2 is solved at four spacings, and for each spacing its own optimal barrier is found by maximising the chain's value at zero over every barrier on the lattice. Nothing is taken from the closed form. The chain's barrier is then compared with the root of the barrier equation of Theorem 1.

Parameters. The house parameters of Example 1; speeds $\theta = 0.05$, 0.3 and 2; spacings $h = 0.04$, 0.02, 0.01 and 0.005.

Result. At $\theta = 0.3$ the equation gives $b^* = 3.0305$, and the chain chooses 3.04, 3.02, 3.03 and 3.03. At $\theta = 0.05$ it gives 3.9892 against 3.96, 3.98, 3.99, 3.99; at $\theta = 2$, 2.0523 against 2.04, 2.06, 2.05, 2.05.

Error. The chain can only choose barriers on its lattice, and its own discretisation error can move its optimum by a lattice point. In all twelve runs its barrier lies within one spacing of the root, and at the two finest spacings within half a spacing: the largest error there is 0.0023, below half of $h = 0.005$. A barrier chosen by brute force over some sixteen hundred candidates lands on the root of a closed-form equation that the chain never saw, which is the verification the optimality claim of Theorem 2 needs.

The value function. *Method.* At each spacing, the value the chain assigns to its own optimal policy at the surplus levels $x = 0$ and $x = 1$ is compared with the closed-form value g_{b^*} of Proposition 1 and Theorem 1, and the errors are extrapolated to zero spacing.

Parameters. As in the previous check: the house parameters, $\theta \in \{0.05, 0.3, 2\}$ and $h \in \{0.04, 0.02, 0.01, 0.005\}$.

Result. At $\theta = 0.3$ the closed form gives $V^*(0) = 5.69097$ and $V^*(1) = 7.71079$. The chain's errors at zero are -0.04631, -0.02323, -0.01162 and -0.00582, and at one they are -0.02409, -0.01214, -0.00608 and -0.00305. Richardson extrapolation from the two finest spacings, $2V^h_{h=0.005} - V^h_{h=0.01}$, gives 5.69096 and 7.71079. At $\theta = 0.05$ it gives 1.46699 against 1.46701, and at $\theta = 2$ it gives 7.67838 against 7.67838.

Error. Each halving of the spacing halves the error, to within one per cent of the ratio, which is the first-order convergence the chain is built to have, and the extrapolated values agree with the closed forms to four or five decimal places. The chain undershoots in every run. That error comes from the discretised generator rather than from the lattice barrier, since the value at an optimum is insensitive to first order to a small error in the barrier.

The critical speed. *Method.* For each spacing, the chain's best value at zero is computed as a function of θ, optimising the barrier afresh at every speed, and the speed at which it crosses zero is found by root bracketing. That speed is the chain's own critical speed, found without the formula of Theorem 3.

Parameters. The house parameters; spacings $h = 0.04$, 0.02 and 0.01; tolerance 10^{-6} in θ.

Result. The chain's critical speeds are 0.029586, 0.029246 and 0.029076, against $\theta^* = \kappa/(\bar{\phi} - \phi) = 0.028906$. The errors are 6.80×10^{-4}, 3.40×10^{-4} and 1.70×10^{-4}, and extrapolation from the last two spacings gives $2 \times 0.029076 - 0.029246 = 0.028906$.

Error. The errors halve exactly with the spacing, and the extrapolated threshold matches the closed form to six decimal places. This is the most demanding of the checks, because the threshold is where the chain's optimised value changes sign, a quantity that depends on the whole optimisation and not on any single solve, and it is the one number in the paper that the formula $\theta^* = \kappa/(\bar{\phi} - \phi)$ asserts without any intermediate step.

The one-over-speed law. *Method.* The barrier and the value at zero are computed from the barrier equation at increasing speeds, and the products $\theta(b^* - b_{LZ})$ and $\theta(V_{LZ}(0) - V^*(0))$ are compared with the constants c_1 and ϕc_1 of Theorem 6.

Parameters. The house parameters; $\theta = 10$, 100, 1000 and 10^4. The constants are $c_1 = 1.7351$ and $\phi c_1 = 2.0821$, with $b_{LZ} = 1.4938$ and $V_{LZ}(0) = 8.4207$.

Result. The scaled barrier gap is 1.5409, 1.7125, 1.7328 and 1.7348, and the scaled value loss is 1.8947, 2.0603, 2.0799 and 2.0818.

Error. The distance to the limit falls by a factor of ten each time the speed rises by ten: it is 0.0226 at $\theta = 100$ and 0.0023 at $\theta = 1000$ for the barrier, which is the $O(\theta^{-2})$ remainder of Theorem 6 appearing as an $O(\theta^{-1})$ error in the scaled quantity. Both constants are reached to within 3×10^{-4} at the largest speed. The ratio of the value loss to the barrier gap is 1.2296, 1.2031, 1.2003 and 1.2000, converging to $\phi = 1.2$, as the theorem's reading requires.

Frozen time and capital raised. *Method.* The discounted frozen time u of Theorem 7 is computed in closed form and, independently, from the lattice chain with the running cost replaced by a unit reward for time at zero. The capital raised $\theta u(0)$ is then traced to large speeds and compared with the discounted injections of the instant-injection policy.

Parameters. The house parameters at $\theta = 0.3$, with $b^* = 3.0305$; spacings $h = 0.02$, 0.01 and 0.005; for the limit, $\theta = 10$ to 10^4.

Result. The closed form gives $u(0) = 3.4560$ and $u(1) = 1.8124$; the chain gives 3.4699, 3.4678 and 3.4621 at zero and 1.8083, 1.8160 and 1.8145 at one. From zero, the firm pays 7.6263 in discounted dividends, spends $0.3 \times 3.4560 \times 1.2 = 1.2442$ on capital and $0.2 \times 3.4560 = 0.6912$ on frozen time, and $7.6263 - 1.2442 - 0.6912 = 5.6909$ recovers the value $V^*(0) = 5.6910$. As the speed grows, the capital raised from zero is 2.8383, 3.2885, 3.3460 and 3.3519, converging to the instant-injection injections 3.3526, while the frozen time falls as 0.2838, 0.0329, 0.0033 and 0.0003.

Error. The chain's errors are below half a per cent and shrink with the spacing, though not monotonically at $x = 1$, because the chain's barrier is the nearest lattice point to b^* and moves by up to half a spacing between runs. The decomposition closes to the fourth decimal, and the limit confirms the reading of Theorem 7: fast capital eliminates the waiting, not the rescues.

Source. The lattice chain of Algorithm 2 at the house parameters of Example 1 and speed $\theta = 0.3$, solved exactly at each spacing, with its barrier optimised by brute force; the last column is the chain's critical speed, found separately as in the numerical check above. The exact values are $b^* = 3.0305$, $V^*(0) = 5.69097$, $V^*(1) = 7.71079$ and $\theta^* = 0.028906$.

Columns. The spacing; the chain's optimal barrier and its error; the chain's value at zero and at one, and their errors; the chain's critical speed and its error.

h	**barrier**	**err**	$V^h(0)$	**err**	$V^h(1)$	**err**	θ^*_h	**err**
0.040	3.04	+0.0095	5.64466	−0.04631	7.68670	−0.02409	0.029586	+0.000680
0.020	3.02	−0.0105	5.66773	−0.02323	7.69865	−0.01214	0.029246	+0.000340
0.010	3.03	−0.0005	5.67934	−0.01162	7.70471	−0.00608	0.029076	+0.000170
0.005	3.03	−0.0005	5.68515	−0.00582	7.70775	−0.00305	n/a	n/a

The value columns halve down the table, each ratio within one per cent of two, and so does the critical-speed column; the barrier column cannot, because the chain can only choose lattice points, and once the spacing is below the distance to the nearest one it stops improving at −0.0005. The threshold was not computed at the finest spacing, where each of its evaluations would need a full barrier search at sixteen hundred candidates; the three coarser spacings already fix it by extrapolation.

Script. `fig_barrier.py`, which draws the optimal barrier and the value at zero from Algorithm 1 over speeds from 0.004 to 1000, and overlays the lattice chain's own optimal barrier and value at $h = 0.01$ at six speeds.

Shows. De Finetti's plateau, the kink at the critical speed, and the fall to the instant-injection barrier.

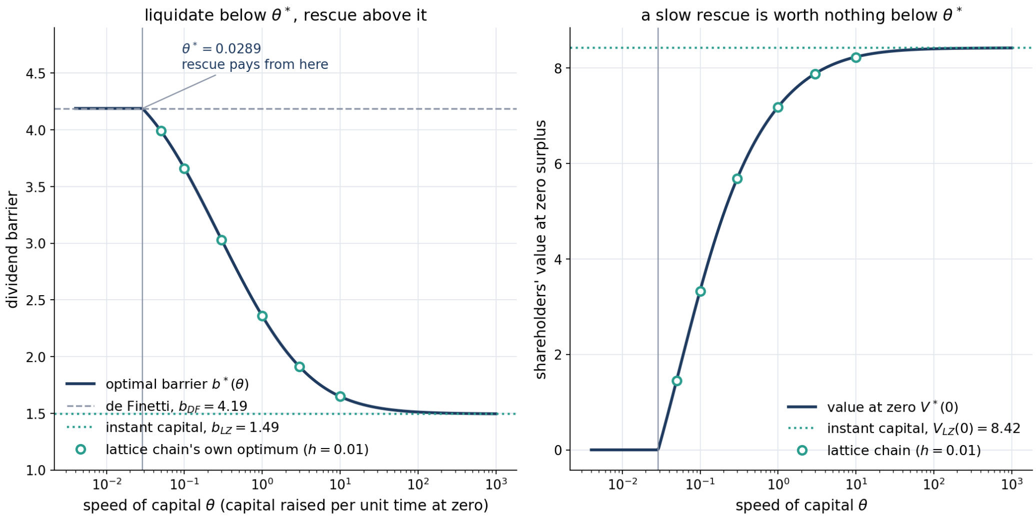

Figure 1: The optimal dividend barrier and the shareholders' value at zero surplus against the speed of capital on a logarithmic axis: the barrier is flat at de Finetti's level until the critical speed, then falls with a kink towards the instant-injection level, while the value at zero is zero until the critical speed and then rises towards the instant-injection value, with the lattice chain's own optima sitting on both curves.

The left panel is Theorem 3, Proposition 2 and Theorem 5 in one curve. The barrier sits at $b_{DF} = 4.19$ for every speed up to $\theta^* = 0.0289$, turns down with the slope −10.8 of Theorem 5, and descends through 2.36 at $\theta = 1$ towards $b_{LZ} = 1.49$. The right panel shows why the plateau exists: below the critical speed a rescue is worth nothing, so the firm is valued as though it would be liquidated, and above it the value at zero climbs towards 8.42. The six markers are the chain's own optima, found without the formula, and they lie on both curves.

Script. `fig_value.py`, which draws the optimal value function and its slope for three speeds of capital, with de Finetti's and the instant-injection value for reference.

Shows. How the whole value function, not only its value at zero, moves between the two classical problems.

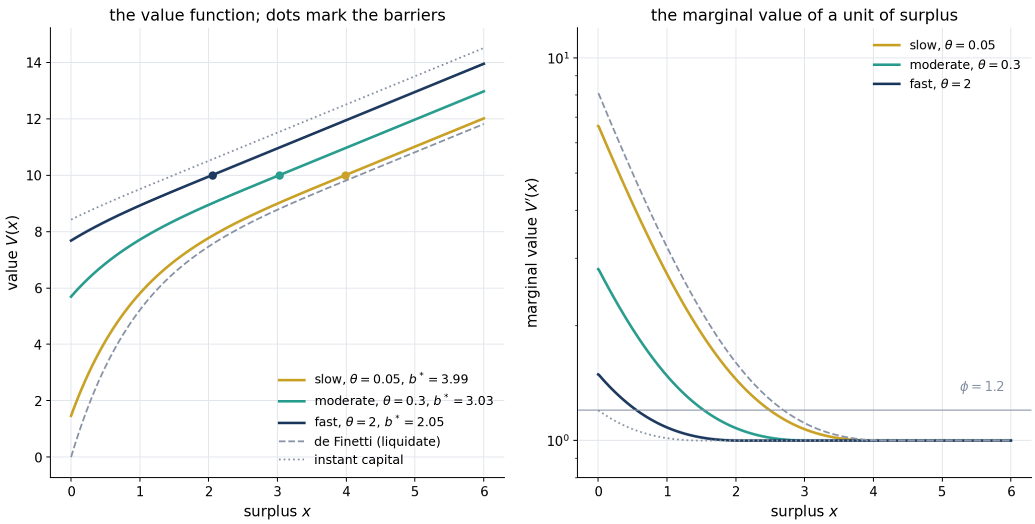

Figure 2: The value function and its slope against the surplus for slow, moderate and fast capital, lying between de Finetti's value below and the instant-injection value above, with dots marking where each reaches its barrier and the slope falling to one there.

The curves are ordered by speed everywhere, not only at zero, and they are ordered in the way Proposition 2 predicts: faster capital gives a higher value and an earlier barrier, marked by the dots. The right panel shows the marginal value on a logarithmic scale. At zero it ranges from $\bar{\phi} = 8.12$ for de Finetti's firm to $\phi = 1.2$ for instant capital, and for every speed it falls monotonically to one at the barrier and stays there, which is the concavity of Lemma 2 and the smooth fit of Theorem 1. The slow firm's marginal value at zero stays far above the price of capital, nearer de Finetti's than the instant-injection value, because at $\theta = 0.05$ it is barely above the critical speed.

Script. `fig_regions.py`, which draws the rescue and liquidation regions in the plane of speed and cost of frozen time for several prices, and the critical speed as a function of the price.

Shows. Theorem 3 and Theorem 4 as maps.

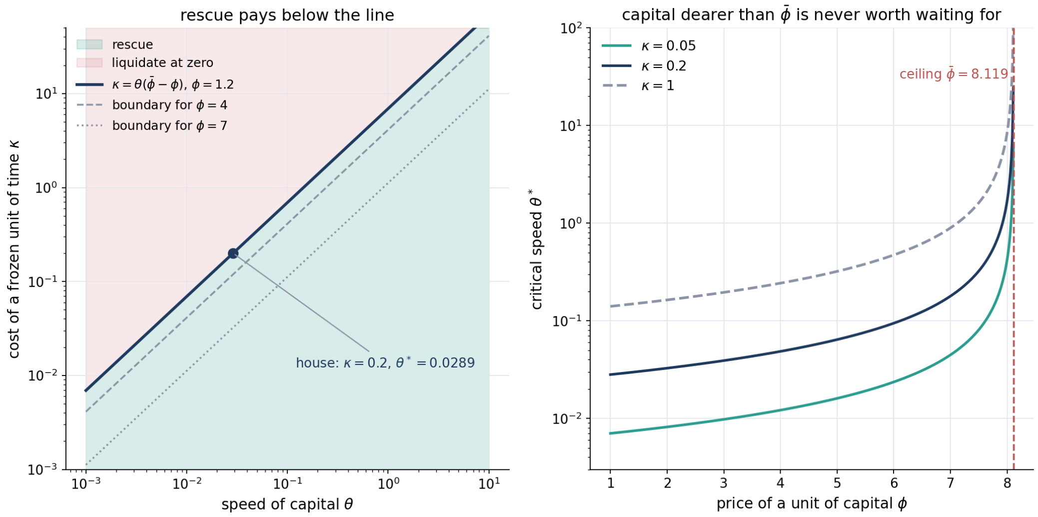

Figure 3: Left: on logarithmic axes of speed and cost of a frozen unit of time, the straight line kappa equals theta times the gap between the ceiling and the price divides the plane into a rescue region below and a liquidation region above, with the lines for dearer capital lying lower. Right: the critical speed against the price of capital for three costs of frozen time, each curve rising slowly and then exploding at the ceiling.

The boundary $\kappa = \theta(\bar{\phi} - \phi)$ is a straight line of slope one on logarithmic axes, so the rescue region is a half-plane: doubling the cost of frozen time is exactly offset by doubling the speed of capital. Dearer capital shifts the line down, shrinking the rescue region, and at $\phi = \bar{\phi}$ the line disappears entirely. The right panel shows how abruptly that happens. For each cost the critical speed grows slowly over most of the range of prices and then explodes at the ceiling $\bar{\phi} = 8.119$, the pole of $\kappa/(\bar{\phi} - \phi)$.

Script. `fig_paths.py`, which simulates one path of the lattice chain at $h = 0.02$ under the optimal policy for slow capital, $\theta = 0.05$, and for fast capital, $\theta = 2$, with the same random seed, and marks every moment the surplus spends at zero.

Shows. What a surplus frozen at zero looks like, and how the speed of capital changes the time the firm spends there.

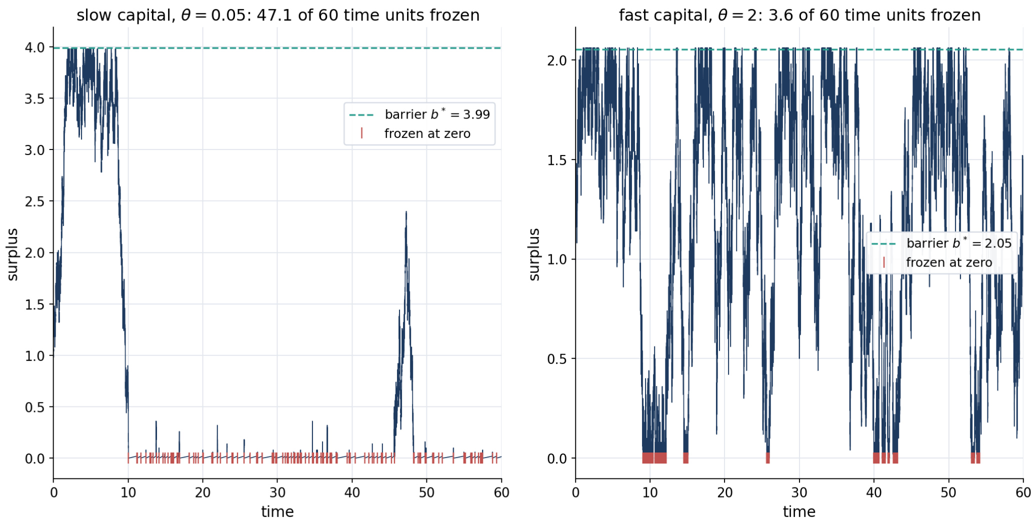

Figure 4: Two simulated surplus paths over sixty units of time under the optimal barrier, for slow and for fast capital, with the moments spent frozen at zero marked along the bottom.

The slow firm pays dividends at its barrier early on, falls to zero, and then spends most of the remaining time frozen: the panel's title reports 47.1 of 60 units of time at zero. It does not stay at zero on any interval; it leaves and returns constantly, and the marks crowd into a dense band, which is the sticky point of Definition 2 seen through a fine lattice. Capital trickles in at the rate θ while it is there, and the path escapes only when the noise carries it clear, as it does briefly near $t = 47$. The fast firm reaches zero repeatedly, but each visit is brief, 3.6 units of time in total, and it spends almost all of its history working below its barrier. The same seed drives both panels, so the difference is the speed of capital alone.

Script. `fig_flows.py`, which reruns the two paths of the previous figure, with the same seed and spacing, and accumulates the dividends paid at the barrier and the capital raised at zero.

Shows. The flows of Theorem 7 along single histories: what each firm pays out, and what it has to raise.

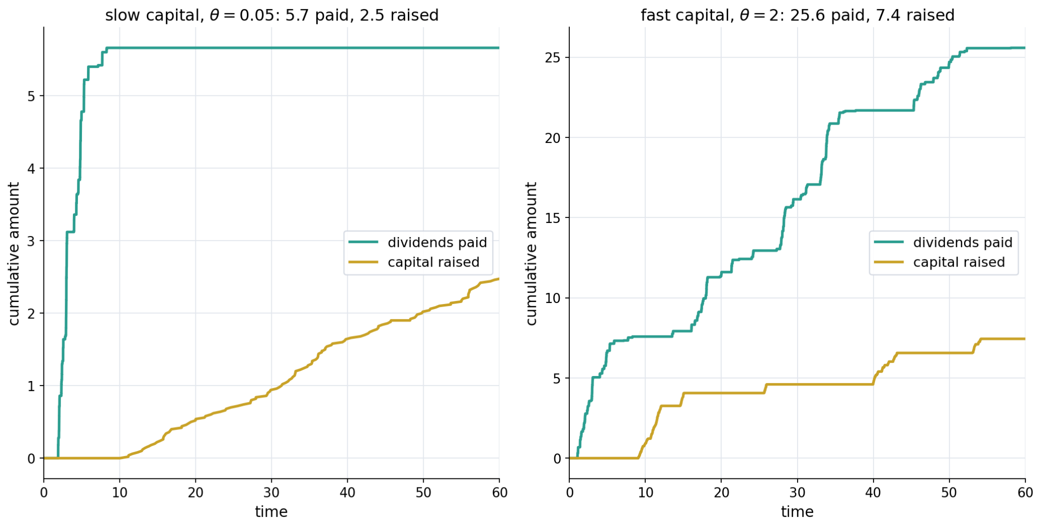

Figure 5: The cumulative dividends paid and capital raised along the two simulated paths over sixty units of time, for slow capital, which pays out early and then only raises capital, and for fast capital, which keeps paying throughout.

The two panels tell the story of the previous figure in money. The slow firm pays 5.7 in dividends, almost all of it in the first ten units of time, and then pays nothing for the remaining fifty while it raises capital in a thin, steady stream, 2.5 in all: the stream is the rate $\theta = 0.05$ at which capital arrives while the firm is frozen. The fast firm pays 25.6, in steps each time its surplus reaches the barrier, and raises 7.4 in short bursts, one per visit to zero. Fast capital does not make rescues rarer; this firm raises three times as much capital as the slow one. What it removes is the waiting, which is the reading of Theorem 7: the frozen time vanishes with the speed while the capital raised converges to the instant-injection amount.

Script. `fig_loss.py`, which computes the extra buffer and the value lost at zero from the barrier equation for speeds from 0.5 to 10^4, and draws both on logarithmic axes with the lines c_1/θ and $\phi c_1/\theta$ of Theorem 6.

Shows. The one-over-speed law.

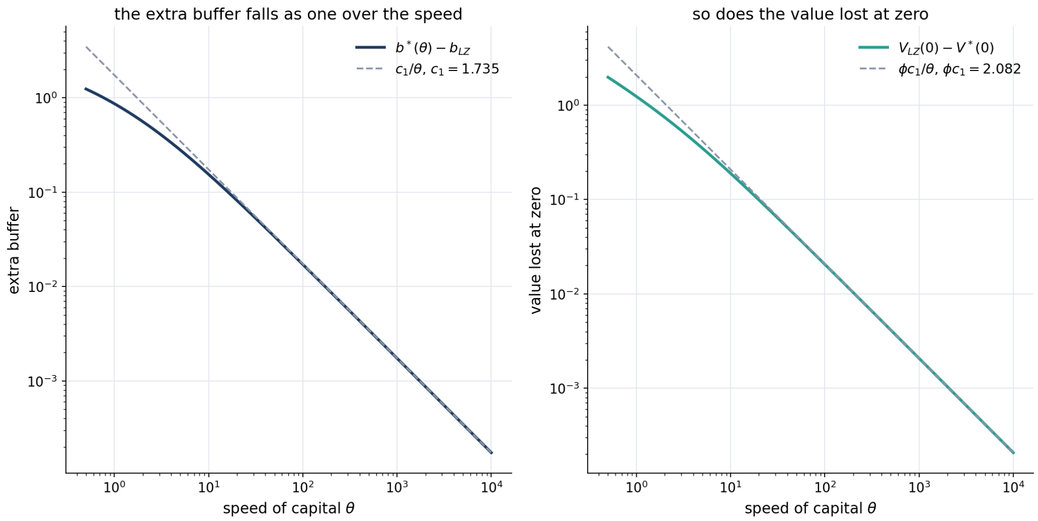

Figure 6: The extra buffer over the instant-injection barrier and the value lost at zero against the speed of capital on logarithmic axes, both converging onto straight lines of slope minus one with intercepts given by the constants of the fast-capital theorem.

Both curves become straight lines of slope -1 once the speed exceeds about ten, and they lie on the predicted lines $1.735/\theta$ and $2.082/\theta$. Below that they bend under the lines, because the expansion describes fast capital only and the barrier cannot rise above de Finetti's. The two panels differ by the factor $\phi = 1.2$ at every speed in the linear range, which is the statement that slow capital costs the market price of the buffer it forces the firm to hold.

7. Discussion

Remark 5 (What the critical speed says about capital in a crisis). In the model, a firm is worth rescuing exactly when $\theta(\bar{\phi} - \phi) > \kappa$. A crisis typically lowers θ, raises ϕ and raises κ at once, and all three move a firm towards liquidation.

Models in which financing conditions switch between good and bad states [19], or in which investors must be searched for [18], make the same point about costly and uncertain capital; the sticky model isolates the waiting at zero and prices it. The decomposition is useful because the three quantities are controlled by different parties. The price ϕ is set in the capital market, the cost κ largely by the legal and administrative process, and the speed θ by both, and by any facility that can advance capital while a rescue is arranged. The model says how they trade off: a facility that doubles the speed of capital is worth exactly as much, for the rescue decision, as halving the cost of administration. It also says what a firm should do in anticipation. At the house parameters its optimal buffer is 1.49 if capital is instant, and it rises to de Finetti's 4.19, nearly three times as much, as the speed falls towards the critical value; a firm that expects capital to be slow in a crisis should retain more and pay less before it. These are statements about the model, and their use in practice rests on how well a constant speed and a constant cost describe the process of raising capital.

Remark 6 (Jumps, other boundaries and a random speed). Three extensions keep the structure of the paper. Surplus with downward jumps, as in the Cramér-Lundberg model [4]; a mean-reverting surplus with the same sticky boundary, as in the sticky Ornstein-Uhlenbeck process [10]; and a speed of capital that depends on the state of the market.

With jumps, a firm can cross zero by an overshoot, and the sticky boundary has to be combined with a rule for negative surplus: either the overshoot is covered by an instant injection, or it is liquidated. The barrier structure is known to be delicate for jump processes: in the Cramér-Lundberg model the optimal policy can be a band rather than a barrier [28], and for spectrally negative Lévy processes a barrier is known to be optimal under conditions on the jumps [29], as the review [5] describes. The analogue of Theorem 1 with jumps is therefore a genuine question. Mean reversion changes the characteristic roots into confluent hypergeometric functions but leaves the sticky boundary condition as it is, so the analysis of Theorems 1 to 3 should carry over with the same reasoning and less explicit formulas. A random speed, switching between a normal and a crisis regime as financing conditions do in [19], would give a critical speed in each regime and a coupled pair of barrier equations; the kink of Theorem 5 would then become a region of speeds in which the optimal decision depends on which regime the firm expects to be in when it next reaches zero.

8. Conclusion

Capital that arrives at a finite speed turns the boundary of the dividend problem into a sticky point, and the sticky point turns out to be tractable. The surplus is held at zero while capital trickles in; the capital raised is its local time there and the time frozen is that local time divided by the speed. The equation has a weak but no strong solution, so the problem is posed on weak solutions and checked by a Markov chain rather than a pathwise scheme.

The optimal policy is a dividend barrier determined by one equation, the sticky boundary condition applied to the smooth-fit function. A rescue is worth paying for exactly when the speed of capital times the gap between its value at the brink and its price exceeds the cost of a frozen unit of time, $\theta(\bar{\phi} - \phi) > \kappa$, and the value at the brink is the ceiling $\bar{\phi} = \rho^{(\rho-1)/(\rho+1)}$, fixed by one ratio of the model's rates. Below the critical speed the firm behaves exactly as de Finetti's and is liquidated at zero; above it the buffer falls, leaving de Finetti's level with a kink and approaching the instant-injection barrier at the rate $1/\theta$, and the value lost to slow capital is the market price of the extra buffer.

At the house parameters the critical speed is 0.028906, the ceiling 8.1189, the kink slope -10.800 and the constant of the fast-capital law 1.7351. A lattice chain that never uses the closed forms finds its own barriers within one lattice step of the true ones, reproduces the value function with errors that halve with the spacing, and recovers the critical speed by extrapolation to six decimal places.

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