

TECHNICAL NOTE

The Square-Root Wall

When a Brownian Path Reaches c Standard Deviations of Its Own Age

Supervised by Dr. T. Zamrik

08-SEP-2025



Sample paths of stochastic logistic growth: $dX_t = r X_t(1 - X_t/K) dt + \sigma X_t dW_t$, $r = 7.89$, $K = 1$, $\sigma = 0.32$, $X_0 = 0.03$



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When a Brownian Path Reaches c Standard Deviations of Its Own Age

Supervised by Dr. T. Zamrik¹

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Abstract. Stop a Brownian path the first time it reaches c standard deviations of its own age, $|B_t| = c\sqrt{1+t}$. In the clock $s = \log(1+t)$ the wall becomes the fixed interval $(-c, c)$ for an Ornstein–Uhlenbeck process, and every moment of the wait follows: $\mathbb{E}[(1+\tau_c)^p] = 1/M(-p, \frac{1}{2}, c^2/2)$, with M Kummer’s function. The mean wait $c^2/(1-c^2)$ is finite exactly when $c < 1$, the variance exactly when $c < 0.741964$, and the tail is a power law whose exponent is the first zero of Kummer’s function in p . A one-sided wall has an exponent below one half at every height, so its mean wait is always infinite. A bracket that widens with $\sqrt{1+t}$, stop at one standard deviation and target at two, wins 20.17 per cent of the time, not a third, and its exponent 0.3633 is below one half, so its outcome has no mean. A bridge-corrected log-time simulator confirms every formula whose estimator has a finite variance, within 1.9 standard errors; a once-a-day check lengthens the mean wait up to 7.7 times, and the tail exponent recovers only slowly.

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1. Introduction

A Brownian path started at zero has standard deviation $\sqrt{t}$ at time t . A wall $c\sqrt{1+t}$ asks when the path first stands c standard deviations from its start, measured against its own age. The question is classical. Breiman studied the exit from a square-root boundary through the clock used here; Shepp [1] and Novikov [2] obtained the moments of the exit time and the conditions for them to be finite; Uchiyama [3] studied exit from, and time spent above, a one-sided moving boundary.

The 1 in $\sqrt{1+t}$ fixes the clock’s origin: a wall at $c\sqrt{t}$ is crossed at once, because near time zero a Brownian path exceeds every multiple of $\sqrt{t}$.

Traders place walls of this shape. A price’s standard deviation over a holding time t is $\sigma\sqrt{t}$, so a stop at two standard deviations of the expected move, or a target that widens as the trade ages, is a wall $c\sigma\sqrt{1+t}$ once time is counted in periods from one period before entry. In units of σ the price is, in this model, a Brownian motion: how long does a trade last on average, does the average exist, how often does the target come first?

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The answers turn abruptly on c . At $c = 2$ the tail exponent is 0.1215: the chance that a trade is still open after t periods decays like $t^{-0.12}$, and a reported average holding time grows without limit with the sample. Walls are also checked discretely; Broadie, Glasserman and Kou [8] showed that a barrier monitored at spacing Δ acts like a continuous one moved out by $0.5826 \sigma \sqrt{\Delta}$.

The classical theorems are not new here. The moment formula goes back to the papers above and is derived again for trapped particles in Grebenkov's review [5]; Denisov, Hinrichs, Sakhanenko and Wachtel [4] sharpened the tail asymptotics and extended them to boundaries only asymptotically of square-root form.

The note adds one route and four consequences. The route is the time change $s = \log(1 + t)$: every result follows from an eigenfunction of the Ornstein–Uhlenbeck generator, and Wald's identity alone settles both the two-sided mean and the one-sided infinite mean. The consequences are a table of the thresholds at which each moment is lost; a one-sided exponent below one half at every height; the win rate of a bracket that widens with $\sqrt{t}$, far below the fixed $a/(a+b)$, with a corollary that its outcome has no mean once its exponent drops to one half; and a measurement of a once-a-day check.

2. The Wall and Its Clock

Definition 1 (The wall times). Let B_t , $t \geq 0$, be a standard Brownian motion with $B_0 = 0$, adapted to a filtration satisfying the usual conditions [6]. For a wall height $c > 0$, a stop $a > 0$ and a target $b > 0$,

$$\begin{aligned}\tau_c &= \inf\{t : |B_t| \geq c\sqrt{1+t}\}, & \tau_c^+ &= \inf\{t : B_t \geq c\sqrt{1+t}\}, \\ \tau_{a,b} &= \inf\{t : B_t \notin (-a\sqrt{1+t}, b\sqrt{1+t})\}\end{aligned}$$

are the two-sided wall, the one-sided wall and the bracket, which ends at the target when $B_{\tau_{a,b}} = b\sqrt{1+\tau_{a,b}}$ and at the stop otherwise.

Each is a stopping time, the first entry of (t, B_t) into a closed set. In the trading reading B is the price change in units of its volatility per period and t the holding time in periods.

Assumption 1 (Standing hypotheses). B is the Brownian motion of Definition 1: no drift, unit variance per unit of time, $B_0 = 0$. The levels c , a and b are fixed positive constants, and every wall is monitored continuously except where the computation section measures a discrete check.

Unit variance is a choice of units; a drift outgrows $\sqrt{t}$ and changes every answer, as the closing remark explains.

Definition 2 (Kummer's function and the parabolic cylinder function). For real a , real $b \notin \{0, -1, -2, \dots\}$ and real z , Kummer's function

$$M(a, b, z) = \sum_{n \geq 0} \frac{(a)_n z^n}{(b)_n n!}, \quad (a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1),$$



solves $zw'' + (b - z)w' - aw = 0$ with $w(0) = 1$. For real ν , D_ν is the solution of $D'' + (\nu + \frac{1}{2} - x^2/4)D = 0$ that decays like $x^\nu e^{-x^2/4}$ as $x \rightarrow +\infty$ [7].

For $a = -n$ the series stops: $M(-1, \frac{1}{2}, z) = 1 - 2z$, $M(-2, \frac{1}{2}, z) = 1 - 4z + \frac{4}{3}z^2$. Also $D_1(x) = x e^{-x^2/4}$. Both functions are catalogued by Borodin and Salminen [7], and SciPy's `hyp1f1` and `pbdv` evaluate every exact number below.

Lemma 1 (The clock). Under Assumption 1, $U_s = e^{-s/2} B_{e^s-1}$ satisfies

$$dU_s = -\frac{1}{2} U_s ds + dW_s, \quad U_0 = 0,$$

for a standard Brownian motion W : an Ornstein–Uhlenbeck process with generator $\mathcal{L}f(u) = \frac{1}{2}f''(u) - \frac{1}{2}uf'(u)$. Hence $\tau_c = e^{S_c} - 1$ with $S_c = \inf\{s : |U_s| \geq c\}$, and τ_c^+ , $\tau_{a,b}$ are $e^S - 1$ for the exit time S of U from $(-\infty, c)$ and from $(-a, b)$.

Proof. $X_s = B_{e^s-1}$ is a continuous martingale in $\mathcal{G}_s = \mathcal{F}_{e^s-1}$ with $\langle X \rangle_s = e^s - 1$. Itô's formula gives the equation with $W_s = \int_0^s e^{-r/2} dX_r$, a continuous local martingale with $\langle W \rangle_s = \int_0^s e^{-r} d(e^r - 1) = s$, hence a standard Brownian motion by Lévy's characterisation [6]. As $t \mapsto \log(1+t)$ increases and $|B_t| \geq c\sqrt{1+t}$ exactly when $|U_{\log(1+t)}| \geq c$, the first such t is $e^{S_c} - 1$; likewise for $[c, \infty)$ and $\mathbb{R} \setminus (-a, b)$. $\square$

A wait S in log time is $e^S - 1$ in clock time, so an exponential tail in s is a power law in t ; and $\mathcal{L}$ is symmetric for the speed measure $2e^{-u^2/2} du$, so the killed problem is a regular Sturm–Liouville problem [7]. Figure 1 shows both clocks.

3. The Moments and the Tail

Theorem 1 (Every moment in closed form). Under Assumption 1, let $\lambda(c)$ be the smallest $p > 0$ with $M(-p, \frac{1}{2}, c^2/2) = 0$. For $0 \leq p < \lambda(c)$,

$$\mathbb{E}[(1 + \tau_c)^p] = \frac{1}{M(-p, \frac{1}{2}, c^2/2)},$$

and for $p \geq \lambda(c)$ the moment is infinite.

Proof. Put $f_p(u) = w(u^2/2)$, $w(z) = M(-p, \frac{1}{2}, z)$. Then $f'_p = u w'$, $f''_p = w' + u^2 w''$, and with $z = u^2/2$

$$\mathcal{L}f_p = z w'' + (\frac{1}{2} - z)w' = -p f_p$$

by Kummer's equation. By Itô's formula, $e^{ps} f_p(U_s)$ is a local martingale, bounded on finite intervals once stopped at S_c , so $\mathbb{E}[e^{p(S_c \wedge s)} f_p(U_{S_c \wedge s})] = 1$. For $p < \lambda(c)$, $f_p > 0$ on $[-c, c]$: by Sturm comparison for $(e^{-u^2/2} f')' + 2p e^{-u^2/2} f = 0$, the first positive zero of f_p moves continuously inward as p grows from 0, where it is at infinity. With $m_p = \min_{[-c, c]} f_p > 0$, $\mathbb{E}[e^{p(S_c \wedge s)}] \leq 1/m_p$, so $\mathbb{E}[e^{pS_c}] < \infty$ by monotone convergence and $S_c < \infty$ almost surely, and dominated convergence gives $\mathbb{E}[e^{pS_c} f_p(U_{S_c})] = 1$. As $U_{S_c} = \pm c$ and f_p is even, $\mathbb{E}[e^{pS_c}] = 1/f_p(c)$, the claim by Lemma 1. As $p \uparrow \lambda(c)$, $f_p(c) \downarrow 0$, and the moment, increasing in p , is infinite from $\lambda(c)$ on. $\square$



Equivalently, the moment of order p is finite exactly when $c < c_p$, the first zero of $M(-p, \frac{1}{2}, c^2/2)$ in c , as λ decreases strictly with $\lambda(c_p) = p$. The formula is classical [1], [2]; Grebenkov derives it for trapped particles [5].

Corollary 1 (The mean and the variance). Under Assumption 1,

$$\mathbb{E}[\tau_c] = \frac{c^2}{1-c^2} \quad (c < 1), \quad \mathbb{E}[(1+\tau_c)^2] = \frac{3}{3-6c^2+c^4} \quad (c^2 < 3-\sqrt{6}),$$

each infinite outside its range; $\lambda(1) = 1$, and τ_c has finite variance exactly when $c < \sqrt{3-\sqrt{6}} = 0.741964$.

Proof. At $z = c^2/2$ the polynomials of Definition 2 are $1 - c^2$ and $(3 - 6c^2 + c^4)/3$, with first zeros $c_1 = 1$ and $c_2 = \sqrt{3-\sqrt{6}}$; Theorem 1 applies exactly when $c < c_p$. $\square$

The mean wait is a third of a unit at $c = 0.5$ and 4.2632 units at $c = 0.9$; the variance is 0.1422 at $c = 0.5$ and 6.152 at $c = 0.7$. Between 0.741964 and 1 averages of waits converge more slowly than at the usual square-root rate.

The mean needs no special function. For $t > 0$ the stopped process $B_{\tau_c \wedge t}$ is bounded, so Wald's second identity [6] and the wall give

$$\mathbb{E}[\tau_c \wedge t] = \mathbb{E}[B_{\tau_c \wedge t}^2] \leq c^2(1 + \mathbb{E}[\tau_c \wedge t]). \quad (3.1)$$

For $c < 1$ this bounds $\mathbb{E}[\tau_c \wedge t]$ by $c^2/(1-c^2)$, so $\mathbb{E}[\tau_c] < \infty$ by monotone convergence; Wald's identity then holds at τ_c , where $B_{\tau_c}^2 = c^2(1 + \tau_c)$:

$$\mathbb{E}[\tau_c] = c^2(1 + \mathbb{E}[\tau_c]), \quad \mathbb{E}[\tau_c] = \frac{c^2}{1-c^2}. \quad (3.2)$$

For $c \geq 1$ a finite mean would force $(1-c^2)\mathbb{E}[\tau_c] = c^2 > 0$, which is impossible. The squared wall grows at rate c^2 and the path's quadratic variation at rate one; the path catches the wall in finite mean time exactly when the wall's square grows more slowly.

Proposition 1 (A power-law tail). Under Assumption 1, for some $C(c) > 0$,

$$\mathbb{P}(1 + \tau_c > t) = C(c) t^{-\lambda(c)} (1 + o(1)), \quad t \rightarrow \infty,$$

and $\lambda(c)$ decreases strictly, equals 1 at $c = 1$, behaves like $\pi^2/(8c^2)$ as $c \rightarrow 0$, and tends to 0 as $c \rightarrow \infty$.

Proof. By Lemma 1, $\mathbb{P}(1 + \tau_c > t) = \mathbb{P}(S_c > \log t)$. Killed at $\pm c$, $\mathcal{L}$ is a regular Sturm-Liouville operator, self-adjoint in L^2 of $m(du) = 2e^{-u^2/2} du$ [7], with simple eigenvalues $\lambda_1 < \lambda_2 < \dots$, normalised eigenfunctions φ_k , each even or odd, and $\varphi_1 > 0$. Expanding the killed density,

$$\mathbb{P}(S_c > s) = \sum_{k \geq 1} e^{-\lambda_k s} \varphi_k(0) \langle \varphi_k, 1 \rangle_m,$$

led by $C(c) = \varphi_1(0) \langle \varphi_1, 1 \rangle_m > 0$. Even eigenfunctions are multiples of f_p , so the even eigenvalues are the zeros of $p \mapsto f_p(c)$; φ_1 has no zero, so it is even and $\lambda_1 = \lambda(c)$.



Monotonicity is domain monotonicity, and $\lambda(1) = 1$ is Corollary 1. As $c \rightarrow 0$ the drift is negligible against $\frac{1}{2}f''$, whose eigenvalue on $(-c, c)$ is $\pi^2/(8c^2)$. As $c \rightarrow \infty$, for $p \in (0, 1)$, $M(-p, \frac{1}{2}, z) \sim \Gamma(\frac{1}{2})e^z z^{-p-1/2}/\Gamma(-p) \rightarrow -\infty$ as $\Gamma(-p) < 0$, so $\lambda(c) < p$ eventually. $\square$

Denisov, Hinrichs, Sakhanenko and Wachtel [4] sharpen this asymptotic for boundaries only asymptotically of square-root form. Numerically $\lambda(c) \cdot 8c^2/\pi^2$ is 0.99949 at $c = 0.05$ and 0.99998 at $c = 0.01$, and $\lambda(3) = 0.01197$.

Source. Theorem 1 and Propositions 1 and 2, evaluated with SciPy's `hyp1f1` and `pbdv` by a scan for the first sign change refined by Brent's method. Each exponent agrees with an independent finite-difference eigenvalue solve to a relative 1.2×10^{-6} .

Columns. Left: the order p and the threshold c_p below which $\mathbb{E}[(1 + \tau_c)^p]$ is finite. Right: a wall height c , the two-sided exponent $\lambda(c)$ and the one-sided exponent $\lambda^+(c)$.

order p	threshold c_p	wall height c	two-sided $\lambda(c)$	one-sided $\lambda^+(c)$
1/2	1.306930	0.5	4.6889	0.3244
1	1.000000	1.0	1.0000	0.1941
2	0.741964	1.5	0.3347	0.1043
3	0.616707	2.0	0.1215	0.0486
4	0.539080	3.0	0.0120	0.0058

The blocks are one function read two ways, $\lambda(c_p) = p$. The one-sided column never reaches one half.

Proposition 2 (The one-sided wall). Under Assumption 1, for every $c > 0$, $\tau_c^+ < \infty$ almost surely and $\mathbb{E}[\tau_c^+] = \infty$. Moreover $\mathbb{P}(1 + \tau_c^+ > t) \asymp t^{-\lambda^+(c)}$, where $\lambda^+(c)$ is the smallest $\lambda > 0$ with $D_{2\lambda}(-c) = 0$, and $0 < \lambda^+(c) < \frac{1}{2}$.

Proof. By the law of the iterated logarithm $\tau_c^+ < \infty$. If $\mathbb{E}[\tau_c^+] < \infty$, Wald's identity and $B_{\tau_c^+} = c\sqrt{1 + \tau_c^+}$ give $(1 - c^2)\mathbb{E}[\tau_c^+] = c^2$, so $c < 1$ and $\mathbb{E}[\tau_c^+] = \mathbb{E}[\tau_c]$ by the Wald derivation; but $\tau_c^+ \geq \tau_c$, strictly on $\{B_{\tau_c} < 0\}$, of probability $\frac{1}{2}$, a contradiction. For the exponent, substituting $D = e^{-x^2/4}h$, $x = -u$, in Definition 2 shows that $h_\lambda(u) = e^{u^2/4}D_{2\lambda}(-u)$ solves $h'' - uh' + 2\lambda h = 0$, that is $\mathcal{L}h_\lambda = -\lambda h_\lambda$, with $h_\lambda(u) \sim |u|^{2\lambda}$ as $u \rightarrow -\infty$, so $h_\lambda \in L^2(m)$. Killed at c on $(-\infty, c)$, $\mathcal{L}$ has a discrete spectrum, the drift confining the process; its principal eigenvalue is the smallest λ with $h_\lambda(c) = 0$, and the expansion of Proposition 1 gives the tail. At $c = 0$, $h_{1/2}(u) = -u > 0$ on $(-\infty, 0)$, so $\lambda^+(0) = \frac{1}{2}$, and the principal eigenvalue decreases strictly as the domain grows. $\square$

Since $\lambda^+(c) < \frac{1}{2}$, even $\mathbb{E}[(1 + \tau_c^+)^{1/2}]$ is infinite, and with it the expected excursion $\mathbb{E}[B_{\tau_c^+}]$; a fixed level's tail exponent is one half. Uchiyama [3] studies one-sided moving boundaries of this kind.



4. The Widening Bracket

Proposition 3 (The widening bracket's win rate). Under Assumption 1, $\tau_{a,b} < \infty$ almost surely and the bracket ends at its target with probability

$$\mathbb{P}(B_{\tau_{a,b}} = b\sqrt{1 + \tau_{a,b}}) = \frac{g(a)}{g(a) + g(b)}, \quad g(x) = \int_0^x e^{v^2/2} dv,$$

which is below the fixed bracket's $a/(a+b)$ when $b > a$ and above it when $b < a$.

Proof. Extend g to an odd function. As $g' = e^{u^2/2}$ and $g'' = u e^{u^2/2}$, $\mathcal{L}g = 0$, so $g(U_s)$ is a local martingale. The exit time of U from $(-a, b)$ is at most $S_{\max(a,b)}$, finite by Theorem 1, and the stopped $g(U)$ is bounded, so optional stopping gives $0 = g(0) = q g(b) - (1-q) g(a)$, with q the probability of ending at the target. Next, $q < a/(a+b)$ exactly when $g(a)/a < g(b)/b$, and $g(x)/x$, the average of the increasing $e^{v^2/2}$ over $[0, x]$, increases strictly with x . The fixed bracket is the same argument with scale function u . $\square$

This is the Ornstein–Uhlenbeck scale function [7] read as a trade: a fixed bracket's win rate depends only on b/a , a widening one on both levels, since the path is pulled back towards zero in log time.

Corollary 2 (When the outcome has a mean). Under Assumption 1, let $\lambda(a, b)$, the principal Dirichlet eigenvalue of $\mathcal{L}$ on $(-a, b)$, be the smallest $\lambda > 0$ with

$$f_e(-a) f_o(b) - f_e(b) f_o(-a) = 0, \quad f_e(u) = M(-\lambda, \tfrac{1}{2}, \tfrac{u^2}{2}), \quad f_o(u) = u M(\tfrac{1}{2} - \lambda, \tfrac{3}{2}, \tfrac{u^2}{2}).$$

Then $\mathbb{P}(1 + \tau_{a,b} > t) \asymp t^{-\lambda(a,b)}$ on a win and on a loss separately. If $\lambda(a, b) > \frac{1}{2}$, $B_{\tau_{a,b}}$ is integrable with mean zero; if $\lambda(a, b) \leq \frac{1}{2}$, its positive and negative parts both have infinite mean, and it has no mean.

Proof. f_e, f_o are the even and odd solutions of $\mathcal{L}f = -\lambda f$, the second from Kummer's second solution $z^{1/2} M(\frac{1}{2} - \lambda, \frac{3}{2}, z)$. Write $\tau = \tau_{a,b}$, S for the exit time of U from $(-a, b)$, and $w(u) > 0$ for the probability of leaving through b from u . By the Markov property and the expansion of Proposition 1,

$$\mathbb{P}(S > s, U_S = b) = \mathbb{E}[\mathbf{1}\{S > s\} w(U_s)] = \sum_{k \geq 1} e^{-\lambda_k s} \varphi_k(0) \langle \varphi_k, w \rangle_m,$$

with $\lambda_1 = \lambda(a, b)$ and a positive leading coefficient, as $\varphi_1, w > 0$; likewise with $1 - w$. On a win $B_\tau = b\sqrt{1 + \tau}$, and $\mathbb{E}[\sqrt{1 + \tau}; \text{win}] = \int_0^\infty \mathbb{P}(1 + \tau > x^2, \text{win}) dx$ is finite exactly when $2\lambda(a, b) > 1$; likewise on a loss. If $\lambda(a, b) > \frac{1}{2}$, $\mathbb{E}[\sqrt{\tau}] < \infty$ and Wald's first identity gives $\mathbb{E}[B_\tau] = 0$. $\square$

So the mean holding time exists exactly when $\lambda(a, b) > 1$ and the mean outcome exactly when $\lambda(a, b) > \frac{1}{2}$: the bracket $(1, 1)$, with $\lambda = 1$, has the second but not the first.

Example 1 (The 2:1 bracket, read off). Stop at $a = 1$, target at $b = 2$: risk one standard deviation of the holding time to make two. With $g(1) = 1.1950$ and $g(2) = 4.7289$,



Proposition 3 gives the win rate $1.1950/(1.1950 + 4.7289) = 0.2017$, and Corollary 2 gives $\lambda(1, 2) = 0.3633$.

The fixed-bracket rule says $1/3$, too high by a factor of 1.65, and since $0.3633 < \frac{1}{2}$ neither the outcome nor the holding time has a mean. The bracket $(1, 3)$ wins 0.0327 against the fixed 0.25, with $\lambda = 0.2167$. The bracket $(0.5, 1)$ has the same ratio as $(1, 2)$ but wins 0.3039, with $\lambda = 1.9602$ and a mean-zero outcome: a small bracket stays where $e^{v^2/2}$ is close to one. Reversed, $(2, 1)$ wins 0.7983, above the fixed $2/3$, with the same exponent and the same missing mean.

5. Computation

Algorithm 1 — The wall time in log time, with a bridge-corrected crossing test

Inputs: walls $\ell_- < 0 < \ell_+$ in units of $\sqrt{1+t}$ ($\ell_- = -\infty$ for one wall), the step Δ , a cap $s_{\max}$, the number of paths n , a seed. *Output:* each path's exit time S and wall; the wait is $\tau = e^S - 1$.

1. Set $U = 0$ and $s = 0$ for all n paths, held as one vector of live paths.
2. Advance each live path exactly: $U' = e^{-\Delta/2}U + \sqrt{1 - e^{-\Delta}}\xi$, ξ standard normal.
3. If $U' \geq \ell_+$ or $U' \leq \ell_-$, the path has left by that wall.
4. Otherwise, with uniforms $v_{\pm}$, it left by the upper wall if $v_+ < \exp(-2(\ell_+ - U)(\ell_+ - U')/\Delta)$, by the lower if $v_- < \exp(-2(U - \ell_-)(U' - \ell_-)/\Delta)$; if both, choose in proportion.
5. Record $S = s + \Delta/2$ for each path that left and remove it; add Δ to s and repeat until no path is live or $s \geq s_{\max}$, reporting survivors as censored.

Cost. Three random draws per live path per step; the note's thirty runs take three minutes.

The bridge test, measured. Without step 4, crossings between grid points are missed and exits are late by order $\sqrt{\Delta}$, as for discrete barriers [8]. At $c = 0.5$, a million paths, a common seed, the uncorrected mean is 0.44995 at $\Delta = 0.01$ and 0.38715 at $\Delta = 0.0025$ against the exact $1/3$, a bias that halves as the step is quartered; the corrected means are 0.33295 ± 0.00038 and 0.33353 ± 0.00038 , within 1.1 standard errors.

Method. Algorithm 1 with walls $\pm c$, scored on τ and $(1 + \tau)^2$ against Corollary 1. Where the next moment is infinite the standard error is invalid, and a lower moment is checked instead.

Parameters. $c = 0.3, 0.5, 0.7, 0.9, 1.2$; $n = 10^6$; $\Delta = c^2/100$; $s_{\max} = 60$, no path censored.

Result. The mean wait is 0.09895 ± 0.00009 against 0.09890 at $c = 0.3$, 0.33335 ± 0.00038 against 0.33333 at $c = 0.5$, 0.95821 ± 0.00207 against 0.96078 at $c = 0.7$; the second moment is 1.21561 ± 0.00022 against 1.21551 at $c = 0.3$ and 1.92063 ± 0.00198 against 1.92 at $c = 0.5$.

Error. The deviations are +0.51, +0.05, -1.25, +0.47 and +0.32 standard errors. At $c = 0.9$, where τ has infinite variance, the mean is 4.0155 against 4.2632, 5.8 per cent



low, with batch means from 3.682 to 4.751: the long waits that carry the mean are under-sampled. The same paths give $\mathbb{E}[(1+\tau)^{1/2}] = 1.76971 \pm 0.00137$ against 1.76904, within 0.48 standard errors. At $c = 0.7$ the second moment, whose estimator has infinite variance, is 8.10 against 9.997. At $c = 1.2$, where the mean is infinite, $\mathbb{E}[(1+\tau)^{1/4}] = 1.77486 \pm 0.00143$ against 1.77756, within 1.9 standard errors.

Method. In log time $\mathbb{P}(S > s) \approx Ce^{-\lambda s}$, so the exponent is the least-squares slope of the empirical $\log \mathbb{P}(S > s)$ on a window, opened at $s_1 = \log(1000)/(\lambda_{\text{next}} - \lambda)$ where the next contributing eigenvalue has decayed a thousandfold, and closed where about two thousand paths remain, at most $s = 80$. Errors from twenty batches.

Parameters. $n = 10^6$ per wall; $\Delta = 0.01$, and 0.0025 for the two-sided $c = 0.5$.

Result. Observed orders against the exact exponents:

wall	c	window in s	exponent	observed slope	deviation
two-sided	0.5	[0.17, 1.33]	4.6889	4.6933 ± 0.0127	+0.35
two-sided	1.0	[0.70, 6.21]	1.0000	0.9981 ± 0.0031	-0.63
two-sided	1.5	[1.56, 18.57]	0.3347	0.3338 ± 0.0008	-1.11
two-sided	2.0	[2.70, 51.15]	0.1215	0.1213 ± 0.0004	-0.60
one-sided	0.5	[7.66, 19.16]	0.3244	0.3255 ± 0.0031	+0.36
one-sided	1.0	[8.57, 32.01]	0.1941	0.1923 ± 0.0011	-1.70
one-sided	1.5	[9.63, 59.56]	0.1043	0.1043 ± 0.0004	-0.15
one-sided	2.0	[10.81, 80.00]	0.0486	0.0485 ± 0.0001	-0.93

Error. All eight are within 1.7 standard errors, the largest relative error 0.96 per cent. One-sided windows open late: there the next eigenvalue is only 0.64 to 0.90 higher.

Method. Algorithm 1 with walls $-a$ and b ; the win rate, with its binomial error, against Proposition 3 and against $a/(a+b)$; and the outcome B_τ .

Parameters. The brackets of Example 1; $n = 10^6$; $\Delta = \min(a, b)^2/100$; $s_{\max} = 80$, none censored.

Result.

bracket	simulated	Proposition 3	deviation	fixed $a/(a+b)$
(1, 1)	0.50050 ± 0.00050	0.50000	+1.00	0.5000
(1, 2)	0.20116 ± 0.00040	0.20172	-1.40	0.3333
(1, 3)	0.03248 ± 0.00018	0.03266	-1.01	0.2500
(0.5, 1)	0.30441 ± 0.00046	0.30388	+1.14	0.3333
(2, 1)	0.79802 ± 0.00040	0.79828	-0.66	0.6667

Error. Each win rate is within 1.4 standard errors; the fixed rule is 330 away at (1, 2). For (0.5, 1) the twenty batch means of B_τ lie between -0.0066 and 0.0075; for (1, 2) between



−3838 and 1250, one trade in a million carrying 19 per cent of the absolute total. Of the 160,336 trades of (1, 2) open at $s = 5$, 0.28551 ± 0.00113 end at the target, against 0.28551 from the principal eigenfunction’s exit flux: both exits carry the tail.

Method. B is simulated exactly on the grid $t = \Delta, 2\Delta, \dots$ and stopped at the first grid time with $|B_t| \geq c\sqrt{1+t}$, with no bridge test. The mean is compared with $c^2/(1-c^2)$ and with the wall moved out by $0.5826\sqrt{\Delta}$ [8], simulated by Algorithm 1; the tail slope with $\lambda(c)$ and with the shifted wall’s local exponent $\lambda(c + 0.5826\sqrt{\Delta/(1+t)})$ over the window.

Parameters. $c = 0.5$ and 0.8 ; $\Delta = 1$ and 0.1 ; 4×10^6 paths at $c = 0.5$ and 2×10^6 at $c = 0.8$; window from ten per cent survival to a thousand survivors.

Result. At $c = 0.5$ the mean wait is 2.5622 at $\Delta = 1$, 7.7 times the continuous 0.3333, and 0.7157 at $\Delta = 0.1$; the shifted wall gives 2.0725 and 0.6904, 78 and 93 per cent of the change. At $c = 0.8$: 9.71 and 3.349 against 1.7778, shifted 9.19 and 3.289, 93 and 96 per cent. The slopes are 2.85 and 3.53 against $\lambda(0.5) = 4.6889$, and 1.499 and 1.597 against $\lambda(0.8) = 1.6881$; the shifted wall predicts 2.84, 3.51, 1.480 and 1.571.

Error. Each slope agrees with its prediction to a relative 1.7 per cent while lying 5 to 39 per cent below $\lambda(c)$. As the shift decays like $(1+t)^{-1/2}$ the exponent returns to $\lambda(c)$ only in the limit; at $c = 0.8$, $\Delta = 1$ the slope climbs from 1.448 to 1.632 ± 0.049 across the window. At $c = 0.8$ the waits have infinite variance, and one path per run was censored.

Script. `fig_clock.py`: six paths to $t = 30$ from a seeded batch of four hundred, chosen so their stops spread across log time, one still running at $t = 30$.

Shows. The time change of Lemma 1.

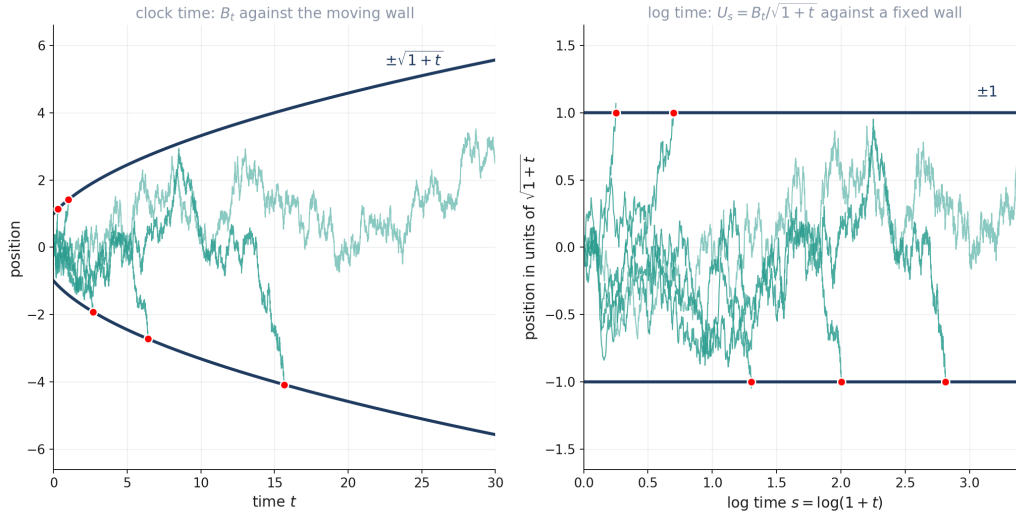


Figure 1: Six Brownian paths against the wall at one standard deviation of their age, in clock time against the curved wall on the left and in log time against flat walls at plus and minus one on the right, each stop marked.

In clock time the stops crowd towards the origin, at $t = 0.29, 1.01, 2.68, 6.42$ and 15.65 ; in log time they sit at $s = 0.25, 0.70, 1.30, 2.00$ and 2.81 , evenly spread against a wall that does not move. In the batch, 3.5 per cent of paths were still inside at $t = 30$.



Script. `fig_lambda.py`: both exponents on 320 wall heights from 0.05 to 3.

Shows. $\lambda(c)$ and $\lambda^+(c)$, with the moment thresholds marked.

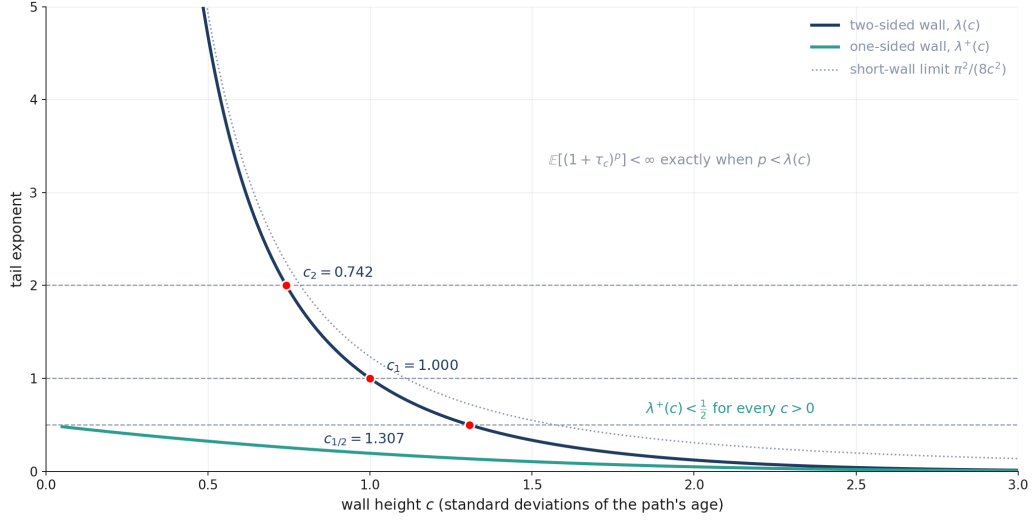


Figure 2: The tail exponent of the two-sided wall and of the one-sided wall against the wall height, with the short-wall limit dotted and the thresholds for the half moment, the mean and the variance marked on the two-sided curve.

The two-sided exponent falls from 493.2 at $c = 0.05$ to 0.0120 at $c = 3$; a moment of order p exists left of where it crosses the guide at p , at 1.307, 1 and 0.742. The one-sided curve starts at $\frac{1}{2}$ and never returns, its largest value on the grid 0.4803.

Script. `fig_bracket.py`: Proposition 3 for stops 0.5, 1 and 2, the fixed bracket at stop 1, and the five simulated brackets.

Shows. The widening bracket's win rate against the fixed-bracket rule.

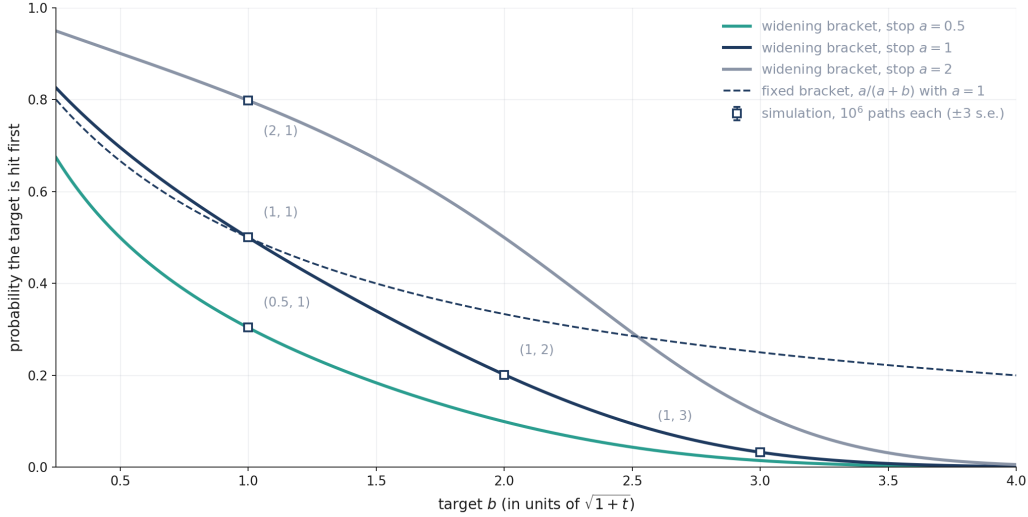


Figure 3: The probability that a widening bracket hits its target before its stop, against the target, for stops at one half, one and two, with the fixed bracket at stop one dashed and five simulated brackets marked with three-standard-error bars.

At stop 1 the two rules agree at $b = 1$. Beyond it the widening bracket wins less, 0.2017 against 0.3333 at $b = 2$ and 0.0015 against 0.2 at $b = 4$; below it, more. The simulated points sit on their curves, their error bars inside the markers.

6. Discussion

Remark 1 (A stop at c standard deviations of the holding time). For $c < 1$ the mean wait is finite: a third of a period at $c = 0.5$, 4.26 periods at $c = 0.9$. Between 0.742 and 1 the variance is infinite, so averages over many trades converge slowly and their error bars mislead. At $c \geq 1$ the mean wait is infinite, and a one-sided target scaled to $\sqrt{t}$ never has a finite mean wait. A 2:1 widening bracket wins about one time in five, not one in three, and its outcome has no mean, so no backtest average of its profit converges.

The model leaves out drift, which removes the power law (next remark), volatility clustering, gaps through stops and the cost of holding; each changes the numbers. What survives is the structure: whether an average exists is decided by a tail exponent, set by how fast the wall grows against the path's spread.

Remark 2 (What the note leaves out: drift, dimension, the full density). With a drift $\mu \neq 0$, $\mathbb{P}(\tau_c > t) \leq \mathbb{P}(|B_t + \mu t| < c\sqrt{1+t})$ decays like $e^{-\mu^2 t/2 + O(\sqrt{t})}$, so every moment of the two-sided and the bracket's wait is finite; a one-sided wall has every moment finite if the drift points towards it, and may never be reached if it points away. For a d -dimensional path and the radial wall $|B_t| = c\sqrt{1+t}$ the same route gives $\mathbb{E}[(1+\tau)^p] = 1/M(-p, d/2, c^2/2)$ [5]. The full law of the wait is the series of Proposition 1, of which only the first term is used.

Boundaries only asymptotically of square-root form, such as a wall set on a smoothed estimate of the spread, are treated by Denisov, Hinrichs, Sakhanenko and Wachtel [4]. A



discretely sampled path appears here only through the once-a-day check.

7. Conclusion

In the clock $s = \log(1 + t)$, a Brownian path stopped at c standard deviations of its own age is an Ornstein–Uhlenbeck process leaving the fixed interval $(-c, c)$, and that one change gives the rest. Every moment is exact, $\mathbb{E}[(1 + \tau_c)^p] = 1/M(-p, \frac{1}{2}, c^2/2)$: the mean $c^2/(1 - c^2)$ is finite exactly when $c < 1$, the variance exactly when $c < 0.741964$, and the tail exponent is the first zero of Kummer’s function in p . A one-sided wall has an exponent below one half at every height, so its mean wait is always infinite. The scale function gives the widening bracket’s win rate, 0.2017 at 2:1 rather than a third, and its exponent 0.3633 leaves its outcome without a mean. A bridge-corrected log-time simulator confirms the formulas within 1.9 standard errors, and a once-a-day check lengthens the mean wait up to 7.7 times while lowering the observed tail exponent on every reachable horizon.



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About the Desk

Stochastic Analysis Desk

Zamrik Quant Research

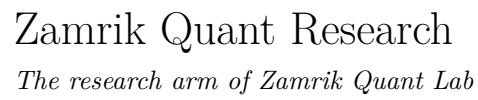
How random processes behave — Itô calculus, diffusions, local time, Fokker–Planck — in theory and in application.

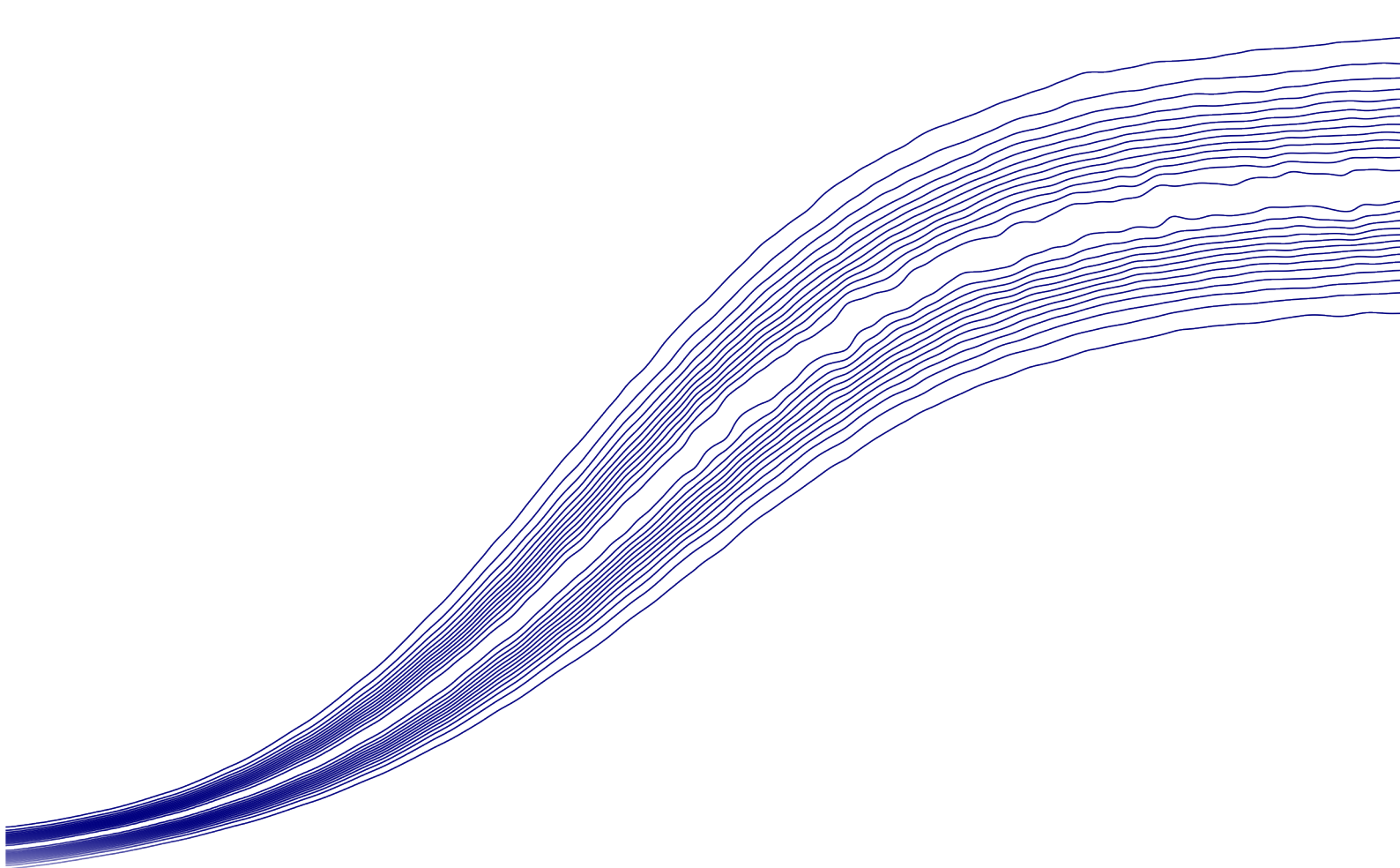
Stochastic analysis is the grammar of every other desk: a price, an order queue, a population or a value function is a functional of a random path, and what can be said about it depends on what can be proved about the path. The Stochastic Analysis desk works on those proofs and on the exact formulas they produce.

Its core questions are the classical ones, asked with care: when a diffusion first reaches a boundary, fixed or moving; how long it spends at each level before it does; and what happens at the boundary itself — reflection, killing, a sticky or skewed point, a reset to the start. The answers come through Itô calculus and martingale methods, local time and Tanaka’s formula, Ray–Knight theorems, time changes, and the Feynman–Kac and Fokker–Planck equations that turn questions about paths into equations for expectations and densities. Wherever the problem allows, the desk derives the law in closed form and then checks it against an independently built simulation.

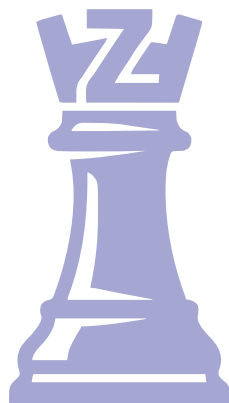
The results travel. The same calculus corrects a barrier that is checked once a day, explains why most stocks lose to the average stock, separates real correlation from the noise of random matrices, sets the acceptance rate of a Metropolis sampler, and models a neuron’s membrane and a day’s rainfall. The desk also studies the numerics that decide whether a simulated path tells the truth: the strong and weak convergence of Euler–Maruyama and Milstein schemes, Brownian-bridge corrections for barriers monitored in discrete time, and the error left when a continuous process is sampled on a grid.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Fokker-Planck density of stochastic logistic growth: $\partial_t p = -\partial_x [r x (1 - x/K) p] + \frac{1}{2} \sigma^2 \partial_{xx} (x^2 p)$



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