

TECHNICAL NOTE

Saddle-Point Structures for Two-Player Zero-Sum SDEs with Lévy Noise

Existence, coincidence of values, and a non-local first-order condition

Supervised by Dr. T. Zamrik

02-MAY-2023

Matérn RBF interpolant: $s(\mathbf{x}) = \sum_{i=1}^n \lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|)$, $A\lambda = f$, $\varphi(r) = (1 + \sqrt{5}\varepsilon r + \frac{5}{3}\varepsilon^2 r^2) e^{-\sqrt{5}\varepsilon r}$, $n = 36$



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Saddle-Point Structures for Two-Player Zero-Sum SDEs with Lévy Noise

Existence, coincidence of values, and a non-local first-order condition

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Abstract. We extend the zero-sum stochastic differential game to a state process driven by Lévy noise. With the generator and standing assumptions of the resolvent paper, the lower and upper values have at most linear growth and are Lipschitz in the state, and under the Isaacs condition they coincide as the unique viscosity solution of at most linear growth of a Hamilton–Jacobi–Isaacs integro-differential equation. Saddle strategies satisfy a first-order condition that is non-local in the attacker’s control: it integrates the value against the jump measure the attacker controls. A central-bank-versus-speculator game on a Kou state has a constant saddle pair and an affine value in closed form, which a numerical evaluation of the Isaacs equation and an exact Monte Carlo at an exponential time both confirm.

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1. Introduction

Two-player zero-sum stochastic differential games arise whenever a decision-maker faces a worst-case-adversarial environment whose state is itself stochastic — a central bank defending a peg against a speculator, a risk manager allocating capital against an adversarial market, a planner pricing carbon against an adversarial emitter. The classical Brownian formulation is well developed: Bensoussan and Frehse [1] introduced the elliptic-PDE characterisation; Fleming and Souganidis [2] gave the modern viscosity-solution treatment.

This technical note extends both threads to a state process X driven by a **Lévy noise**. The generator acquires a non-local jump term, and the value functions satisfy an integro-differential Hamilton–Jacobi–Isaacs equation rather than a local one. The resolvent paper [3] fixes the operator register used here and proves a comparison principle for the optimal-stopping variational inequality under the same generator [3, Theorem 5]; the present note adapts that argument to the Isaacs equation and assembles a self-contained zero-sum framework.

¹Correspondence: Optimal Control & Decision Desk, Zamrik Quant Research, info@zamrik.com.



1.1 Contributions

Claim 1 (Value-function regularity). *The lower value V^- and the upper value V^+ of the zero-sum Lévy game have at most linear growth and are Lipschitz in x with constant L_ℓ/q , under assumptions (A1)–(A2) of [3] and a running cost Lipschitz in x with constant L_ℓ .*

Claim 2 (Coincidence of values). *Under the Isaacs minimax condition*

$$\sup_{u \in U} \inf_{v \in A} [H(x, p, M, u, v) + I_0 + vI_a] = \inf_{v \in A} \sup_{u \in U} [H(x, p, M, u, v) + I_0 + vI_a],$$

the values coincide and are the unique viscosity solution of at most linear growth of the HJI integro-differential equation $qV = H^*[V]$.

Claim 3 (Saddle-point first-order condition). *When the running cost is concave in the defender's control and convex in the attacker's, an interior saddle at state x is characterised by*

$$\partial_x V(x) + \partial_u \ell(x, u^*, v^*) = 0, \quad \mathcal{J}_a[V](x) + \partial_v \ell(x, u^*, v^*) = 0,$$

*and a feedback that selects it is a saddle pair of the game. The second condition is **non-local**: $\mathcal{J}_a[V](x) = \int (V(x+z) - V(x)) \nu_a(dz)$ integrates the value against the jump measure ν_a the attacker controls.*

Claim 4 (Worked example: drift-and-intensity game). *In a **central-bank-versus-speculator** game on a Kou state, in which the bank controls the drift and the speculator the rate of up-jumps, the saddle pair is constant and the value is affine, both in closed form in the mean drift $\psi'(0)$ and the expected up-jump displacement per unit time $\lambda p/\eta_+$.*

Claim 5 (Two-way check). *The closed form of Claim 4 is confirmed by evaluating the Isaacs equation numerically — the jump integrals by quadrature, the sup-inf and inf-sup by direct search — and by Monte Carlo of the payoff from the exact law of the controlled state at an independent exponential time.*

1.2 Paper map

§2 fixes notation, adopting the operator register of [3, §2]. §3 proves the regularity claim. §4 states the HJI integro-differential equations and establishes value coincidence. §5 derives the non-local first-order condition. §6 specialises to the central-bank-versus-speculator game, solves it in closed form and checks the solution numerically. §7 concludes.

2. Setup and notation

We adopt the operator register of [3, §2] and summarise it here, adding the game-specific notation.



2.1 State process

The uncontrolled state $X = (X_t)_{t \geq 0}$ is a càdlàg Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (b, σ^2, ν) , cumulant exponent $\psi(r) = \log \mathbb{E}_0[e^{rX_1}]$ on the analyticity strip $(-\eta_-, \eta_+)$, and infinitesimal generator

$$\mathcal{L}f(x) = b \partial_x f(x) + \frac{1}{2} \sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} (f(x+z) - f(x) - z \partial_x f(x) \mathbb{1}_{\{|z| \leq 1\}}) \nu(dz). \quad (2.1)$$

We use assumptions **(A1)** (finite exponential moments) and **(A2)** (small jumps) of [3, §2]. Assumption (A3) there concerns the payoff of a stopping problem; this note has none.

2.2 Controls and strategies

Two players act on the state. Player I (the **defender**) shifts the drift by u ; Player II (the **attacker**) adds jumps at rate v times a finite **attack measure** ν_a on $\mathbb{R} \setminus \{0\}$ with $\int (1 + |z|) \nu_a(dz) < \infty$ and $\int e^{rz} \nu_a(dz) < \infty$ for r near 0, so that (A1) holds for $\nu + v \nu_a$ as well.

Definition 1 (Admissible controls and strategies). Let $U = [u_{\min}, u_{\max}]$ and $A = [0, \bar{v}]$ be compact intervals and $\bar{u} := \max(|u_{\min}|, |u_{\max}|)$. $\mathcal{U}$ is the set of $\mathcal{F}$ -progressively-measurable processes with values in U , and $\mathcal{V}$ the same with values in A . A **strategy** of the defender is a map $\alpha : \mathcal{V} \rightarrow \mathcal{U}$ that is non-anticipative — $v = v'$ on $[0, t]$ implies $\alpha[v] = \alpha[v']$ on $[0, t]$ — and Γ is the set of such maps; Δ is the set of non-anticipative strategies $\beta : \mathcal{U} \rightarrow \mathcal{V}$ of the attacker, as in [2].

Under controls (u, v) the state started at x is $X_t^{u,v} = x + Y_t^{u,v}$, where $Y_t^{u,v} = \int_0^t (b + u_s) ds + \sigma W_t + J_t + J_t^a$: W is a Brownian motion, J the jump part of the uncontrolled process (small jumps compensated as in (2.1)), and $J_t^a = \int_0^t \int \int z \mathbb{1}_{\{\theta \leq v_s\}} N^a(ds, dz, d\theta)$ for an independent Poisson random measure N^a on $[0, \infty) \times \mathbb{R} \times [0, \bar{v}]$ with intensity $ds \nu_a(dz) d\theta$. The attack jumps arrive at rate $v_s \nu_a(dz)$ and, ν_a being finite with a first moment, need no compensation. At frozen controls the generator is

$$\mathcal{L}^{u,v} f = \mathcal{L}f + u \partial_x f + v \mathcal{I}_a f, \quad \mathcal{I}_a f(x) := \int_{\mathbb{R}} (f(x+z) - f(x)) \nu_a(dz). \quad (2.2)$$

2.3 Costs and value functions

The running cost $\ell : \mathbb{R} \times U \times A \rightarrow \mathbb{R}$ is continuous and Lipschitz in x with constant L_ℓ , uniformly in (u, v) . The discount rate is $q > 0$.

Definition 2 (Payoff and value functions). The **payoff** at initial state x under controls (u, v) is

$$J(x; u, v) := \mathbb{E}_x \left[\int_0^\infty e^{-qt} \ell(X_t^{u,v}, u_t, v_t) dt \right]; \quad (2.3)$$

the defender maximises it and the attacker minimises it. The **lower value** and the **upper**



value are

$$V^-(x) := \inf_{\beta \in \Delta} \sup_{u \in \mathcal{U}} J(x; u, \beta[u]), \quad V^+(x) := \sup_{\alpha \in \Gamma} \inf_{v \in \mathcal{V}} J(x; \alpha[v], v). \quad (2.4)$$

When $V^- = V^+$ we write V for the **common value** and call the game **value-defined**.

In the lower value the attacker answers the defender's control; in the upper value the defender answers the attacker's.

2.4 Hamiltonian

Definition 3 (Hamiltonian and non-local terms). For $(x, p, M) \in \mathbb{R}^3$ and $(u, v) \in U \times A$,

$$\begin{aligned} H(x, p, M, u, v) &:= (b + u)p + \frac{1}{2}\sigma^2 M + \ell(x, u, v), \\ \mathcal{J}[V](x; v) &:= \mathcal{J}_0[V](x) + v \mathcal{J}_a[V](x), \\ \mathcal{J}_0[V](x) &:= \int_{\mathbb{R}} (V(x + z) - V(x) - z \partial_x V(x) \mathbb{1}_{\{|z| \leq 1\}}) \nu(dz). \end{aligned} \quad (2.5)$$

The **lower** and **upper Hamiltonians** are

$$\begin{aligned} H^-[V](x) &:= \sup_{u \in U} \inf_{v \in A} \{H(x, \partial_x V, \partial_{xx} V, u, v) + \mathcal{J}[V](x; v)\}, \\ H^+[V](x) &:= \inf_{v \in A} \sup_{u \in U} \{H(x, \partial_x V, \partial_{xx} V, u, v) + \mathcal{J}[V](x; v)\}, \end{aligned} \quad (2.6)$$

and $H^- \leq H^+$ always.

2.5 Typographic register

All notation conventions are inherited from [3, §2]: script $\mathcal{L}, \mathcal{R}_q, \mathcal{J}$ for operators and $\mathcal{U}, \mathcal{V}$ for control classes; blackboard $\mathbb{E}, \mathbb{P}, \mathbb{R}$ for probability and reals; ∂_x rather than subscript for partial derivatives.

3. Value-function regularity

Theorem 1 (Linear growth and Lipschitz regularity). Under **(A1)**–**(A2)** and a running cost ℓ Lipschitz in x with constant L_ℓ , uniformly in (u, v) , the lower and upper values of (2.4) satisfy

$$\begin{aligned} |V^\pm(x)| &\leq \frac{\ell_0 + L_\ell |x|}{q} + L_\ell C_Y \left(\frac{1}{q} + \frac{1}{q^2} \right), \\ |V^\pm(x_1) - V^\pm(x_2)| &\leq \frac{L_\ell}{q} |x_1 - x_2|, \end{aligned} \quad (3.1)$$

where $\ell_0 := \max_{U \times A} |\ell(0, u, v)|$ and C_Y depends only on $b, \sigma, \nu, \nu_a, \bar{u}$ and $\bar{v}$. If ℓ is bounded, $\|V^\pm\|_\infty \leq \|\ell\|_\infty / q$.

Proof. Three steps.



Step 1. (Coupling.) In the construction of §2 the state started at x is $x + Y^{u,v}$, and $Y^{u,v}$ is built from W , the jump measures and the controls alone: no coefficient depends on the state. Strategies do not change this, since $\alpha[v]$ and $\beta[u]$ are functions of the opponent's control, not of x .

Step 2. (Growth.) Under (A1)–(A2) and controls bounded by $\bar{u}$ and $\bar{v}$,

$$\begin{aligned} \mathbb{E}|Y_t^{u,v}| &\leq (|b| + \bar{u})t + \left(\sigma + \left(\int_{|z|\leq 1} z^2 \nu(dz)\right)^{1/2}\right)\sqrt{t} \\ &\quad + t \int_{|z|>1} |z| \nu(dz) + \bar{v}t \int |z| \nu_a(dz) \leq C_Y(1+t), \end{aligned}$$

so $|J(x; u, v)| \leq \int_0^\infty e^{-qt}(\ell_0 + L_\ell|x| + L_\ell C_Y(1+t)) dt$, which is the first bound in (3.1), for every pair of controls; taking infima and suprema over controls and strategies preserves it. For bounded ℓ the same integral gives $\|\ell\|_\infty/q$, the computation of [3, Lemma 1].

Step 3. (Lipschitz.) By Step 1, for every pair of controls,

$$|J(x_1; u, v) - J(x_2; u, v)| \leq \mathbb{E} \int_0^\infty e^{-qt} |\ell(x_1 + Y_t, u_t, v_t) - \ell(x_2 + Y_t, u_t, v_t)| dt \leq \frac{L_\ell}{q} |x_1 - x_2|, \quad (3.2)$$

in particular for the pairs $(\alpha[v], v)$ and $(u, \beta[u])$. Since $|\sup_\alpha \inf_v f - \sup_\alpha \inf_v g| \leq \sup |f - g|$, and likewise for $\inf \sup$, the bound passes to $V^\pm$. $\square$

Remark 1 (Sharpness). The constant L_ℓ/q is attained: in the game of §6 the value is affine with slope $-\alpha/q$, and $L_\ell = \alpha$. Lipschitz continuity comes from Step 1: the controls act on the drift and the jump rate, not through the state, so two copies of the state started apart stay exactly $x_1 - x_2$ apart. With state-dependent coefficients the copies separate, and the same argument then needs q large relative to the Lipschitz constants of the coefficients; this note does not treat that case.

Remark 2 (Discount-rate dependence). The Lipschitz constant scales as $1/q$ and the growth bound as $1/q^2$, deteriorating as $q \downarrow 0$. The limit $q = 0$ requires an ergodic-cost reformulation beyond the scope of this note.

4. The HJI integro-differential equation

This section states the Hamilton–Jacobi–Isaacs equations satisfied by the two values, the Isaacs condition, and the value-coincidence theorem.

4.1 The HJI equations

By the dynamic-programming principle for games with non-anticipative strategies — the argument of Fleming and Souganidis [2], with Itô's formula for semimartingales with jumps [4] in place of the diffusion one — the lower value solves the **lower HJI equation**

$$qV^-(x) = H^-[V^-](x), \quad (4.1)$$

and the upper value solves the **upper HJI equation**



$$qV^+(x) = H^+[V^+](x), \quad (4.2)$$

in the viscosity sense — the approach of [5] — with the non-local convention of [3, Definition 3]: the test function is used near the contact point and the solution itself in the jump integrals away from it.

4.2 The Isaacs condition

Definition 4 (Isaacs minimax condition). The **Isaacs condition** holds if for every $x \in \mathbb{R}$ and $(p, M, I_0, I_a) \in \mathbb{R}^4$

$$\sup_{u \in U} \inf_{v \in A} [H(x, p, M, u, v) + I_0 + vI_a] = \inf_{v \in A} \sup_{u \in U} [H(x, p, M, u, v) + I_0 + vI_a]. \quad (4.3)$$

Then $H^- = H^+ =: H^*$, and (4.1) and (4.2) are the same equation.

The Isaacs condition holds whenever the running cost separates, $\ell(x, u, v) = \ell_1(x, u) + \ell_2(x, v)$: the bracket is then a function of u plus a function of v , and both sides equal the sum of the separate extrema. It also holds whenever ℓ is concave in u and convex in v , by the minimax theorem for such functions on compact intervals [6]; the worked example of §6 has both properties.

4.3 Coincidence of values

Theorem 2 (Value coincidence). Under **(A1)**–**(A2)**, the hypotheses of Theorem 1 and the Isaacs condition (4.3), the lower and upper values coincide,

$$V^-(x) = V^+(x) =: V(x), \quad x \in \mathbb{R}, \quad (4.4)$$

and V is the unique viscosity solution of at most linear growth of

$$qV(x) = H^*[V](x). \quad (4.5)$$

Proof. Three steps.

Step 1. (Viscosity solutions.) Each value is a viscosity solution of its own HJI equation, by the dynamic-programming argument of [2] with the jump terms handled as in §4.1, and has at most linear growth by Theorem 1.

Step 2. (Comparison.) Under the Isaacs condition both equations are (4.5). Its comparison principle in the class of functions of at most linear growth follows the proof of [3, Theorem 5] for the optimal-stopping variational inequality, with the obstacle removed and the linear operator $q - \mathcal{L}$ replaced by $q - H^*$. That proof uses three properties of the operator, and H^* has all three: it is Lipschitz in p with constant $\max_{u \in U} |b + u|$, being a sup–inf of functions affine in p with these slopes; it is non-decreasing in M and in the two non-local terms, since $\sigma^2 \geq 0$ and $v \geq 0$; and it is continuous in x , uniformly in the other arguments, because ℓ is Lipschitz in x . The jump integral of ν is split at $|z| = 1$ exactly as there, and the attack term, a finite measure with a first moment, is treated like the large jumps.



Hence a viscosity sub-solution of at most linear growth lies below a super-solution of at most linear growth.

Step 3. (Identification.) V^- and V^+ are viscosity solutions of (4.5) of at most linear growth, so by Step 2 each lies below the other: (4.4) holds, and the common V is the unique such solution. $\square$

4.4 Consequences

The value-coincidence theorem reduces the **two-player zero-sum game** with Lévy noise to **one HJI equation**, of the same type as the single-player optimal-stopping variational inequality of [3, §5]. The saddle pair, characterised in §5 by a non-local first-order condition, is then constructed by verification once V is known.

5. The non-local first-order condition

This section characterises saddle strategies. The first-order condition in the attacker's control is **non-local**: it picks up an integral of the value against the attack measure, which has no analogue when the attacker acts on a local coefficient.

5.1 Saddle points of the Hamiltonian and of the game

Theorem 3 (Non-local first-order condition for saddle strategies). Assume the hypotheses of Theorem 2, that $\ell(x, \cdot, v)$ is concave on U and $\ell(x, u, \cdot)$ convex on A and both differentiable, and that the common value V is C^2 with $V, \partial_x V, \partial_{xx} V$ of at most linear growth.

(i) For every x , a point (u°, v°) in the interior of $U \times A$ is a saddle point of

$$(u, v) \mapsto H(x, \partial_x V(x), \partial_{xx} V(x), u, v) + \mathcal{J}[V](x; v), \quad (5.1)$$

a maximum in u at v° and a minimum in v at u° , if and only if

$$\begin{aligned} \partial_x V(x) + \partial_u \ell(x, u^\circ, v^\circ) &= 0, \\ \mathcal{J}_a[V](x) + \partial_v \ell(x, u^\circ, v^\circ) &= 0. \end{aligned} \quad (5.2)$$

(ii) Let $(u^*(x), v^*(x))$ be a measurable selection of saddle points of (5.1), and suppose the state equation with $u_t = u^*(X_{t-})$ against any $v \in \mathcal{V}$, and with $v_t = v^*(X_{t-})$ against any $u \in \mathcal{U}$, has a unique solution. Then

$$J(x; u, v^*) \leq J(x; u^*, v^*) = V(x) \leq J(x; u^*, v), \quad u \in \mathcal{U}, v \in \mathcal{V}. \quad (5.3)$$

The second line of (5.2) is the **non-local first-order condition**: $\mathcal{J}_a[V](x) = \int (V(x+z) - V(x)) \nu_a(dz)$ depends on V at every point of $x + \text{supp } \nu_a$.

Proof. In (5.1), H is affine in u apart from ℓ and $\mathcal{J}$ is affine in v , so the map is concave in u and convex in v . For a differentiable concave (convex) function on an interval, an



interior point is a maximiser (minimiser) if and only if the derivative vanishes there, and $\partial_u(H + \mathcal{J}) = \partial_x V + \partial_u \ell$, $\partial_v(H + \mathcal{J}) = \mathcal{J}_a[V] + \partial_v \ell$.

(ii) Fix $v \in \mathcal{V}$ and let X be the state under $(u^*(X_{t-}), v_t)$. By Itô's formula for $e^{-qt}V(X_t)$ [4], after localisation,

$$\mathbb{E}_x[e^{-qT}V(X_T)] = V(x) + \mathbb{E}_x \int_0^T e^{-qt} (\mathcal{L}^{u^*, v_t} V - qV)(X_t) dt. \quad (5.4)$$

The bracket in (5.1) equals $\mathcal{L}^{u, v} V + \ell(\cdot, u, v)$. At $(u^*(x), v^*(x))$ it equals $H^*[V](x) = qV(x)$ by (4.5), and at $(u^*(x), v)$ it is no smaller, since $v^*(x)$ minimises it. Hence $\mathcal{L}^{u^*, v} V - qV \geq -\ell(\cdot, u^*, v)$ and $V(x) \leq \mathbb{E}_x \int_0^T e^{-qt} \ell dt + \mathbb{E}_x[e^{-qT}V(X_T)]$. The last term vanishes as $T \rightarrow \infty$ by the linear growth of V and Step 2 of Theorem 1, so $V(x) \leq J(x; u^*, v)$. The other inequality is symmetric, and both are equalities at (u^*, v^*) . $\square$

When the attacker acts on a local coefficient instead — the drift or the volatility — its condition involves $\partial_x V$ or $\partial_{xx} V$ at x only. With jumps it involves V on the whole support of $x + \nu_a$: the attacker weighs where its jumps land.

5.2 Existence of a saddle pair

The first-order condition (5.2) characterises interior saddles; existence needs structure.

Corollary 1 (Saddle existence under concave–convex costs). Under the hypotheses of Theorem 3, with $\ell(x, \cdot, v)$ strictly concave and $\ell(x, u, \cdot)$ strictly convex, the Isaacs condition holds, and for every x the map (5.1) has exactly one saddle point $(u^*(x), v^*(x))$ in $U \times A$. When it is interior it is the unique solution of (5.2).

Proof. The bracket of (4.3) is continuous on the compact intervals U and A , strictly concave in u and strictly convex in v , for every (x, p, M, I_0, I_a) ; by the minimax theorem [6] its sup–inf and inf–sup agree, which is the Isaacs condition, and by compactness and continuity both are attained, which gives a saddle point. Saddle points of a zero-sum game are interchangeable — if (u_1, v_1) and (u_2, v_2) are saddle points, so is (u_1, v_2) — so strict concavity in u and strict convexity in v leave only one. The last statement is Theorem 3 (i). $\square$

6. Worked example: a central-bank-versus-speculator game

This section specialises §§2–5 to a zero-sum game between a central bank (the defender, maximising) and a speculator (the attacker, minimising). The state is the log exchange rate, a Kou jump-diffusion, and a rise is a depreciation. The bank chooses an intervention drift $u \in U = [-\bar{u}, \bar{u}]$; the speculator chooses an intensity $v \in A = [0, \bar{v}]$ that scales up the rate of up-jumps — a speculative attack.



6.1 The bank-versus-speculator setup

Let $X = \log S$, with S_t the spot exchange rate in domestic currency per unit of foreign currency, be the Kou process of [3, §4]: drift b of the compound-Poisson representation (the convention of [3, §2] for finite-activity families), volatility σ , jump rate λ , up-jump probability p , and exponential jump sizes with rates η_+ and η_- . The attack measure is the up-jump part of ν , $\nu_a(dz) = \lambda p \eta_+ e^{-\eta_+ z} \mathbb{1}_{\{z > 0\}} dz$, so under constant controls (u, v) the state is again a Kou process, with cumulant exponent

$$\psi^{u,v}(r) = (b+u)r + \frac{1}{2}\sigma^2 r^2 + (1+v)\lambda p \frac{\eta_+}{\eta_+ - r} + \lambda(1-p) \frac{\eta_-}{\eta_- + r} - (1+v)\lambda p - \lambda(1-p). \quad (6.1)$$

The bank's running payoff is

$$\ell(x, u, v) = -\alpha x - \frac{1}{2}\gamma u^2 + \frac{1}{2}\delta v^2, \quad \alpha, \gamma, \delta > 0 : \quad (6.2)$$

depreciation costs the bank α per unit of log rate per unit time, the bank pays a quadratic cost for intervening, and the speculator pays a quadratic cost for attacking, which in a zero-sum game is a gain to the bank. The payoff is concave in the bank's control and convex in the speculator's, as Theorem 3 and Corollary 1 require, and separable, so the Isaacs condition holds. We write

$$a := \frac{\alpha}{q}, \quad m_+ := \int z \nu_a(dz) = \frac{\lambda p}{\eta_+}, \quad \bar{\mu} := \psi'(0) = b + \frac{\lambda p}{\eta_+} - \frac{\lambda(1-p)}{\eta_-}, \quad (6.3)$$

for the marginal value of a unit of log rate, the expected up-jump displacement per unit time, and the mean drift of the uncontrolled state.

6.2 Closed-form saddle

Theorem 4 (Closed-form saddle for the bank-versus-speculator game). In the game of (6.1)–(6.2) the saddle pair is constant,

$$u^* = \Pi_U(-a/\gamma), \quad v^* = \Pi_A(a m_+/\delta), \quad (6.4)$$

with Π the projection onto the interval; the game is value-defined, and its value is affine,

$$V(x) = -a x + c, \quad c = \frac{1}{q} \left[-a \bar{\mu} - a u^* - \frac{1}{2}\gamma (u^*)^2 - a m_+ v^* + \frac{1}{2}\delta (v^*)^2 \right], \quad (6.5)$$

which for interior controls reads $c = q^{-1}[-a \bar{\mu} + a^2/(2\gamma) - a^2 m_+^2/(2\delta)]$. For all admissible u and v , $J(x; u, v^*) \leq V(x) \leq J(x; u^*, v)$.

Proof. Through the Isaacs equation. For $V(x) = -a x + c$, $\partial_x V = -a$, $\partial_{xx} V = 0$, and in the finite-activity convention $\mathcal{J}_0[V](x) = -a \int z \nu(dz) = -a(\lambda p/\eta_+ - \lambda(1-p)/\eta_-)$ and



$\mathcal{J}_a[V](x) = -a m_+$. The bracket of Definition 3 becomes

$$-\alpha x - a\bar{\mu} + g(u) + h(v), \quad g(u) := -au - \frac{1}{2}\gamma u^2, \quad h(v) := -a m_+ v + \frac{1}{2}\delta v^2. \quad (6.6)$$

It separates; g is strictly concave with maximiser $\Pi_U(-a/\gamma)$ on U , and h strictly convex with minimiser $\Pi_A(am_+/\delta)$ on A , so $H^*[V](x) = -\alpha x - a\bar{\mu} + g(u^*) + h(v^*)$. Since $qV(x) = -\alpha x + qc$, (4.5) holds for every x exactly when c is as in (6.5). V has linear growth, so it is the value by Theorem 2, and Theorem 3 gives the saddle pair; for interior controls (5.2) reads $-a - \gamma u^* = 0$ and $-a m_+ + \delta v^* = 0$, the second being the non-local condition.

Directly. For admissible (u, v) , $\mathbb{E}_x[X_t] = x + \bar{\mu}t + \mathbb{E} \int_0^t (u_s + m_+ v_s) ds$, because the attack adds jumps with compensator $v_s \nu_a(dz) ds$. Integrating against e^{-qt} , with $\int_0^\infty e^{-qt} \int_0^t f_s ds dt = q^{-1} \int_0^\infty e^{-qs} f_s ds$,

$$J(x; u, v) = -a x - \frac{a\bar{\mu}}{q} + \mathbb{E} \int_0^\infty e^{-qt} [g(u_t) + h(v_t)] dt. \quad (6.7)$$

The integrand is at most $g(u^*) + h(v_t)$ and at least $g(u_t) + h(v^*)$, pointwise, so $J(x; u, v^*) \leq J(x; u^*, v^*) = V(x) \leq J(x; u^*, v)$ for all admissible u and v . The same bounds hold against strategies — $\alpha \equiv u^*$ guarantees $V(x)$ to the bank, and $\inf_v J(x; \alpha[v], v) \leq J(x; \alpha[v^*], v^*) \leq V(x)$ for every $\alpha \in \Gamma$ — so $V^- = V^+ = V$. $\square$

Neither control depends on the state, because the marginal value of the log rate, $-a$, is the same everywhere. The attack is worth $a m_+$ per unit of intensity to the speculator — the expected up-jump displacement per unit time times the marginal value — and this is where the jump law enters the saddle.

Remark 3 (No threshold, no root). The game has no stopping time and hence no free boundary, and with a payoff linear in the state its Isaacs equation is solved on the whole line by an affine function: neither a root of $\psi^{u,v}(r) = q$ nor a stopping threshold enters V . The fluctuation theory of [3] would enter if the bank could also abandon the peg at a stopping time for a put-type payoff $(K - x)^+$: for each constant control pair the state is a Kou process, whose stopping threshold is $b^* = K + \mathbb{E}[I]$ of [3, Theorem 7], with I its infimum at an independent exponential time, set by the two negative roots of $\psi^{u,v}(r) = q$. The joint game of control and stopping is not treated here.

6.3 Numerical check of the closed form

We take the Kou benchmark of [3] — $b = 0$, $\sigma = 0.2$, $\lambda = 1$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$, so that $\bar{\mu} = -0.02$ and $m_+ = 0.06$ — and the costs $\alpha = 0.01$, $\gamma = 2$, $\delta = 0.1$ with $\bar{u} = 0.5$, $\bar{v} = 1$. Theorem 4 gives $u^* = -0.10$ and $v^* = 0.12$, both interior, and $V(0) = 0.2656$. The script `python/saddle_check.py` checks this two ways.

The Isaacs equation, evaluated numerically. At nine states $x \in \{-2, -1.5, \dots, 2\}$ we evaluate the bracket of Definition 3 on the closed-form V , with the jump integrals by adaptive quadrature and the derivatives by central differences, and compute its sup–inf and inf–sup over $U \times A$ by nested bounded Brent searches, which never use (6.4). The residual



$qV(x) - H^-[V](x)$ is at most 1.1×10^{-12} , so V solves the equation of Theorem 2; the sup-inf and the inf-sup differ by at most 2.2×10^{-18} , the Isaacs condition; the searched saddle controls agree with (6.4) to 1.5×10^{-9} in u and 7.1×10^{-9} in v , the first-order conditions of Theorem 3; and the non-local term $\mathcal{J}_a[V]$ by quadrature equals $-a m_+ = -0.012$ to 4.9×10^{-17} .

Monte Carlo at an exponential time. For constant controls the controlled state is a Lévy process, so the payoff is a resolvent, and the killed representation in the proof of [3, Theorem 1] gives $J(x; u, v) = q^{-1} \mathbb{E}_0[\ell(x + X_{\mathbf{e}(q)}^{u,v}, u, v)]$, with $\mathbf{e}(q)$ an independent exponential time of rate q . We sample $X_{\mathbf{e}(q)}^{u,v}$ under $\mathbb{P}_0$ exactly: $\mathbf{e}(q)$, a Gaussian increment, Poisson numbers of up- and down-jumps given $\mathbf{e}(q)$, and Gamma-distributed sums of their sizes. Figure 1 shows the two one-sided profiles through the saddle, $J(0; u, v^*)$ at 11 values of u and $J(0; u^*, v)$ at 11 values of v , with 4×10^5 paths per point. Against the closed-form curves the 22 points give $\chi^2 = 28.8$ on 22 degrees of freedom ($p = 0.15$; largest $|z| = 2.76$). With 2×10^6 paths the saddle value is $J(0; u^*, v^*) = 0.26520 \pm 0.00037$ against $V(0) = 0.26560$ ($z = -1.08$), and the slope of Theorem 1, from independent runs at $x = 0$ and $x = 1$, is -0.19926 ± 0.00052 against $-\alpha/q = -0.2$ ($z = +1.42$): the bound of Theorem 1 is attained, as Remark 1 says.

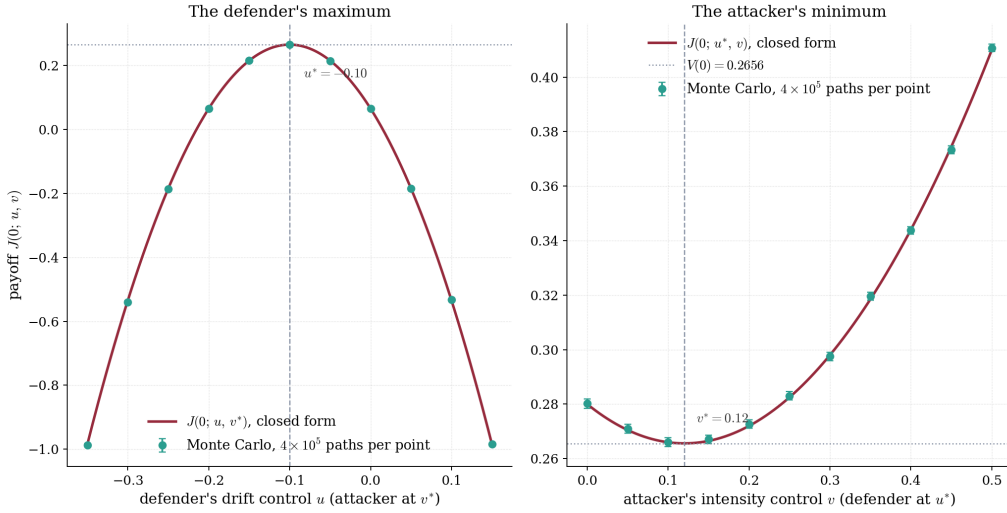


Figure 1: The two one-sided profiles through the saddle of Theorem 4 at $x = 0$, benchmark of §6. Left: the bank's payoff $J(0; u, v^*)$ against its drift control, maximal at $u^* = -0.10$. Right: the payoff $J(0; u^*, v)$ against the speculator's intensity control, minimal at $v^* = 0.12$. Curves: closed form. Points: Monte Carlo from the exact law of the controlled state at an independent exponential time, 4×10^5 paths each, with 95 % error bars. The dotted line is $V(0) = 0.2656$ in both panels.

7. Conclusion

This technical note extended the zero-sum stochastic differential game of Bensoussan and Frehse [1] and Fleming and Souganidis [2] to a state process driven by a Lévy noise satisfying assumptions (A1)–(A2) of [3].



We proved that the values have at most linear growth and are Lipschitz in the state (Theorem 1), that under the Isaacs condition they coincide as the unique viscosity solution of at most linear growth of the HJI integro-differential equation (Theorem 2), and that saddle strategies satisfy a **non-local first-order condition** (Theorem 3). The non-locality is the main qualitative difference from the Brownian case: when the attacker controls a jump intensity, its condition integrates the value against the measure of the jumps it controls. In the central-bank-versus-speculator game of §6 the saddle pair is constant and the value affine (Theorem 4), and the closed form is confirmed by a numerical evaluation of the Isaacs equation and by Monte Carlo.

7.1 What this note deliberately omits

Four directions are left for future work:

1. **Leader–follower games.** If the bank commits to a policy first and the speculator best-responds, the game is a Stackelberg game rather than a zero-sum one; the leader’s problem is constrained by the follower’s best response, and the Isaacs-condition arguments above do not apply.
2. **Non-zero-sum (Nash) games.** The N-player Nash equilibrium with Lévy noise reduces to a coupled system of N HJB equations, for which the comparison argument of §4 does not by itself give uniqueness.
3. **Mean-field limits.** The $N \rightarrow \infty$ continuum-of-players limit produces a master equation, a research-paper-length topic.
4. **Stopping and asymmetric information.** A bank that may abandon the peg adds a stopping time to the game (Remark 3), and worst-case filtering under Knightian uncertainty changes the state space; each needs a separate treatment.



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About the Desk

Optimal Control & Decision Desk

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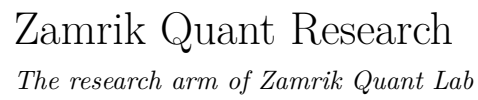
What to do under uncertainty: optimal control and HJB, mean-field games, agent-based models, irreversible decisions, betting, lotteries and insurance.

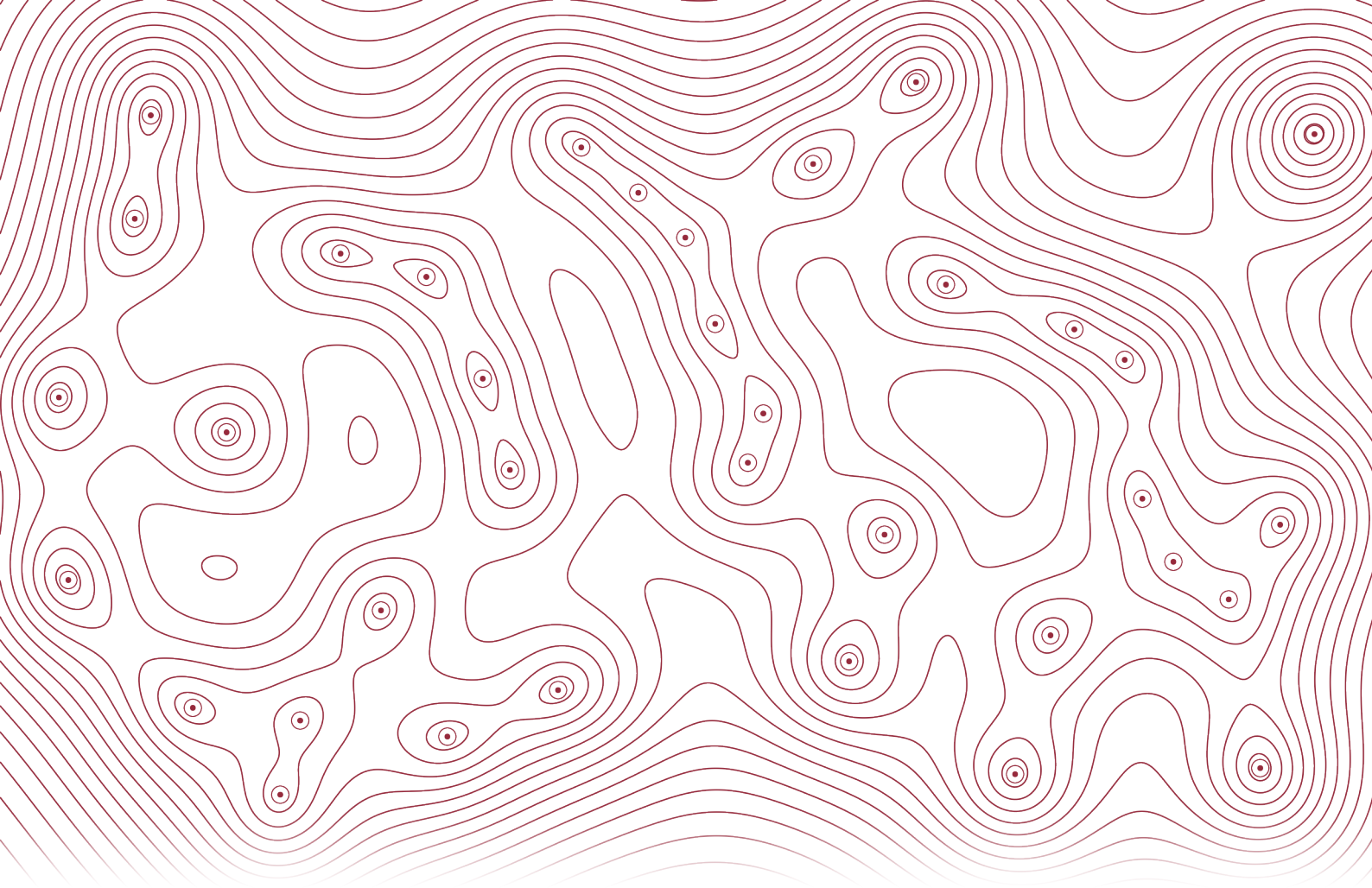
Most of this desk's questions have the same shape: someone must act now, without knowing what comes next, and live with the result. The Optimal Control & Decision desk turns such questions into Hamilton–Jacobi–Bellman equations, optimal stopping and singular control problems, and solves them.

When several parties optimise at once, the problem becomes a game: Dynkin games and wars of attrition between two or three players, and mean field games, in which a large population and its representative member shape each other through coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations. Where the equations give out, agent-based models take over, and the desk says plainly which tool fits which case.

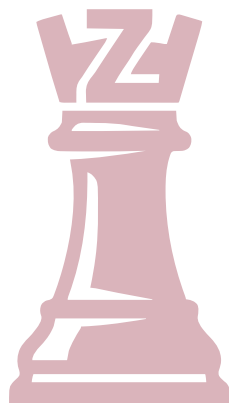
The applications are concrete: when to build an irreversible plant, how a company should pay dividends, how to run a home battery against a time-of-use tariff, when a wind turbine should turn into the wind, how a central bank should answer stagflation, and how much to bet when an edge is small. Its notebooks and essays carry the same reasoning to retirement, market timing, lotteries, insurance and prize competitions, each worked through to a decision a reader could actually make.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Power function of the Matérn interpolant: $P(\mathbf{x})^2 = \varphi(0) - \varphi_X(\mathbf{x})^\top A^{-1} \varphi_X(\mathbf{x})$, zero at every data point $\mathbf{x}_i$



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