

PREPRINT

Resolvent Operators for One-Dimensional Lévy Processes: A Unified Treatment

Wiener–Hopf factorisation and rational families

Supervised by Dr. T. Zamrik

01-APR-2023

Matérn RBF interpolant: $s(\mathbf{x}) = \sum_{i=1}^n \lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|)$, $A\lambda = f$, $\varphi(r) = (1 + \sqrt{3}\varepsilon r) e^{-\sqrt{3}\varepsilon r}$, $n = 42$



PR-2026-68980035
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Suggested Citation

Stochastic Analysis Desk, Zamrik Quant Research (2023). 'Resolvent Operators for One-Dimensional Lévy Processes: A Unified Treatment: Wiener–Hopf factorisation and rational families'. *Zamrik Quant Research Preprint PR-2026-68980035*. Available at <https://zamrik.com/research-items/pr-2026-68980035/>.

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PREPRINT



Resolvent Operators for One-Dimensional Lévy Processes: A Unified Treatment

Wiener–Hopf factorisation and rational families

Supervised by Dr. T. Zamrik¹

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PR-2026-68980035

Abstract. We give a unified treatment of the resolvent operator $\mathcal{R}_q$ of a one-dimensional Lévy process, organised around its Wiener–Hopf factorisation $\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^-$ into the resolvents of the running supremum and the running infimum. For three rational families — Kou double-exponential, hyperexponential and meromorphic — the factors act on exponentials through the Wiener–Hopf factors $\Phi_q^\pm$, finite or convergent infinite products over the roots of $\psi(r) = q$ and the jump rates, and the supremum and infimum at an exponential time are mixtures of exponential laws. We prove a comparison principle for the optimal-stopping variational inequality, identify its solution with the stopping value, and show that the perpetual put is governed by the infimum: the optimal threshold is $b^* = K + \mathbb{E}[X_{\mathbf{e}(q)}]$ for the payoff $(K - x)^+$ and $e^{b^*} = K \mathbb{E}[e^{X_{\mathbf{e}(q)}}]$ for $(K - e^x)^+$. Under Kou jumps both close in the two negative roots of $\psi(r) = q$; at the benchmark $b^* = -0.0897$. Every closed form is checked by an independent method — exact Monte Carlo of the process killed at an exponential time, and a finite-difference solve of the integro-differential equations — and agrees within the Monte Carlo error and at the grid’s second-order rate.

PR-2026-68980035 · Stochastic Analysis Desk · Preprint · Published 01-APR-2023

· Reading time: about 35 min

1. Introduction

The resolvent operator $\mathcal{R}_q$ of a one-dimensional Lévy process governs nearly every quantitative problem in stochastic analysis and control. Perpetual optimal stopping, perpetual control, free-boundary problems with non-local generators — all of them reduce, at some critical step, to the action of $\mathcal{R}_q$ on a test function. The purpose of this paper is to develop $\mathcal{R}_q$ in unified form, establish the Wiener–Hopf factorisation $\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^-$, and tabulate its closed form on exponential test functions for three rational families: the Kou double-exponential jump model, hyperexponential mixtures, and the meromorphic class.

The decision to treat the resolvent as a stand-alone subject is deliberate. In the existing literature, the Wiener–Hopf factorisation is presented variously as a fluctuation identity in the probabilistic tradition [1], as a pseudo-differential operator-symbol decomposition in the analytic tradition [2], and as a computational device in pricing applications [3], [4].

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None of these treatments quite suffices as a reference that later work can cite for “the closed form of $\mathcal{R}_q^+ e_r$ under Kou jumps”, together with the optimal-stopping threshold it produces. This paper supplies that reference.

Closed forms in this area are easy to get subtly wrong — a root taken on the wrong side of the origin, one exponential where the jumps require two, a normalising factor of q — and an error of that kind is invisible to a reader who only checks the algebra. We therefore check every closed form twice: once by evaluating it, and once by a method that does not use it — an exact Monte Carlo simulation of the process killed at an independent exponential time (Algorithm 2), or a finite-difference solve of the integro-differential equation or variational inequality (Algorithm 3). The checks are reported next to the results they test (§§3.4, 4.5, 6.3 and 7).

1.1 Contributions

We make five claims, summarised here and proved in §§3–6.

Claim 1 (*Wiener–Hopf factorisation*). *For every Lévy process satisfying (A1)–(A2) and every $q > 0$, the resolvent factorises as $\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^-$, where $\mathcal{R}_q^\pm$ are the resolvents of the supremum process $\bar{X}_t := \sup_{0 \leq s \leq t} X_s$ and the infimum process $\underline{X}_t := \inf_{0 \leq s \leq t} X_s$; equivalently $q/(q - \psi(r)) = \Phi_q^+(r) \Phi_q^-(r)$. Theorem 1.*

Claim 2 (*Rational closed forms for $\mathcal{R}_q^\pm$*). *For the Kou, hyperexponential and meromorphic families with a diffusive part, $\mathcal{R}_q^\pm e_r = q^{-1} \Phi_q^\pm(r) e_r$ with Φ_q^+ a finite or infinite product over the positive roots of $\psi(r) = q$ and the up-jump rates, and Φ_q^- the same over the negative roots and the down-jump rates; the supremum and minus the infimum at an exponential time are mixtures of exponential laws whose rates are those roots. Theorems 2, 3 and 4.*

Claim 3 (*Comparison principle*). *Under (A1)–(A3), the optimal-stopping variational inequality $\min\{(q - \mathcal{L})u, u - g\} = 0$ has a unique viscosity solution of at most linear growth, and it coincides with the optimal-stopping value $V(x) = \sup_\tau \mathbb{E}_x[e^{-q\tau} g(X_\tau)]$. Theorems 5 and 6.*

Claim 4 (*The threshold is set by the infimum*). *For the perpetual put the stopping region is a half-line $(-\infty, b^*]$ with $b^* = K + \mathbb{E}[\underline{X}_{e(q)}]$ for $g(x) = (K - x)^+$ and $b^* = K \mathbb{E}[e^{\underline{X}_{e(q)}}]$ for $g(x) = (K - e^x)^+$; continuous fit always holds and smooth fit holds exactly when 0 is regular for $(-\infty, 0)$. Under Kou jumps $b^* = K - 1/\beta_1 - 1/\beta_2 + 1/\eta_-$, where $-\beta_1, -\beta_2$ are the two negative roots of $\psi(r) = q$, and the value is a sum of two exponentials. Proposition 1, Theorem 7, Corollary 2.*

Claim 5 (*Operator register*). *The notation and conventions established in §2 — script $\mathcal{L}$, $\mathcal{R}_q$ for operators; blackboard $\mathbb{E}$, $\mathbb{P}$, $\mathbb{R}$ for probability and reals; the*



standing assumptions (A1)–(A3) — are fixed once here and used unchanged throughout.

1.2 Paper map

§2 fixes notation and the operator register. §3 defines $\mathcal{R}_q$, proves the factorisation theorem and checks it by simulation. §4 derives the three rational closed forms and checks them against the simulated laws of the supremum and infimum. §5 proves the comparison principle and the verification theorem for the optimal-stopping variational inequality. §6 proves that the perpetual put's threshold is set by the infimum, gives its closed form under Kou jumps, and checks threshold and value against a grid solve of the variational inequality and against Monte Carlo of the stopped process. §7 collects the worked examples — the perpetual American put under Kou jumps and a hyperexponential first-passage time. §8 closes. Appendix A gives Algorithms 1–3 (the root-finder and closed forms, the exact Monte Carlo, the policy-iteration grid solver); Appendix B tabulates the closed forms of §4 for ready reference.

2. Notation, conventions, and the operator register

This section establishes the notation used throughout the paper.

2.1 State process and Lévy triplet

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space carrying a càdlàg one-dimensional Lévy process $X = (X_t)_{t \geq 0}$ with $X_0 = x \in \mathbb{R}$ under $\mathbb{P}_x$. We denote expectation under $\mathbb{P}_x$ by $\mathbb{E}_x$, and write $\mathbb{P} = \mathbb{P}_0$, $\mathbb{E} = \mathbb{E}_0$. The Lévy–Khintchine triplet (b, σ^2, ν) characterises X via the cumulant exponent

$$\psi(r) := \log \mathbb{E}_0[e^{rX_1}] = br + \frac{1}{2}\sigma^2 r^2 + \int_{\mathbb{R}} (e^{rz} - 1 - rz \mathbb{1}_{\{|z| \leq 1\}}) \nu(dz), \quad (2.1)$$

defined and analytic on a maximal open strip $\{r \in \mathbb{C} : -\eta_- < \operatorname{Re} r < \eta_+\}$ with $\eta_{\pm} \in (0, \infty]$. We refer to ψ as the **cumulant exponent** and to $(-\eta_-, \eta_+)$ as the **analyticity strip**. For the finite-activity families of §4 the truncation term is absorbed into the drift, and b there denotes the drift of the compound-Poisson representation. We write $e_r(x) := e^{rx}$ for the exponential test function, $\mathbf{e}(q)$ for an exponential time of rate q independent of X , and $S := \overline{X}_{\mathbf{e}(q)}$, $I := \underline{X}_{\mathbf{e}(q)}$ for the supremum and infimum of X up to $\mathbf{e}(q)$ under $\mathbb{P}_0$.

2.2 Standing assumptions

Assumption 1 (A1). (Finite exponential moments). The Lévy measure satisfies $\int_{|z| > 1} e^{rz} \nu(dz) < \infty$ for r in a non-trivial neighbourhood of 0; equivalently, $\eta_+ > 0$ and $\eta_- > 0$.

Assumption 2 (A2). (Small jumps). The Lévy measure satisfies $\int_{0 < |z| < 1} z^2 \nu(dz) < \infty$. Every Lévy measure does; the condition is recorded because the generator (2.2) and the viscosity arguments of §5 use it explicitly.



Assumption 3 (A3). (Single-crossing payoff). The payoff $g : \mathbb{R} \rightarrow [0, \infty)$ used from §5 onward is Lipschitz, non-increasing and of at most linear growth, and there is a threshold x^* such that g is strictly decreasing on $(-\infty, x^*)$ and zero on $[x^*, \infty)$.

Assumption (A1) is the analyticity condition needed for the closed forms on exponentials. (A2) secures a well-defined infinitesimal generator. (A3) covers the two payoffs of §6, the linear put $g(x) = (K - x)^+$ with $x^* = K$ and the exponential put $g(x) = (K - e^x)^+$ with $x^* = \log K$; linear growth is allowed because the linear put is unbounded.

2.3 Infinitesimal generator

For $f \in C^2(\mathbb{R})$ of at most linear growth the **infinitesimal generator** of X acts as

$$\mathcal{L}f(x) = b \partial_x f(x) + \frac{1}{2} \sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} (f(x+z) - f(x) - z \partial_x f(x) \mathbb{1}_{\{|z| \leq 1\}}) \nu(dz), \quad (2.2)$$

the third term being the **non-local jump term**. Under (A1)–(A2) the integral converges absolutely for every such f with bounded second derivative, and $\mathcal{L} : C_b^2 \rightarrow C_b$ is bounded.

2.4 Resolvent operators

Definition 1 (Resolvent). For $q > 0$, the **resolvent of X at discount q** is the operator

$$\mathcal{R}_q f(x) := \mathbb{E}_x \left[\int_0^\infty e^{-qt} f(X_t) dt \right], \quad f \in L^\infty(\mathbb{R}). \quad (2.3)$$

Equivalently, $\mathcal{R}_q = (q - \mathcal{L})^{-1}$ on suitable test-function classes. Under (A1) the same formula defines $\mathcal{R}_q f$ for measurable f of at most linear growth and for $f = e_r$ with $\psi(r) < q$.

Definition 2 (Supremum / infimum resolvents). Let $\bar{X}_t := \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t := \inf_{0 \leq s \leq t} X_s$ under $\mathbb{P}_0$. The **supremum-process resolvent** $\mathcal{R}_q^+$ and the **infimum-process resolvent** $\mathcal{R}_q^-$ are

$$\begin{aligned} \mathcal{R}_q^+ f(x) &:= \mathbb{E}_0 \left[\int_0^\infty e^{-qt} f(x + \bar{X}_t) dt \right], \\ \mathcal{R}_q^- f(x) &:= \mathbb{E}_0 \left[\int_0^\infty e^{-qt} f(x + \underline{X}_t) dt \right]. \end{aligned} \quad (2.4)$$

Their **Wiener–Hopf factors** are $\Phi_q^+(r) := \mathbb{E}[e^{rS}]$ and $\Phi_q^-(r) := \mathbb{E}[e^{rI}]$, finite at least for $r \leq 0$ and $r \geq 0$ respectively.

The pair $(\mathcal{R}_q^+, \mathcal{R}_q^-)$ is the Wiener–Hopf factorisation pair. Theorem 1 establishes $\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^-$.

2.5 Typographic register

We fix the following notation:



- **Operators** (single capital scripts): $\mathcal{L}$ generator, $\mathcal{R}_q$ resolvent, $\mathcal{R}_q^\pm$ Wiener–Hopf factors, $\mathcal{F}$ filtration. Never $\mathcal{L}$ or $\mathcal{R}_q$.
- **Probability** (blackboard): $\mathbb{E}$ expectation, $\mathbb{P}$ probability, $\mathbb{R}, \mathbb{C}, \mathbb{Z}, \mathbb{N}$.
- **Partial derivatives** in PDEs and SDEs: $\partial_r P, \partial_x V, \partial_{xx} u$. Never subscript notation P_r, V_x, u_{xx} in math-mode body text (subscripts are reserved for indices).
- **Roots** of $\psi(r) = q$: positive roots $\alpha_1 < \alpha_2 < \dots$, negative roots $-\beta_1 > -\beta_2 > \dots$ with $\beta_i > 0$; the roots inside the strip are $r_+ := \alpha_1$ and $r_- := -\beta_1$.
- **Sets and stopping times** (standard): τ, σ for stopping times; $\mathcal{T}$ for the class of stopping times of $\mathcal{F}$.

3. The resolvent and its Wiener–Hopf factorisation

This section defines the resolvent $\mathcal{R}_q$ as a bounded operator on $L^\infty(\mathbb{R})$, establishes its action on exponential test functions, and proves the central factorisation theorem of the paper.

3.1 Boundedness and integral representation

Lemma 1 (Resolvent boundedness). Under **(A1)**–**(A2)**, for every $q > 0$ the operator $\mathcal{R}_q : L^\infty(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ is bounded with operator norm $\|\mathcal{R}_q\|_{\infty \rightarrow \infty} \leq 1/q$.

Proof. For $f \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$, the integrand in (2.3) satisfies $|e^{-qt}f(X_t)| \leq e^{-qt}\|f\|_\infty$ uniformly in $t \geq 0$ and $\omega \in \Omega$. By Tonelli,

$$|\mathcal{R}_q f(x)| \leq \int_0^\infty e^{-qt} \|f\|_\infty dt = \frac{1}{q} \|f\|_\infty, \quad (3.1)$$

uniformly in x , which proves the claim. $\square$

On exponential test functions the action of $\mathcal{R}_q$ is explicit.

Lemma 2 (Resolvent on exponentials). For real r in the analyticity strip with $\psi(r) < q$,

$$\mathcal{R}_q e_r(x) = \frac{e^{rx}}{q - \psi(r)}. \quad (3.2)$$

Proof. By stationarity and independence of increments $\mathbb{E}_x[e^{rX_t}] = e^{rx+t\psi(r)}$, so by Tonelli

$$\mathcal{R}_q e_r(x) = \int_0^\infty e^{-qt} \mathbb{E}_x[e^{rX_t}] dt = e^{rx} \int_0^\infty e^{(\psi(r)-q)t} dt = \frac{e^{rx}}{q - \psi(r)}, \quad (3.3)$$

the integral being finite exactly when $\psi(r) < q$. By Lemma 3 below this is the interval (r_-, r_+) whenever both roots exist. $\square$

Lemma 3 (Strict convexity of ψ). Under **(A1)**, the cumulant exponent ψ is real-analytic and strictly convex on the open analyticity strip $(-\eta_-, \eta_+)$. For $q > 0$ the equation



$\psi(r) = q$ has at most one root r_+ in $(0, \eta_+)$ and at most one root r_- in $(-\eta_-, 0)$; r_+ exists if and only if $\lim_{r \uparrow \eta_+} \psi(r) > q$, and r_- if and only if $\lim_{r \downarrow -\eta_-} \psi(r) > q$. Both exist when $\sigma > 0$ and $\eta_{\pm} = \infty$, and for every family of §4, where ψ has poles at the ends of the strip.

Proof. Differentiating (2.1) twice on the strip:

$$\psi''(r) = \sigma^2 + \int_{\mathbb{R}} z^2 e^{rz} \nu(dz). \quad (3.4)$$

The integral on the right is finite under (A1) and (A2), and strictly positive whenever $\nu \neq 0$ or $\sigma > 0$ — which we always assume, to avoid the deterministic case. Hence $\psi'' > 0$ on the strip. If $0 < r_1 < r_2 < \eta_+$ were two roots, convexity and $\psi(0) = 0$ would give $\psi(r_1) \leq (r_1/r_2) \psi(r_2) < q$, a contradiction; the negative side is the same. Existence follows from the intermediate value theorem, since $\psi(0) - q = -q < 0$. $\square$

3.2 The factorisation theorem

Theorem 1 (Wiener–Hopf factorisation of $\mathcal{R}_q$). Under (A1)–(A2) and $q > 0$, the resolvent of the Lévy process X factorises as

$$\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^- = q \mathcal{R}_q^- \mathcal{R}_q^+ \quad (3.5)$$

as bounded operators on $L^\infty(\mathbb{R})$, and the identity extends to measurable functions of at most linear growth. Equivalently, in cumulant-exponent form,

$$\frac{q}{q - \psi(r)} = \Phi_q^+(r) \Phi_q^-(r) \quad \text{for real } r \text{ with } \psi(r) < q, \quad (3.6)$$

where $\Phi_q^+(r) = \mathbb{E}[e^{rS}]$ and $\Phi_q^-(r) = \mathbb{E}[e^{rI}]$ are the transforms of the supremum and infimum at the independent exponential time $\mathbf{e}(q)$.

Proof. The proof has three steps.

Step 1. (Killed-process representation.) For $f \geq 0$ or bounded, Tonelli and $\int_0^\infty q e^{-qt}(\cdot) dt = \mathbb{E}[\cdot]$ at $t = \mathbf{e}(q)$ give

$$\mathcal{R}_q f(x) = \frac{1}{q} \mathbb{E}[f(x + X_{\mathbf{e}(q)})], \quad \mathcal{R}_q^+ f(x) = \frac{1}{q} \mathbb{E}[f(x + S)], \quad \mathcal{R}_q^- f(x) = \frac{1}{q} \mathbb{E}[f(x + I)]. \quad (3.7)$$

Step 2. (Decomposition into supremum and infimum.) By the classical Wiener–Hopf factorisation [1], [5], for any Lévy process and an independent exponential time, S and $X_{\mathbf{e}(q)} - S$ are independent and $X_{\mathbf{e}(q)} - S$ has the law of I ; by the same result applied to $-X$, I and $X_{\mathbf{e}(q)} - I$ are independent and $X_{\mathbf{e}(q)} - I$ has the law of S .

Step 3. (Operator identity.) Write $X_{\mathbf{e}(q)} = S + (X_{\mathbf{e}(q)} - S)$. By Step 2 and Fubini, $\mathbb{E}[f(x + X_{\mathbf{e}(q)})] = \mathbb{E}[h(x + S)]$ with $h(y) := \mathbb{E}[f(y + I)] = q \mathcal{R}_q^- f(y)$. By (3.7), $q \mathcal{R}_q f(x) = q \mathcal{R}_q^+ h(x) = q^2 \mathcal{R}_q^+ \mathcal{R}_q^- f(x)$, which is (3.5); the reversed order follows from the second half of Step 2. Applying (3.5) to $f = e_r$ and using Lemma 2 on the left and $\mathcal{R}_q^\pm e_r = q^{-1} \Phi_q^\pm(r) e_r$ (from (3.7)) on the right gives (3.6). $\square$



3.3 Numerical illustration

Figure 1 plots $\psi(r) - q$ for the Kou family (§4) at the benchmark parameters $b = 0$, $\sigma = 0.20$, $\lambda = 1$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$, which are used throughout. The convexity proved in Lemma 3 is visible inside the strip $(-5, 10)$, where the curve dips below zero between the two inside-strip roots $r_- = -\beta_1 = -0.8563$ and $r_+ = \alpha_1 = 1.4071$. Beyond the poles at $-\eta_-$ and η_+ the rational continuation of ψ has two further real roots, $\alpha_2 = 12.6503$ and $-\beta_2 = -8.2010$; they are not roots of the defining integral, which diverges there, but they are poles of the Wiener–Hopf factors (Theorem 2) and enter every closed form below. The four roots were computed twice — by bisection with Newton polish on each interval between the poles (Algorithm 1), and as the eigenvalues of the companion matrix of the quartic numerator of $\psi(r) - q$ — and the two agree to 8.9×10^{-15} .

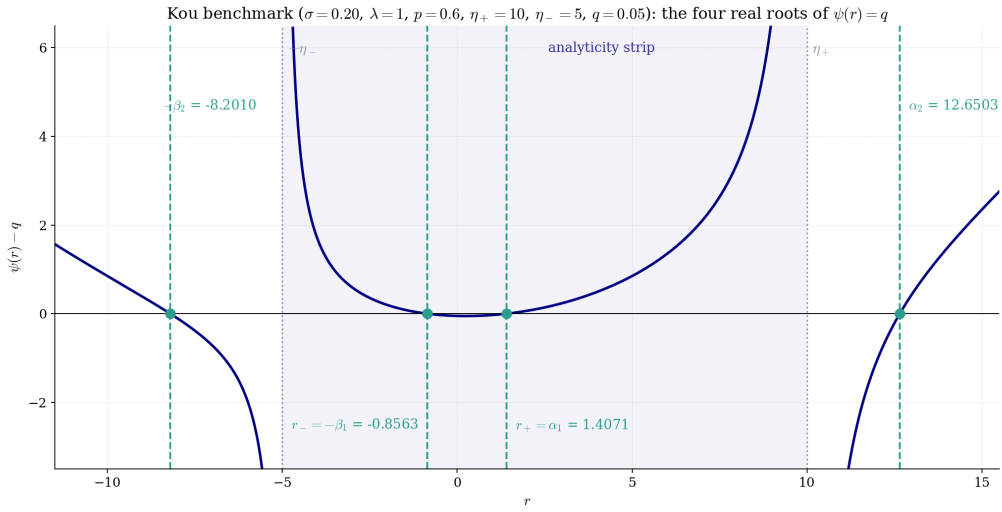


Figure 1: Cumulant exponent $\psi(r) - q$ for the Kou benchmark, with its rational continuation beyond the poles at $-\eta_- = -5$ and $\eta_+ = 10$ (dotted). Dashed lines mark the four real roots of $\psi(r) = q$; the shaded band is the analyticity strip, which holds the two roots $r_{\pm}$ of Lemma 3.

3.4 Numerical check of Theorem 1

Theorem 1 is a statement about the joint law of $(S, I, X_{\mathbf{e}(q)})$, so it can be tested by simulating that law directly, without any root or factor. Algorithm 2 samples the Kou benchmark exactly up to an independent exponential time — Brownian segments between Poisson jump times, with the maximum and minimum of each segment drawn from their exact conditional laws given its end points [6] — for 1,000,000 paths. Table 1 compares $\mathbb{E}[e^{rX_{\mathbf{e}(q)}}]$, the product $\mathbb{E}[e^{rS}] \mathbb{E}[e^{rI}]$ and $q/(q - \psi(r))$; the values of r keep $2r$ inside (r_-, r_+) , so every estimator has finite variance. The sample correlation between S and $X_{\mathbf{e}(q)} - S$ is -0.0006 , and the sample means of $X_{\mathbf{e}(q)} - S$ and of I are -1.0893 ± 0.0012 and -1.0891 ± 0.0012 , as Step 2 of the proof requires. The simulated mean $\mathbb{E}[X_{\mathbf{e}(q)}] = -0.4000 \pm 0.0014$ matches $\psi'(0)/q = -0.4000$.



Table 1: Theorem 1 at the Kou benchmark: the transform of $X_{e(q)}$ and the product of the transforms of S and I , each from 1,000,000 exactly simulated paths ($\pm$ one standard error), against $q/(q - \psi(r))$.

r	$q/(q - \psi(r))$	$\mathbb{E}[e^{rX_{e(q)}}]$, Monte Carlo	$\mathbb{E}[e^{rS}] \mathbb{E}[e^{rI}]$, Monte Carlo
−0.4	1.42473	1.42341 ± 0.00188	1.42329 ± 0.00189
−0.2	1.12872	1.12862 ± 0.00039	1.12857 ± 0.00038
0.2	0.95530	0.95521 ± 0.00024	0.95525 ± 0.00024
0.4	0.97220	0.97170 ± 0.00054	0.97184 ± 0.00052
0.6	1.05576	1.05390 ± 0.00132	1.05421 ± 0.00117

3.5 Consequences

Theorem 1 reduces the analysis of $\mathcal{R}_q$ on any family to the analysis of $\mathcal{R}_q^+$ and $\mathcal{R}_q^-$ separately. For the rational families of §4 the factor Φ_q^+ depends on the up-jump rates and the positive roots only, and Φ_q^- on the down-jump rates and the negative roots only. This decoupling is what makes the closed forms of §4 possible.

A second consequence, used in §5, is that $\mathcal{R}_q$ inherits positivity from its factors.

Corollary 1 (Positivity preservation). Under (A1)–(A2) and $q > 0$, for $f \in L^\infty(\mathbb{R})$ with $f \geq 0$ one has $\mathcal{R}_q^\pm f \geq 0$ and $\mathcal{R}_q f \geq 0$, both pointwise.

Proof. By (3.7) each of $\mathcal{R}_q^\pm$ is $1/q$ times an expectation against a probability law (that of S , respectively I); the product $q\mathcal{R}_q^+\mathcal{R}_q^-$ inherits positivity preservation, and by Theorem 1 it equals $\mathcal{R}_q$. $\square$

4. Rational families and closed-form Wiener–Hopf factors

The Wiener–Hopf factorisation of §3 is operator-theoretic. To extract usable formulas one restricts to Lévy families whose factors $\Phi_q^\pm$ are **rational**, or meromorphic, in the spectral variable r . This section treats three such families — Kou double-exponential, hyperexponential mixtures, and the meromorphic class — and derives their closed forms.

4.1 Spectrally one-sided processes

The simplest case is a process with jumps of one sign only. Here one of the two factors is a single pole.

Lemma 4 (One-sided jumps). Let $q > 0$. If X has no positive jumps and is not the negative of a subordinator, then S is exponentially distributed with rate r_+ , the positive root of $\psi(r) = q$, so that

$$\Phi_q^+(r) = \frac{r_+}{r_+ - r}, \quad r < r_+. \quad (4.1)$$



If X has no negative jumps and is not a subordinator, then $-I$ is exponentially distributed with rate $-r_-$ and $\Phi_q^-(r) = r_-/(r_- - r)$ for $r > r_-$.

Proof. With no positive jumps $\eta_+ = \infty$ and $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$, so r_+ exists by Lemma 3. The process $M_t := e^{r_+ X_t - qt}$ is a martingale, and X passes upward through a level $a > 0$ continuously, $X_{\tau_a^+} = a$ on $\{\tau_a^+ < \infty\}$ with $\tau_a^+ := \inf\{t : X_t \geq a\}$. Optional stopping of M at $\tau_a^+ \wedge t$ and $t \rightarrow \infty$ (bounded convergence, since $M \leq e^{r_+ a}$ up to τ_a^+) give $\mathbb{E}[e^{-q\tau_a^+}] = e^{-r_+ a}$, and $\mathbb{P}(S \geq a) = \mathbb{P}(\tau_a^+ < \infty) = \mathbb{E}[e^{-q\tau_a^+}]$; see also [1], [5]. The second statement is the first applied to $-X$. $\square$

With jumps of both signs neither factor is a single pole: the overshoot of a jump past the level breaks the argument above. The rational families below are the cases in which the factors remain finite products.

4.2 Kou double-exponential family

Theorem 2 (Closed-form $\mathcal{R}_q^\pm$ for Kou). Let X be a Kou jump-diffusion with $\sigma > 0$, drift $b \in \mathbb{R}$ and Lévy measure

$$\nu(dz) = \lambda[p\eta_+ e^{-\eta_+ z} \mathbb{1}_{\{z>0\}} + (1-p)\eta_- e^{\eta_- z} \mathbb{1}_{\{z<0\}}] dz, \quad (4.2)$$

with $\lambda > 0$, $p \in (0, 1)$, $\eta_+, \eta_- > 0$, so that on the strip $(-\eta_-, \eta_+)$

$$\psi(r) = br + \frac{1}{2}\sigma^2 r^2 + \lambda p \frac{\eta_+}{\eta_+ - r} + \lambda(1-p) \frac{\eta_-}{\eta_- + r} - \lambda. \quad (4.3)$$

For each $q > 0$ the equation $\psi(r) = q$, with ψ continued rationally, has exactly four real roots $-\beta_2 < -\eta_- < -\beta_1 < 0 < \alpha_1 < \eta_+ < \alpha_2$. The Wiener–Hopf factors are

$$\begin{aligned} \Phi_q^+(r) &= \frac{\alpha_1 \alpha_2 (\eta_+ - r)}{\eta_+ (\alpha_1 - r)(\alpha_2 - r)}, \quad r < \alpha_1, & \Phi_q^-(r) &= \frac{\beta_1 \beta_2 (\eta_- + r)}{\eta_- (\beta_1 + r)(\beta_2 + r)}, \quad r > -\beta_1, \\ \mathcal{R}_q^\pm e_r(x) &= \frac{1}{q} \Phi_q^\pm(r) e^{rx}. \end{aligned} \quad (4.4)$$

Equivalently, S is the mixture $v_1 \text{Exp}(\alpha_1) + v_2 \text{Exp}(\alpha_2)$ and $-I$ the mixture $w_1 \text{Exp}(\beta_1) + w_2 \text{Exp}(\beta_2)$, with positive weights

$$v_1 = \frac{\alpha_2(\eta_+ - \alpha_1)}{\eta_+(\alpha_2 - \alpha_1)}, \quad v_2 = \frac{\alpha_1(\alpha_2 - \eta_+)}{\eta_+(\alpha_2 - \alpha_1)}, \quad w_1 = \frac{\beta_2(\eta_- - \beta_1)}{\eta_-(\beta_2 - \beta_1)}, \quad w_2 = \frac{\beta_1(\beta_2 - \eta_-)}{\eta_-(\beta_2 - \beta_1)}, \quad (4.5)$$

and in particular $\mathbb{E}_0[e^{-q\tau_a^+}] = \mathbb{P}(S \geq a) = v_1 e^{-\alpha_1 a} + v_2 e^{-\alpha_2 a}$ for $a \geq 0$ [7].

Proof. Roots. $D(r) := (q - \psi(r))(\eta_+ - r)(\eta_- + r)$ is a polynomial of degree four with leading coefficient $\sigma^2/2$. On $(0, \eta_+)$, $\psi - q$ runs from $-q$ to $+\infty$; on (η_+, ∞) from $-\infty$ (just right of the pole) to $+\infty$ (as $\sigma > 0$); the negative side is the same. This gives four real roots in the stated intervals, and the degree allows no others, so $D(r) = \frac{1}{2}\sigma^2(\alpha_1 - r)(\alpha_2 - r)(\beta_1 + r)(\beta_2 + r)$. At $r = 0$ this reads $q\eta_+\eta_- = \frac{1}{2}\sigma^2\alpha_1\alpha_2\beta_1\beta_2$, and dividing gives $q/(q - \psi(r)) = F^+(r)F^-(r)$, with $F^\pm$ the two rational functions displayed in (4.4).



The law of S . Fix $a > 0$, put $u(x) := v_1 e^{\alpha_1(x-a)} + v_2 e^{\alpha_2(x-a)}$ for $x < a$ and $u(x) := 1$ for $x \geq a$. For $x < a$ split the up-jump integral of $\mathcal{L}u(x)$ at $z = a - x$; the part landing below a integrates the exponentials over a finite interval, the part landing above integrates the constant 1. Collecting terms,

$$(q - \mathcal{L})u(x) = \sum_{i=1}^2 v_i (q - \psi(\alpha_i)) e^{\alpha_i(x-a)} + \lambda p \left(\sum_{i=1}^2 \frac{v_i \eta_+}{\eta_+ - \alpha_i} - 1 \right) e^{-\eta_+(a-x)}, \quad x < a, \quad (4.6)$$

with ψ the rational expression (4.3). The first sum vanishes because α_i are roots, and the second term because the weights (4.5) solve $v_1 + v_2 = 1$ (so that u is continuous at a) and $\sum_i v_i \eta_+ / (\eta_+ - \alpha_i) = 1$. Itô's formula for $e^{-qt} u(X_t)$ up to τ_a^+ (u is smooth and bounded on $(-\infty, a)$, and $u(X_{\tau_a^+}) = 1$ whether X creeps onto a or jumps over it) gives $u(x) = \mathbb{E}_x[e^{-q\tau_a^+}]$, hence $\mathbb{P}(S \geq a) = u(0) = v_1 e^{-\alpha_1 a} + v_2 e^{-\alpha_2 a}$. Partial fractions show that the Laplace transform of this mixture is F^+ : the two conditions on the weights say exactly that $F^+(0) = 1$ and $F^+(\eta_+) = 0$. Hence $\Phi_q^+ = F^+$.

The law of I is the same computation for $-X$, which is Kou with the roles of (p, η_+) and $(1 - p, \eta_-)$ exchanged, so $\Phi_q^- = F^-$. The resolvent formulas follow from (3.7), and $F^+ F^- = q/(q - \psi)$ from the first paragraph confirms (3.6). $\square$

4.3 Hyperexponential mixtures

Theorem 3 (Closed-form $\mathcal{R}_q^\pm$ for hyperexponential). Let X have $\sigma > 0$, drift b and Lévy measure of finite-mixture form

$$\nu(dz) = \sum_{j=1}^{N_+} a_j^+ \eta_j^+ e^{-\eta_j^+ z} \mathbb{1}_{\{z>0\}} dz + \sum_{k=1}^{N_-} a_k^- \eta_k^- e^{\eta_k^- z} \mathbb{1}_{\{z<0\}} dz, \quad (4.7)$$

with intensities $a_j^\pm > 0$ and rates $0 < \eta_1^\pm < \dots < \eta_{N_\pm}^\pm$. Then $\psi(r) = q$ has exactly $N_+ + 1$ positive roots, interlacing with the up-rates, $0 < \alpha_1 < \eta_1^+ < \alpha_2 < \dots < \eta_{N_+}^+ < \alpha_{N_++1}$, and $N_- + 1$ negative roots $-\beta_i$ interlacing likewise with $-\eta_k^-$, and

$$\Phi_q^+(r) = \prod_{i=1}^{N_++1} \frac{\alpha_i}{\alpha_i - r} \prod_{j=1}^{N_+} \frac{\eta_j^+ - r}{\eta_j^+}, \quad \Phi_q^-(r) = \prod_{i=1}^{N_-+1} \frac{\beta_i}{\beta_i + r} \prod_{k=1}^{N_-} \frac{\eta_k^- + r}{\eta_k^-}, \quad (4.8)$$

and $\mathcal{R}_q^\pm e_r = q^{-1} \Phi_q^\pm(r) e_r$. The supremum S is the mixture $\sum_i c_i \text{Exp}(\alpha_i)$ with positive weights $c_i = \prod_j (\eta_j^+ - \alpha_i) / \eta_j^+ \prod_{k \neq i} \alpha_k / (\alpha_k - \alpha_i)$, the residues of Φ_q^+ at its poles, and $-I$ is the analogous mixture over the β_i .

Proof. ψ is rational with poles at $\{\eta_j^+\} \cup \{-\eta_k^-\}$. On each interval between consecutive positive poles $\psi - q$ runs from $-\infty$ to $+\infty$, on $(0, \eta_1^+)$ from $-q$ to $+\infty$, and beyond the last pole from $-\infty$ to $+\infty$ since $\sigma > 0$; this gives $N_+ + 1$ positive roots in the stated intervals, likewise $N_- + 1$ negative ones, and the numerator of $\psi - q$ has degree $2 + N_+ + N_-$, so there are no others. The law of S follows exactly as in Theorem 2: the candidate $u(x) = \sum_i c_i e^{\alpha_i(x-a)}$ for $x < a$, $u = 1$ for $x \geq a$, solves $(q - \mathcal{L})u = 0$ on $(-\infty, a)$ if and only if $\sum_i c_i = 1$ and $\sum_i c_i \eta_j^+ / (\eta_j^+ - \alpha_i) = 1$ for every j — one condition per up-rate.



These $N_+ + 1$ linear conditions say that the rational function $\sum_i c_i \alpha_i / (\alpha_i - r)$ equals 1 at $r = 0$ and vanishes at every η_j^+ ; the product in (4.8) is the unique such function with these poles, and its residues are the stated c_i , which alternate in sign in numerator and denominator together and are therefore positive. See [8] and, for phase-type jumps, [4]. $\square$

The Kou family is the case $N_+ = N_- = 1$.

4.4 The meromorphic class

Theorem 4 (Closed-form $\mathcal{R}_q^\pm$ for the meromorphic class). Let X have $\sigma > 0$ and Lévy measure

$$\nu(dz) = \sum_{j \geq 1} a_j^+ \eta_j^+ e^{-\eta_j^+ z} \mathbb{1}_{\{z > 0\}} dz + \sum_{k \geq 1} a_k^- \eta_k^- e^{\eta_k^- z} \mathbb{1}_{\{z < 0\}} dz, \quad (4.9)$$

with rates $0 < \eta_1^+ < \eta_2^+ < \dots \rightarrow \infty$ and intensities $a_j^\pm > 0$ such that $\sum_j a_j^\pm / \eta_j^\pm < \infty$. Then ψ is meromorphic with poles exactly at $\{\eta_j^+\} \cup \{-\eta_k^-\}$, the positive roots of $\psi(r) = q$ interlace with the up-rates, $0 < \alpha_1 < \eta_1^+ < \alpha_2 < \eta_2^+ < \dots$, and

$$\Phi_q^+(r) = \prod_{i \geq 1} \frac{1 - r/\eta_i^+}{1 - r/\alpha_i}, \quad \mathcal{R}_q^+ e_r = \frac{1}{q} \Phi_q^+(r) e_r, \quad (4.10)$$

with the analogous product for Φ_q^- . The product converges absolutely and uniformly on compact subsets of $\mathbb{C}$ away from the α_i .

Proof. Sketch. The interlacing and the product formula are the theorem of Kuznetsov, Kyprianou and Pardo for meromorphic Lévy processes [9], which extends the β -family of [10]. The condition $\sum_j a_j^\pm / \eta_j^\pm < \infty$ makes $\int |z| \nu(dz)$ finite, so ψ is a convergent series of the rational terms of Theorem 3. Truncating the measure after N components gives hyperexponential processes $X^{(N)}$ for which Theorem 3 holds; $X^{(N)} \rightarrow X$ in law on path space, the roots of the truncated equations converge to those of $\psi(r) = q$, and the finite products converge to (4.10). Absolute convergence follows from the interlacing: $\sum_i (1/\alpha_i - 1/\eta_i^+) \leq 1/\alpha_1 + \sum_{i \geq 2} (1/\eta_{i-1}^+ - 1/\eta_i^+) < \infty$. $\square$

For computation the product is better grouped as $\Phi_q^+(r) = (1 - r/\alpha_1)^{-1} \prod_{i \geq 1} (1 - r/\eta_i^+) / (1 - r/\alpha_{i+1})$: the partial products differ from those of (4.10) only by the factor $1 - r/\eta_N^+ \rightarrow 1$, but each regrouped factor is within $O(a_i^+)$ of one, because α_{i+1} sits just above the pole η_i^+ when a_i^+ is small.

4.5 Numerical check of Theorems 2, 3 and 4

Kou (Theorem 2). The same 1,000,000 exactly simulated paths as in §3.4 give the empirical laws of S and I ; Table 2 compares their transforms and tails with the closed forms at the benchmark, where $(v_1, v_2) = (0.9668, 0.0332)$ and $(w_1, w_2) = (0.9254, 0.0746)$. Figure 2 shows the factor $\mathcal{R}_q^+ e_r(0) = \Phi_q^+(r)/q$ at three jump intensities with Monte Carlo points (4×10^5 paths each, all within 1.8 standard errors), and the two tail functions over $a \in [0, 4]$.



Table 2: Theorem 2 at the Kou benchmark: transforms, tails and means of S and I from the closed forms, against 1,000,000 exactly simulated paths. The values of r keep $2r$ inside the domain of the transform, so that the estimators have finite variance.

Quantity	Theorem 2	Monte Carlo	(MC – closed)/s.e.
$\mathbb{E}[e^{rS}], r = -2$	0.42793	0.42807 ± 0.00031	+0.5
$\mathbb{E}[e^{rS}], r = -1$	0.59591	0.59602 ± 0.00027	+0.4
$\mathbb{E}[e^{rS}], r = -0.5$	0.74525	0.74534 ± 0.00020	+0.5
$\mathbb{E}[e^{rS}], r = 0.5$	1.53431	1.53282 ± 0.00098	−1.5
$\mathbb{E}[e^{rS}], r = 0.65$	1.83190	1.82778 ± 0.00242	−1.7
$\mathbb{E}[e^{rI}], r = -0.4$	1.81501	1.81298 ± 0.00237	−0.9
$\mathbb{E}[e^{rI}], r = 0.5$	0.65458	0.65471 ± 0.00025	+0.5
$\mathbb{E}[e^{rI}], r = 1$	0.49339	0.49354 ± 0.00031	+0.5
$\mathbb{E}[e^{rI}], r = 2$	0.33742	0.33759 ± 0.00032	+0.5
$\mathbb{P}(S > 0.25)$	0.68152	0.68156 ± 0.00047	+0.1
$\mathbb{P}(S > 0.5)$	0.47848	0.47840 ± 0.00050	−0.2
$\mathbb{P}(S > 1)$	0.23674	0.23652 ± 0.00042	−0.5
$\mathbb{P}(S > 2)$	0.05797	0.05792 ± 0.00023	−0.2
$\mathbb{P}(-I > 0.25)$	0.75664	0.75652 ± 0.00043	−0.3
$\mathbb{P}(-I > 0.5)$	0.60431	0.60409 ± 0.00049	−0.4
$\mathbb{P}(-I > 1)$	0.39305	0.39312 ± 0.00049	+0.2
$\mathbb{P}(-I > 2)$	0.16693	0.16631 ± 0.00037	−1.7
$\mathbb{E}[S]$	0.68975	0.68927 ± 0.00071	−0.7
$\mathbb{E}[-I]$	1.08975	1.08910 ± 0.00116	−0.6

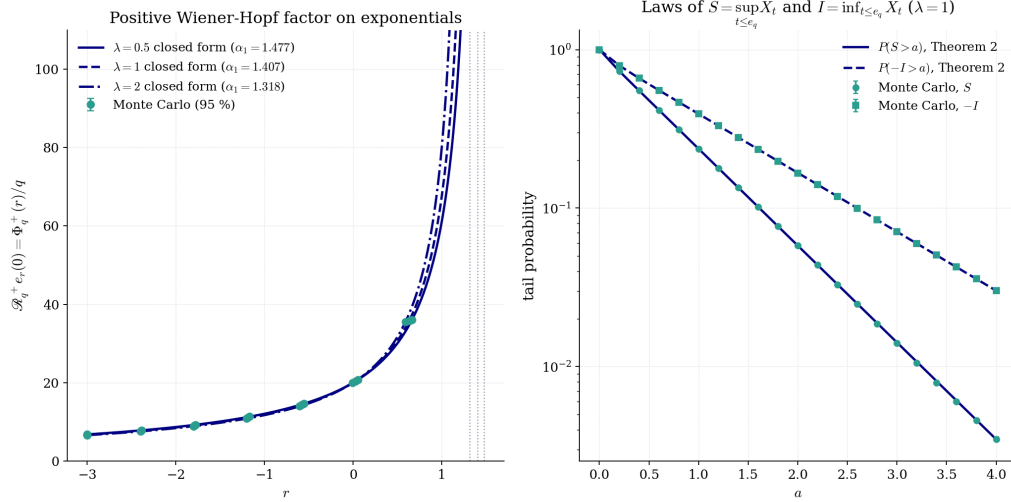


Figure 2: Left: the positive Wiener–Hopf factor on exponentials, $\mathcal{R}_q^+ e_r(0) = \Phi_q^+(r)/q$ from Theorem 2, for the Kou benchmark at $\lambda \in \{0.5, 1, 2\}$ (lines), with Monte Carlo estimates of $\mathbb{E}[e^{rS}]/q$ and 95% intervals (points); dotted lines mark the poles $\alpha_1(\lambda)$. Right: the tails $\mathbb{P}(S > a)$ and $\mathbb{P}(-I > a)$ of the two-exponential mixtures of Theorem 2 at $\lambda = 1$, against the empirical tails of 10^6 simulated paths.

Hyperexponential (Theorem 3) is checked in Example 2 (§7).

Meromorphic (Theorem 4). Take $\sigma = 0.20$, $b = 0$, $q = 0.05$, up-jumps with $a_n^+ = 0.6/n^2$ at rates $\eta_n^+ = 5n$ and down-jumps with $a_n^- = 0.4/n^2$ at rates $\eta_n^- = 5n$, $n \geq 1$. The first roots are $\alpha_1, \dots, \alpha_5 = 0.5825, 7.9651, 11.2659, 15.3274, 20.1175$, each in its interval (η_{n-1}^+, η_n^+) as the interlacing requires (checked for the first 201 roots). Table 3 shows the regrouped product truncated at N factors converging to eight digits by $N = 50$, and the Monte Carlo transform of S (10^6 paths; jump sizes drawn from the mixture over $n \leq 10^6$, which misses less than 10^{-6} of the jump intensity) agreeing with it.

Table 3: Theorem 4 for the meromorphic example: the regrouped product truncated at N factors, and the Monte Carlo transform $\mathbb{E}[e^{rS}]$ from 1,000,000 paths. The mean $\mathbb{E}[S] = 1/\alpha_1 + \sum_{i \geq 1} (1/\alpha_{i+1} - 1/\eta_i^+)$ is 1.6293 from 200 factors and 1.6290 ± 0.0017 by simulation.

N	$\Phi_q^+(-2)$	$\Phi_q^+(-1)$	$\Phi_q^+(-0.5)$	$\Phi_q^+(0.25)$
5	0.25803068	0.39714095	0.56042030	1.71259758
10	0.25806178	0.39716558	0.56043794	1.71257003
20	0.25806412	0.39716740	0.56043923	1.71256803
50	0.25806429	0.39716754	0.56043933	1.71256788
200	0.25806429	0.39716754	0.56043933	1.71256788
Monte Carlo	0.25836 ± 0.00030	0.39736 ± 0.00032	0.56053 ± 0.00029	1.71226 ± 0.00175



5. Comparison principle for the variational inequality

This section establishes the result governing optimal stopping under the Lévy generator $\mathcal{L}$ of §2. We prove a viscosity comparison principle for the **variational inequality**

$$\min\{(q - \mathcal{L})u(x), u(x) - g(x)\} = 0, \quad x \in \mathbb{R}, \quad (5.1)$$

and identify its unique viscosity solution with the optimal-stopping value $V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} g(X_\tau)]$.

5.1 Viscosity solutions for non-local generators

We adopt the standard notion of viscosity solution for integro-differential equations, adapted to the non-local operator $\mathcal{L}$: the test function is used near the contact point and the solution itself away from it, so that the non-local integral is well defined.

Definition 3 (Viscosity sub-/super-solution of (5.1)). An upper-semicontinuous function $u : \mathbb{R} \rightarrow \mathbb{R}$ of at most linear growth is a **viscosity sub-solution** of (5.1) if for every $x_0 \in \mathbb{R}$ and every C^2 test function φ such that $u - \varphi$ attains a local maximum at x_0 ,

$$\min\{(q - \mathcal{L})\varphi(x_0), u(x_0) - g(x_0)\} \leq 0. \quad (5.2)$$

A lower-semicontinuous function v of at most linear growth is a **viscosity super-solution** if the reverse inequality, with $\geq$, holds at every local minimum of $v - \varphi$. A **viscosity solution** is both.

The non-local term in $\mathcal{L}\varphi(x_0)$ is understood with φ replaced by the solution outside a neighbourhood of x_0 — the standard modification for integro-differential equations [11], [12].

5.2 Comparison principle

Theorem 5 (Comparison and uniqueness). Under **(A1)–(A3)**, let u be an upper-semicontinuous viscosity sub-solution of (5.1) and v a lower-semicontinuous viscosity super-solution, both of at most linear growth. Then $u \leq v$ on $\mathbb{R}$. In particular, (5.1) has at most one viscosity solution of at most linear growth, and such a solution is continuous.

Proof. The proof follows the doubling-of-variables strategy for non-local equations [11], [12], with the obstacle handled as in the local case.

Step 1. (Doubling.) Suppose, for contradiction, that $\delta := \sup_{\mathbb{R}}(u - v) > 0$. For $n \geq 1$ and $\varepsilon > 0$ let (x_n, y_n) maximise $u(x) - v(y) - \frac{n}{2}|x - y|^2 - \varepsilon(x^2 + y^2)$; the quadratic penalty dominates the linear growth, so the maximum is attained, and for ε small its value exceeds $\delta/2$. As $n \rightarrow \infty$, $n|x_n - y_n|^2 \rightarrow 0$ and (x_n, y_n) stays bounded.

Step 2. (Obstacle alternative.) If $u(x_n) \leq g(x_n)$, then, since a super-solution satisfies $v \geq g$, $u(x_n) - v(y_n) \leq g(x_n) - g(y_n) \leq \text{Lip}(g)|x_n - y_n| \rightarrow 0$, contradicting $u(x_n) - v(y_n) > \delta/2$. Hence, along a subsequence, the sub-solution inequality is the equation



branch $(q - \mathcal{L})\varphi_n(x_n) \leq 0$, while the super-solution gives $(q - \mathcal{L})\phi_n(y_n) \geq 0$, with φ_n, ϕ_n the quadratic test functions of the doubling.

Step 3. (Local and non-local terms.) Subtracting the two inequalities, the drift and diffusion terms are bounded by $Cn|x_n - y_n|^2$ plus a penalty term that vanishes with ε , by the Ishii lemma [11]. The jump terms are split at $|z| = 1$: the small jumps contribute a term that vanishes as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ by (A2), and the large jumps, evaluated on u and v themselves, contribute $\int_{|z|>1} [(u(x_n + z) - v(y_n + z)) - (u(x_n) - v(y_n))] \nu(dz)$, which is at most the jump integral of the penalty $\varepsilon(x^2 + y^2)$ because (x_n, y_n) maximises the penalised difference; that integral is finite by (A1) and vanishes as $\varepsilon \rightarrow 0$ since $\varepsilon(|x_n| + |y_n|) = O(\sqrt{\varepsilon})$.

Step 4. (Conclusion.) Combining, $q(u(x_n) - v(y_n)) \leq o(1) + \rho(\varepsilon)$ as $n \rightarrow \infty$, with $\rho(\varepsilon) \rightarrow 0$, so $q\delta/2 \leq \rho(\varepsilon)$, a contradiction for ε small. Hence $u \leq v$. Uniqueness follows by applying this in both directions, and continuity because the upper- and lower-semicontinuous envelopes of a solution are sub- and super-solutions. $\square$

5.3 Identification with the optimal-stopping value

Theorem 6 (Verification). Under (A1)–(A3) and $q > 0$: (i) the optimal-stopping value

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} g(X_\tau)] \quad (5.3)$$

is the unique viscosity solution of (5.1) of at most linear growth, where $\mathcal{T}$ is the set of stopping times of the filtration generated by X and $e^{-q\tau} g(X_\tau) := 0$ on $\{\tau = \infty\}$; (ii) if W is of at most linear growth, C^1 on $\mathbb{R}$ and C^2 with bounded second derivative off one point, and satisfies (5.1) pointwise off that point, then $W = V$ and the supremum is attained at the entry time

$$\tau^* = \inf\{t \geq 0 : W(X_t) = g(X_t)\}. \quad (5.4)$$

Proof. By the dynamic-programming principle for optimal stopping, V is a viscosity solution of (5.1); the argument is the one for diffusions [13], with Itô's formula for Lévy processes [14] in place of the diffusion one. V has at most linear growth because g does and $\mathbb{E}_x[\sup_t e^{-qt}|X_t|] < \infty$ under (A1). Uniqueness is Theorem 5.

(ii) For the **upper bound**, apply Itô's formula — in the Itô–Meyer form, which applies because W is C^1 with a bounded second derivative off one point, so no local-time term arises — to $e^{-qt}W(X_t)$ on $[0, \tau \wedge T]$ for any $\tau \in \mathcal{T}$ and $T > 0$. The drift is $-(q - \mathcal{L})W \leq 0$ by the variational inequality, so after localisation and $T \rightarrow \infty$ under (A1),

$$W(x) \geq \mathbb{E}_x[e^{-q\tau} W(X_\tau)] \geq \mathbb{E}_x[e^{-q\tau} g(X_\tau)], \quad (5.5)$$

using the obstacle constraint $W \geq g$. Taking the supremum over τ gives $W \geq V$.

For the **lower bound**, take $\tau = \tau^*$. On $\{t < \tau^*\}$ we have $W(X_t) > g(X_t)$, so the variational inequality gives $(q - \mathcal{L})W(X_t) = 0$, and Itô's formula yields equality in the first inequality of (5.5) at $\tau = \tau^*$, with $W(X_{\tau^*}) = g(X_{\tau^*})$ by construction. Hence $W(x) = \mathbb{E}_x[e^{-q\tau^*} g(X_{\tau^*})] \leq V(x)$, so $W = V$ and τ^* attains the supremum. $\square$



The identification (5.3) is the bridge from the variational-inequality formulation to the probabilistic one. §6 uses the probabilistic side to find the stopping region; the grid solve of §6.3 uses the analytic side.

6. Smooth-pasting and the optimal threshold

Section 5 identified the solution of the variational inequality with the optimal-stopping value. This section determines the stopping region of the perpetual put and its boundary b^* . The answer is governed by the infimum I of the process up to an exponential time, that is by the negative Wiener–Hopf factor Φ_q^- and, for rational families, by the negative roots of $\psi(r) = q$.

6.1 Continuation region as a half-line

The canonical payoff is the perpetual American put on the state itself, $g(x) = (K - x)^+$, with $x^* = K$ in (A3); the exponential put $g(x) = (K - e^x)^+$ is treated in Corollary 2.

Proposition 1 (Half-line structure). Let X satisfy (A1), not be a subordinator, $q > 0$, and $g(x) = (K - x)^+$. Then the stopping region $\mathcal{S} := \{x : V(x) = g(x)\}$ is empty or a closed half-line $(-\infty, b^*]$ with $b^* < K$, and the continuation region is $\mathcal{C} := \{x : V(x) > g(x)\} = (b^*, \infty)$.

Proof. V and g are continuous (Theorems 5–6), so $\mathcal{S}$ is closed. For $y < x$ and any stopping time τ , $g(y + X_\tau) \leq g(x + X_\tau) + (x - y)$ because g is non-increasing and 1-Lipschitz, so $\mathbb{E}[e^{-q\tau}g(y + X_\tau)] \leq \mathbb{E}[e^{-q\tau}g(x + X_\tau)] + (x - y)$ and hence $V(y) \leq V(x) + (x - y)$. If $x \in \mathcal{S}$ and $x \leq K$, this gives $V(y) \leq K - y = g(y)$, so $y \in \mathcal{S}$: $\mathcal{S} \cap (-\infty, K]$ is a down-set. No $x \geq K$ lies in $\mathcal{S}$, since $g(x) = 0 < V(x)$: a process that is not a subordinator reaches $(-\infty, K)$ before $\mathbf{e}(q)$ with positive probability. $\square$

Theorem 7 shows that $\mathcal{S}$ is not empty and identifies b^* .

6.2 The threshold

Theorem 7 (The perpetual put: threshold and value). Let X satisfy (A1), not be a subordinator, $q > 0$, and $g(x) = (K - x)^+$. Let $I = \underline{X}_{\mathbf{e}(q)}$ under $\mathbb{P}_0$. (i) The optimal threshold is

$$b^* = K + \mathbb{E}[I] = K + \partial_r \Phi_q^-(0), \quad (6.1)$$

the stopping time $\tau^* = \inf\{t \geq 0 : X_t \leq b^*\}$ is optimal, $\mathcal{S} = (-\infty, b^*]$, and

$$V(x) = \mathbb{E}[(b^* - x - I)^+], \quad x \in \mathbb{R}. \quad (6.2)$$

(ii) Continuous fit $V(b^*) = g(b^*)$ always holds, and $\partial_x V(b^*) = -\mathbb{P}(I < 0)$. Smooth fit $\partial_x V(b^*) = \partial_x g(b^*) = -1$ holds if and only if 0 is regular for $(-\infty, 0)$; in particular



whenever $\sigma > 0$. (iii) For the Kou process of Theorem 2,

$$b^* = K - \frac{1}{\beta_1} - \frac{1}{\beta_2} + \frac{1}{\eta_-}, \quad V(x) = A_1 e^{-\beta_1(x-b^*)} + A_2 e^{-\beta_2(x-b^*)}, \quad x > b^*, \quad (6.3)$$

with $A_i = w_i/\beta_i$, that is $A_1 = \beta_2(\eta_- - \beta_1)/(\eta_- \beta_1(\beta_2 - \beta_1))$ and $A_2 = \beta_1(\beta_2 - \eta_-)/(\eta_- \beta_2(\beta_2 - \beta_1))$. (iv) For Brownian motion with drift ($\nu = 0$), $b^* = K - 1/\beta$ and $V(x) = \beta^{-1} e^{-\beta(x-b^*)}$ for $x > b^*$, with $\beta = -r_- = (b + \sqrt{b^2 + 2\sigma^2 q})/\sigma^2$.

Proof. The argument is that of Novikov and Shiryaev for random walks [15], in its continuous-time form [16]. $\mathbb{E}|I| < \infty$ by (A1), so b^* is finite. Let $H(x) := \mathbb{E}[(b^* - x - I)^+]$. Since $\mathbb{E}[b^* - x - I] = K - x$, Jensen's inequality gives $H \geq g$. For a stopping time τ , the strong Markov property at τ and the lack of memory of $\mathbf{e}(q)$ show that on $\{\tau < \mathbf{e}(q)\}$ the infimum of X over $[\tau, \mathbf{e}(q)]$ is $X_\tau + I'$ with I' independent of $\mathcal{F}_\tau$ and distributed as I . Hence

$$\mathbb{E}_x[e^{-q\tau} H(X_\tau)] = \mathbb{E}_x[(b^* - \inf_{[\tau, \mathbf{e}(q)]} X)^+ \mathbb{1}_{\{\tau < \mathbf{e}(q)\}}] \leq \mathbb{E}_x[(b^* - \inf_{[0, \mathbf{e}(q)]} X)^+] = H(x), \quad (6.4)$$

so $\mathbb{E}_x[e^{-q\tau} g(X_\tau)] \leq H(x)$ for every τ , and $V \leq H$. For $\tau = \tau^*$ the inequality in (6.4) is an equality: before τ^* the path stays above b^* , so on $\{\tau^* \geq \mathbf{e}(q)\}$ the integrand vanishes and on $\{\tau^* < \mathbf{e}(q)\}$ the two infima coincide. For $y \leq b^*$, $b^* - y - I \geq 0$, so $H(y) = \mathbb{E}[b^* - y - I] = K - y = g(y)$; thus $H(x) = \mathbb{E}_x[e^{-q\tau^*} g(X_{\tau^*})] \leq V(x)$, and $V = H$. For $x > b^*$, $H(x) - (K - x) = \mathbb{E}[(x + I - b^*)^+] > 0$ because $\mathbb{P}(I > b^* - x) > 0$, and $H > 0 = g$ on $[K, \infty)$; so $\mathcal{S} = (-\infty, b^*)$. Finally $\mathbb{E}[I] = \partial_r \Phi_q^-(0)$ by differentiating the transform.

(ii) $V(b^*) = \mathbb{E}[-I] = K - b^* = g(b^*)$. The function $x \mapsto \mathbb{E}[(b^* - x - I)^+]$ is convex with right derivative $-\mathbb{P}(I < b^* - x)$, which at $x = b^*$ is $-\mathbb{P}(I < 0)$. Now $\mathbb{P}(I = 0) = \mathbb{P}(\tau_0^- \geq \mathbf{e}(q))$ with $\tau_0^- := \inf\{t > 0 : X_t < 0\}$, which vanishes if and only if $\tau_0^- = 0$ almost surely, the definition of regularity of 0 for $(-\infty, 0)$; a non-zero Brownian component makes 0 regular [1], [17].

(iii) By Theorem 2, $-I$ is the mixture $w_1 \text{Exp}(\beta_1) + w_2 \text{Exp}(\beta_2)$, and $\mathbb{E}[-I] = w_1/\beta_1 + w_2/\beta_2 = 1/\beta_1 + 1/\beta_2 - 1/\eta_-$, the last form being $-\partial_r \log \Phi_q^-(0)$. For $c := x - b^* > 0$ and $M \sim \text{Exp}(\beta)$, $\mathbb{E}[(M - c)^+] = e^{-\beta c}/\beta$, which gives (6.3). (iv) is (iii) with one exponential: $-I \sim \text{Exp}(\beta)$ by Lemma 4. $\square$

In the Kou case the two coefficients and the threshold can also be read off as the unique solution of three linear conditions on V over (b^*, ∞) : continuous fit $A_1 + A_2 = K - b^*$, smooth fit $\beta_1 A_1 + \beta_2 A_2 = 1$, and the condition that down-jumps landing below b^* are paid g there, $A_1 \eta_-/(\eta_- - \beta_1) + A_2 \eta_-/(\eta_- - \beta_2) = K - b^* + 1/\eta_-$ — the computation of [18] for the exponential put, carried out for the linear payoff. Two features of this solution are easy to lose. The exponentials are decreasing, $e^{-\beta_i x}$ with $\psi(-\beta_i) = q$, so the roots that enter are the negative ones; and the third condition cannot be met by a single exponential, so with down-jumps one root is never enough. By symmetry, the call $g(x) = (x - K)^+$, which stops above a threshold, has $b^* = K + \mathbb{E}[S]$ and is governed by the positive roots.

Corollary 2 (The exponential put). Under the hypotheses of Theorem 7 with $g(x) =$



$(K - e^x)^+$, the stopping region is $(-\infty, b^*]$ with

$$e^{b^*} = K \mathbb{E}[e^I] = K \Phi_q^-(1), \quad V(x) = \frac{\mathbb{E}[(e^{b^*} - e^{x+I})^+]}{\mathbb{E}[e^I]}. \quad (6.5)$$

For the Kou process, $e^{b^*} = K\beta_1\beta_2(\eta_- + 1)/(\eta_-(\beta_1 + 1)(\beta_2 + 1))$ and $V(x) = K \sum_{i=1}^2 \frac{w_i}{\beta_i + 1} e^{-\beta_i(x-b^*)}$ for $x > b^*$; for Brownian motion $e^{b^*} = K\beta/(\beta + 1)$.

Proof. This is Mordecki's theorem [19]; the proof of Theorem 7 (i) carries over with $H(x) := \mathbb{E}[(e^{b^*} - e^{x+I})^+]/\mathbb{E}[e^I]$: $H \geq g$ by Jensen because $\mathbb{E}[e^{b^*} - e^{x+I}]/\mathbb{E}[e^I] = K - e^x$, the map $y \mapsto (e^{b^*} - e^y)^+$ is non-increasing, which gives (6.4), and $H = g$ on $(-\infty, b^*]$. For Kou, $\mathbb{E}[e^I] = \Phi_q^-(1)$ from Theorem 2, and for $M \sim \text{Exp}(\beta)$ and $c > 0$, $\mathbb{E}[(e^{b^*} - e^{b^*+c-M})^+] = e^{b^*} e^{-\beta c}/(\beta + 1)$. $\square$

6.3 Numerical check: the variational inequality on a grid and the stopped process

Theorems 5, 6 and 7, Proposition 1 and Corollary 2 are checked at the Kou benchmark with $K = 1$ by two methods that use neither the roots nor the Wiener–Hopf factors.

The grid solve. Algorithm 3 discretises (5.1) on $[-12, 30]$ with step h — central differences for the diffusion, and the jump integrals of a piecewise-linear V computed exactly by a two-term recursion per exponential component — and decides the stopping set by policy iteration [20]. Table 4 shows the grid values converging to the closed forms of Theorem 7 and Corollary 2 at second order (the error falls by about four each time h halves), and the boundary converging to b^* : read off the grid it is the last stopping node, within h ; refined by the smooth-fit parabola through the first three continuation nodes it is within 5×10^{-5} of the closed form at $h = 0.0025$. For Theorem 5, policy iteration started from the free equation everywhere and from immediate stopping wherever $g > 0$ reaches the same grid function at $h = 0.005$ (maximum difference 0, identical stopping sets, after 60 and 59 iterations). For Proposition 1, at every h the computed stopping set is a single interval $[x_{\min}, b_h]$. Figure 3 repeats the comparison of the threshold across jump intensities $\lambda \in [0.1, 5]$: the threshold falls as λ rises, for both payoffs.



Table 4: The grid solve of the variational inequality (Algorithm 3) against Theorem 7 and Corollary 2 at the Kou benchmark: lin. = the linear put (Theorem 7), exp. = the exponential put (Corollary 2). The error is $\max|V_h - V|$ over $x \in \{0, 0.5, 1, 2\}$ (linear put) and $x \in \{-0.5, 0, 0.5\}$ (exponential put); the refined boundary is the vertex of the smooth-fit parabola through the first three continuation nodes.

h	lin. b_h	lin. refined	lin. error	exp. refined	exp. error
0.02	-0.1000	-0.09190	1.2×10^{-4}	-0.70946	5.6×10^{-5}
0.01	-0.0900	-0.09057	3.3×10^{-5}	-0.70688	1.4×10^{-5}
0.005	-0.0900	-0.08995	8.3×10^{-6}	-0.70665	3.6×10^{-6}
0.0025	-0.0900	-0.08980	2.1×10^{-6}	-0.70648	8.2×10^{-7}
closed form		-0.08975		-0.70645	

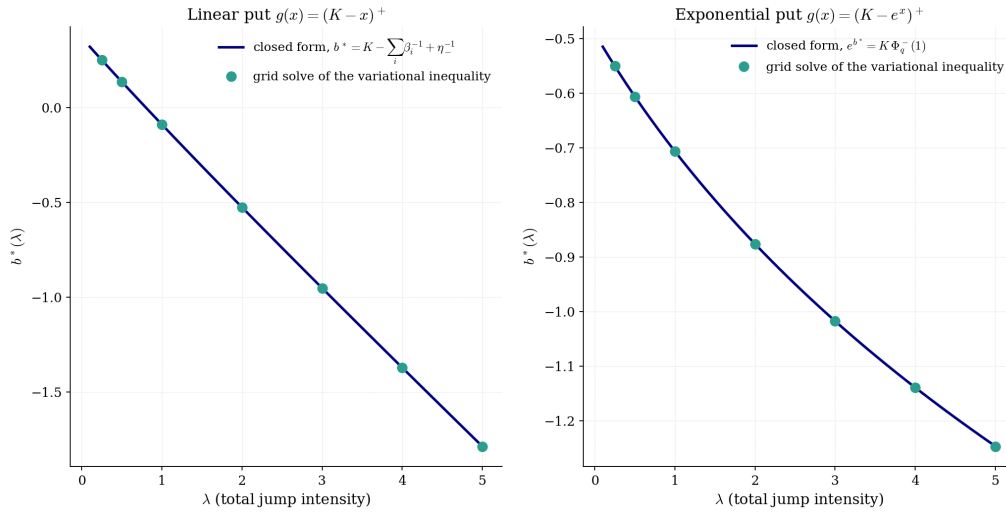


Figure 3: Optimal threshold $b^*(\lambda)$ of the perpetual put under Kou jumps, as the total jump intensity λ varies with $(K, \sigma, p, \eta_+, \eta_-, q) = (1, 0.20, 0.6, 10, 5, 0.05)$. Lines: the closed forms of Theorem 7 (left, linear put) and Corollary 2 (right, exponential put). Points: the boundary of the grid solve of the variational inequality ($h = 0.005$), which knows only the generator and the payoff.

The stopped process. Algorithm 2 simulates the rule “stop at the first entry into $(-\infty, b]$ ” exactly: within a Brownian segment the first passage below b happens at the value b as soon as the segment’s minimum reaches it, and at a jump at the post-jump value. Table 5 gives the discounted payoff of this rule from 8,000,000 paths, with common random numbers across all rows: at $b = b^*$ against V of Theorem 7 and Corollary 2, and at the grid solve’s own boundary b_h against the grid value (Theorem 6: the solution of the variational inequality is the value of the stopping rule it defines). A deeper run of 64,000,000 paths at one point of each payoff gives 0.65228 ± 0.00007 against $V(0.5) =$



0.65225 and 0.41920 ± 0.00003 against $V(-0.5) = 0.41921$ for the exponential put.

Table 5: The discounted payoff of the barrier rule, simulated exactly with 8,000,000 paths (common random numbers across rows), against the closed forms of Theorem 7 and Corollary 2 and against the grid solve at $h = 0.0025$ with its own boundary b_h (Theorem 6).

Payoff	x	barrier b	Monte Carlo	reference	(MC – ref.)/s.e.
linear, Theorem 7	0	$b^* = -0.0897$	1.00526 ± 0.00013	$V = 1.00507$	+1.5
linear, Theorem 7	0.5	$b^* = -0.0897$	0.65238 ± 0.00021	$V = 0.65225$	+0.6
linear, Theorem 7	1	$b^* = -0.0897$	0.42522 ± 0.00020	$V = 0.42503$	+0.9
linear, Theorem 7	2	$b^* = -0.0897$	0.18049 ± 0.00015	$V = 0.18052$	–0.2
linear, Theorem 6	0	$b_h = -0.0898$	1.00526 ± 0.00013	$V_h = 1.00507$	+1.5
linear, Theorem 6	0.5	$b_h = -0.0898$	0.65238 ± 0.00021	$V_h = 0.65225$	+0.6
linear, Theorem 6	1	$b_h = -0.0898$	0.42522 ± 0.00020	$V_h = 0.42503$	+0.9
linear, Theorem 6	2	$b_h = -0.0898$	0.18049 ± 0.00015	$V_h = 0.18053$	–0.2
exponential, Corollary 2	–0.5	$b^* = -0.7065$	0.41940 ± 0.00008	$V = 0.41921$	+2.4
exponential, Corollary 2	0	$b^* = -0.7065$	0.27230 ± 0.00010	$V = 0.27226$	+0.4
exponential, Corollary 2	0.5	$b^* = -0.7065$	0.17751 ± 0.00009	$V = 0.17742$	+1.0

Optimality of b^* is checked the same way. Figure 4 (right) shows the value at $x = 0.5$ of the barrier rule as a function of the barrier, from the simulation and from the grid solve of the linear equation with the stopping set fixed. Both peak at the closed-form threshold: a parabola through the simulated values on $b^* \pm 0.06$ peaks at -0.0924 ± 0.0015 (standard error over 20 independent batches) and the grid curve at -0.0903 , against $b^* = -0.0897$.

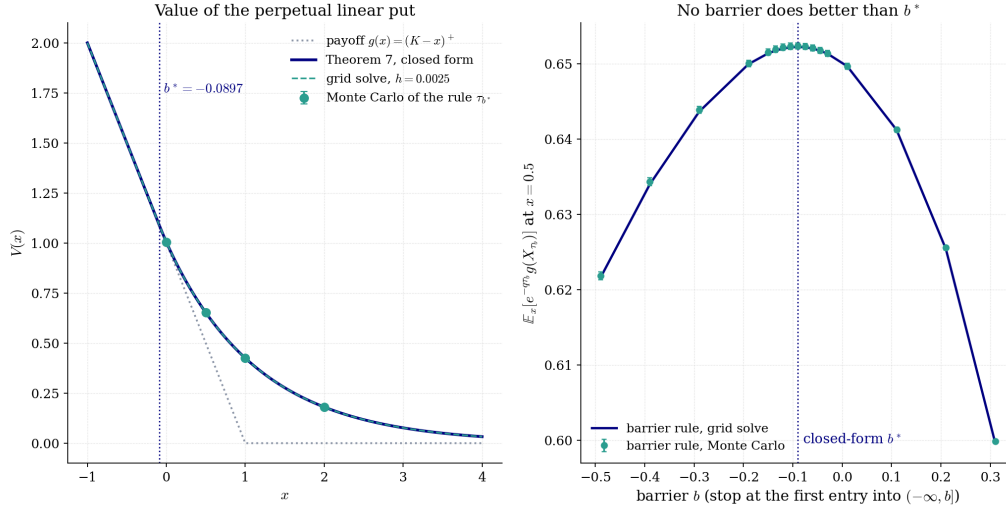


Figure 4: Left: the value V of the perpetual linear put at the Kou benchmark — Theorem 7’s closed form, the grid solve of the variational inequality ($h = 0.0025$, dashed) and the Monte Carlo value of the rule τ_{b^*} with 95% intervals (points); the payoff is dotted. Right: the value at $x = 0.5$ of the rule “stop below b ” as a function of b , by Monte Carlo (points) and by the grid solve with the stopping set fixed (line); both peak at the closed-form b^* of Theorem 7.

7. Worked examples

This section collects the closed forms at the benchmark and a hyperexponential first-passage example.

7.1 Perpetual American put under Kou jumps

Example 1 (The benchmark put). With $b = 0$, $\sigma = 0.20$, $\lambda = 1$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$ and $K = 1$, Algorithm 1 gives the negative roots $\beta_1 = 0.8563$, $\beta_2 = 8.2010$ and the weights $w_1 = 0.9254$, $w_2 = 0.0746$. Theorem 7 then gives $\mathbb{E}[-I] = 1.0897$, the threshold $b^* = -0.0897$ and $A_1 = 1.0806$, $A_2 = 0.0091$, so that $V(0) = 1.0051$, $V(0.5) = 0.6522$, $V(1) = 0.4250$ and $V(2) = 0.1805$; the slope at the boundary is $-(\beta_1 A_1 + \beta_2 A_2) = -1.0000$, smooth fit. For the exponential put, Corollary 2 gives $\mathbb{E}[e^I] = 0.4934$, $b^* = \log 0.4934 = -0.7065$ and $V(0) = 0.2723$. With the jumps switched off ($\lambda \rightarrow 0$) the linear threshold tends to the Brownian value $K - 1/\beta = 0.3675$ of Theorem 7 (iv), $\beta = 1.5811$: jumps lower the threshold by 0.4573 at $\lambda = 1$. All of these are checked in §6.3.

7.2 Hyperexponential first-passage time

Let X be a hyperexponential Lévy process with $N_+ = 2$ up-rates and $N_- = 1$ down-rate. By Theorem 3, the Laplace transform of the first-passage time $\tau_a^+ := \inf\{t : X_t \geq a\}$ at level $a > 0$ is



$$\mathbb{E}_0[e^{-q\tau_a^+}] = \mathbb{P}(S \geq a) = \sum_{i=1}^3 c_i e^{-\alpha_i a}, \quad (7.1)$$

where $\alpha_1 < \alpha_2 < \alpha_3$ are the positive roots of $\psi(r) = q$ and the coefficients c_i are the residues of Φ_q^+ at its poles α_i , given in Theorem 3. They sum to one, $\sum_i c_i = \Phi_q^+(0) = 1$, which is $\mathbb{E}_0[e^{-q\tau_a^+}] \rightarrow 1$ as $a \downarrow 0$.

Example 2 (Residues and simulation). Take $b = 0$, $\sigma = 0.20$, up-jumps of intensity $a_1^+ = 0.6$ at rate $\eta_1^+ = 5$ and $a_2^+ = 0.4$ at rate $\eta_2^+ = 20$, down-jumps of intensity $a_1^- = 1$ at rate $\eta_1^- = 5$, and $q = 0.05$. The positive roots are $\alpha_1 = 1.1983$, $\alpha_2 = 9.1633$, $\alpha_3 = 21.1580$, interlacing with 5 and 20 as Theorem 3 requires (the negative roots are -0.4877 and -11.0318 ; bisection and the companion matrix agree to 1×10^{-14}). The residues are $c = (0.8717, 0.1197, 0.0086)$, summing to one. Table 6 compares (7.1) with the empirical tail of S from 1,000,000 exactly simulated paths: the agreement is within two standard errors at every level, which checks Theorem 3.

Table 6: Example 2: the first-passage transform $\mathbb{E}_0[e^{-q\tau_a^+}] = \mathbb{P}(S \geq a)$ from the residues of Theorem 3, against 1,000,000 exactly simulated paths.

Level a	$\sum_i c_i e^{-\alpha_i a}$ (Theorem 3)	$\mathbb{P}(S > a)$, Monte Carlo	(MC – closed)/s.e.
0.25	0.65820	0.65898 ± 0.00047	+1.6
0.5	0.48004	0.48072 ± 0.00050	+1.4
1	0.26302	0.26331 ± 0.00044	+0.7
2	0.07935	0.07942 ± 0.00027	+0.2
$\mathbb{E}[S] = \sum_i c_i / \alpha_i$	0.74094	0.74114 ± 0.00082	+0.2

7.3 Uses of the closed forms

The closed forms are meant to be cited, and two uses are worth naming. On a bounded interval the boundary conditions turn the Wiener–Hopf factorisation into a Fredholm equation of the second kind, whose Neumann series iterates $\mathcal{R}_q$; Theorems 2–4 give each term in closed form on exponentials. In a stopping problem whose payoff is driven by a controlled process, the stopping side of Theorem 7 applies with the infimum factor of the controlled process: the threshold of a put-type stop is $K + \mathbb{E}[I]$ and requires the negative roots, that of a call-type stop is $K + \mathbb{E}[S]$ and requires the positive ones.

8. Conclusion

This paper has developed the resolvent operator $\mathcal{R}_q$ of a one-dimensional Lévy process as a unified subject. We established the Wiener–Hopf factorisation $\mathcal{R}_q = q \mathcal{R}_q^+ \mathcal{R}_q^-$ (Theorem 1); derived the closed forms of $\mathcal{R}_q^\pm e_r$ for the Kou double-exponential, hyperexponential



and meromorphic families, together with the mixture laws of the supremum and infimum (Theorems 2, 3, 4); proved a viscosity comparison principle for the optimal-stopping variational inequality (Theorem 5) and the verification result identifying its unique solution with the stopping value (Theorem 6); and showed that the perpetual put stops on a half-line whose boundary is set by the infimum, $b^* = K + \mathbb{E}[\underline{X}_{e(q)}]$, with smooth fit exactly when 0 is regular downward (Proposition 1, Theorem 7, Corollary 2). Under Kou jumps the threshold needs both negative roots of $\psi(r) = q$. Every closed form was checked against exact simulation of the process or a grid solve of the equations that does not use it.

8.1 What this paper deliberately omits

Three directions are left for future work:

1. **Higher dimensions.** The Wiener–Hopf factorisation is intrinsically one-dimensional; the analogue in $\mathbb{R}^d$ for $d \geq 2$ requires the matrix and operator Wiener–Hopf theory surveyed in [2] and remains an active area.
2. **Non-rational Lévy families.** Tempered stable, CGMY and self-decomposable families have non-rational cumulant exponents; the factorisation exists by Theorem 1 but does not close in finite terms, and its factors must be computed numerically.
3. **Time-inhomogeneous and Markov-modulated extensions.** The Markov-modulated Lévy resolvent is a vector-valued generalisation of $\mathcal{R}_q$ with a regime-dependent factorisation.

8.2 Heavy tails

Assumption (A1) is essential to the closed forms. An α -stable process has no exponential moments, so the analyticity strip collapses to the imaginary axis, ψ is available only as a Fourier symbol, and the rational calculus of §4 has no counterpart; the comparison principle of §5, by contrast, needs only the integrability that the growth class of the solutions requires. The two regimes call for different tools, and we have treated only the first.

A. Algorithm pseudocode

This appendix gives pseudocode for the numerical routines behind every number in the paper. They are implemented in the supplementary files `python/resolvent_kou.py` (Algorithm 1), `python/levy_mc.py` (Algorithm 2) and `python/vi_solver.py` (Algorithm 3); `python/checks.py` runs every check and writes the values printed in the tables, and `python/make_figures.py` draws the figures from them.

Algorithm 1 — Roots of the cumulant equation and the closed forms

```

1 Input:
2   sigma > 0, drift b, up-jumps (a_j, eta_j), j = 1..N, rates increasing,
3   down-jumps (c_k, theta_k), k = 1..M, discount q > 0, strike K
    
```



```

4  psi(r) = b r + sigma^2 r^2 / 2 + sum_j a_j (eta_j/(eta_j - r) - 1)
5          + sum_k c_k (theta_k/(theta_k + r) - 1)
6
7  Output:
8  positive roots alpha_1 < ... < alpha_{N+1}, moduli beta_1 < ... <
   beta_{M+1}
9  of the negative roots; weights of S and -I; thresholds of both puts
10
11 Step 1. For each interval (0, eta_1), (eta_1, eta_2), ..., (eta_N, R):
12         (R doubled until psi(R) > q)
13         bisection on psi(r) - q to relative width 1e-10,
14         then Newton with the analytic psi'(r) until |step| < 1e-15.
15 Step 2. The same on (-theta_1, 0), (-theta_2, -theta_1), ... for beta_i.
16 Step 3. Check: the real roots of the numerator polynomial
17         (psi(r) - q) prod_j (eta_j - r) prod_k (theta_k + r)
18         (companion-matrix eigenvalues) must agree with Steps 1-2.
19 Step 4. v_i = prod_j (eta_j - alpha_i)/eta_j * prod_{k != i}
   alpha_k/(alpha_k - alpha_i)
20         w_i = prod_k (theta_k - beta_i)/theta_k * prod_{l != i}
   beta_l/(beta_l - beta_i)
21 Step 5. Linear put:      b* = K - sum_i w_i/beta_i
22                       V(x) = sum_i (w_i/beta_i) exp(-beta_i (x -
   b*)), x > b*
23         Exponential put: b* = log(K sum_i w_i beta_i/(beta_i + 1))
24                       V(x) = K sum_i w_i/(beta_i + 1) exp(-beta_i (x
   - b*))
25
26 Complexity:
27 about 40 bisection and 5 Newton evaluations of psi per root.

```

Remark 1 (Why bisection + Newton). Between two consecutive poles of ψ there is exactly one root (Theorems 2 and 3), so the brackets are known in advance and bisection cannot miss or duplicate a root. Pure Newton from a naive initial guess (e.g., the diffusion-only root $\sqrt{2q}/\sigma$) can jump across a pole and converge to a root of the wrong interval; bisection inside the bracket prevents that, and Newton then converges quadratically.

Algorithm 2 — Exact Monte Carlo of the process killed at an exponential time

```

1  Input:
2  sigma, b, q, total jump intensity lambda, jump-size sampler, paths n,
3  optional levels c_1 > ... > c_L (all < 0) for first passage below
4  Output:
5  per path: S, I, X at the killing time; for each level: hit or not,
6  and the value of X at the first passage
7
8  For each path (vectorised over paths):
9  T ~ Exp(q); t = 0; x = 0; S = 0; I = 0
10 repeat

```



```

11 dt = min(Exp(lambda), T - t); last = (dt = T - t)
12 y = x + b dt + sigma sqrt(dt) Z,          Z ~ N(0, 1)
13 max = (x + y + sqrt((y - x)^2 - 2 sigma^2 dt log U1)) / 2
14 min = (x + y - sqrt((y - x)^2 - 2 sigma^2 dt log U2)) / 2
15 S = max(S, max); I = min(I, min)
16 each level c not yet hit: if min <= c then hit, X_tau = c
17 if not last: y = y + jump; S = max(S, y); I = min(I, y)
18   each level c not yet hit: if y <= c then hit, X_tau = y
19 x = y; t = t + dt
20 until last (or every level hit, in a first-passage run)
21
22 Cost: about (lambda/q + 1) Brownian segments per path.

```

Algorithm 3 — Policy iteration for the variational inequality on a grid

```

1 Input:
2   sigma, b, q, jump components, payoff g, grid x_0 < ... < x_N with step
   h,
3   tail integrals of g below x_0, starting policy (continue or stop)
4 Output:
5   grid values V_i, stopping set, boundary b_h
6
7 Unknowns: V_i and, per exponential component with rate eta,
8   P_i = int_0^inf V(x_i + z) eta e^{-eta z} dz (up) or the mirror
   (down),
9   linked by P_i = e^{-eta h} P_{i+1} + w0 V_i + w1 V_{i+1} (exact for
10  piecewise-linear V), so the linear system is sparse.
11 Rows: continuation: (q + lambda) V_i - b D1 V - sigma^2/2 D2 V - sum a
   P_i = 0
12   stopping: V_i = g_i
13   ends: V_0 = g(x_0), V_N = 0
14
15 repeat
16   solve the sparse system for the current policy (sparse LU)
17   new policy at node i: stop if V_i - g_i < ((q - L_h) V)_i, else continue
18 until the policy does not change
19 b_h = last stopping node; refined: vertex of the parabola through
20   (V - g) at the first three continuation nodes
21
22 Cost: one sparse LU per policy step.

```

Remark 2 (Reproducibility). `python/checks.py` runs Algorithms 1–3 for every table and figure in about two minutes and writes `results.json`; every random stream is seeded, so a re-run reproduces every printed digit. `python/make_figures.py` reads that file and evaluates the closed forms afresh for the curves.



B. Tabulated closed forms

This appendix collects the closed forms of §4 in a single reference table, for direct quotation.

B.1 Notation reminder

Throughout the table:

- X is a one-dimensional Lévy process with $\sigma > 0$ and cumulant exponent $\psi(r) = \log \mathbb{E}_0[e^{rX_1}]$, continued rationally (or meromorphically) beyond the analyticity strip $(-\eta_-, \eta_+)$.
- $q > 0$ is the discount rate, S and I the supremum and infimum of X up to an independent $\text{Exp}(q)$ time.
- α_i are the positive roots of $\psi(r) = q$ in increasing order, and $-\beta_i$ the negative roots, $0 < \beta_1 < \beta_2 < \dots$.
- η_j^+, η_k^- are the up- and down-jump rates, with intensities a_j^+, a_k^- .
- In every row $\mathcal{R}_q^\pm e_r = q^{-1} \Phi_q^\pm(r) e_r$ and $\Phi_q^+(r) \Phi_q^-(r) = q/(q - \psi(r))$; Φ_q^+ is finite for $r < \alpha_1$ and Φ_q^- for $r > -\beta_1$.

B.2 Reference table

Family	$\psi(r)$ — cumulant exponent	Positive factor $\Phi_q^+(r) = \mathbb{E}[e^{rS}]$	Negative factor $\Phi_q^-(r) = \mathbb{E}[e^{rI}]$
Kou	$br + \frac{1}{2}\sigma^2 r^2 + \lambda p \frac{\eta_+}{\eta_+ - r} + \lambda(1 - p) \frac{\eta_-}{\eta_- + r} - \lambda$	$\frac{\alpha_1 \alpha_2 (\eta_+ - r)}{\eta_+ (\alpha_1 - r)(\alpha_2 - r)}$	$\frac{\beta_1 \beta_2 (\eta_- + r)}{\eta_- (\beta_1 + r)(\beta_2 + r)}$
Hyperexp (N_+, N_- rates)	$br + \frac{1}{2}\sigma^2 r^2 + \sum_{j=1}^{N_+} a_j^+ \left(\frac{\eta_j^+}{\eta_j^+ - r} - 1 \right) + \sum_{k=1}^{N_-} a_k^- \left(\frac{\eta_k^-}{\eta_k^- + r} - 1 \right)$	$\prod_{i=1}^{N_++1} \frac{\alpha_i}{\alpha_i - r} \prod_{j=1}^{N_+} \frac{\eta_j^+ - r}{\eta_j^+}$	$\prod_{i=1}^{N_-+1} \frac{\beta_i}{\beta_i + r} \prod_{k=1}^{N_-} \frac{\eta_k^- + r}{\eta_k^-}$
Meromorphic	the same with countably many components, $\sum_j a_j^\pm / \eta_j^\pm < \infty$	$\prod_{i \geq 1} \frac{1 - r/\eta_i^+}{1 - r/\alpha_i}$	$\prod_{i \geq 1} \frac{1 + r/\eta_i^-}{1 + r/\beta_i}$

Table: The Wiener–Hopf factors of the three rational families (Theorems 2, 3 and 4). The supremum S is the mixture $\sum_i c_i \text{Exp}(\alpha_i)$ with c_i the residues of Φ_q^+ ; the same holds for $-I$ with the β_i .



B.3 Special cases

Remark 3 (Brownian limit). When $\lambda = 0$ in the Kou row, $\psi(r) = br + \frac{1}{2}\sigma^2 r^2$ has one positive and one negative root, $\alpha = (-b + \sqrt{b^2 + 2\sigma^2 q})/\sigma^2$ and $-\beta = (-b - \sqrt{b^2 + 2\sigma^2 q})/\sigma^2$: as $\lambda \rightarrow 0$ the outer roots α_2, β_2 tend to the poles η_+, η_- and cancel against them. The factors reduce to $\Phi_q^+(r) = \alpha/(\alpha - r)$ and $\Phi_q^-(r) = \beta/(\beta + r)$, the case of Lemma 4.

Remark 4 (Spectrally one-sided). When $p = 1$ in the Kou row there are no down-jumps: $\psi(r) = q$ has two positive roots and a single negative root $-\beta$, $\Phi_q^-(r) = \beta/(\beta + r)$ by Lemma 4 (β a root of the full ψ , jumps included), and Φ_q^+ keeps its two-root form.

Remark 5 (Roots as $q \rightarrow 0$). As $q \downarrow 0$ the roots of $\psi(r) = q$ tend to the roots of $\psi(r) = 0$. If $\psi'(0) = \mathbb{E}[X_1] > 0$ the positive root tends to 0 and the negative one to the non-zero root of ψ ; if $\psi'(0) < 0$ the roles are exchanged; if $\psi'(0) = 0$ both tend to 0 like $\pm\sqrt{2q/\psi''(0)}$. At the benchmark $\psi'(0) = -0.02$, so it is $r_- = -\beta_1$ that tends to 0. A pole at the origin signals a factor that is no longer the transform of a probability law: the extremum it describes is then infinite with positive probability.

The closed forms in this table are the **primary deliverable** of the paper for downstream citation. Later work that invokes “the Wiener–Hopf factor under Kou jumps” should cite Theorem 2 and reproduce the relevant row of the table; a stopping threshold should be taken from Theorem 7 or Corollary 2, with the factor on the side on which the process stops.



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Zamrik Quant Research
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About the Desk

Stochastic Analysis Desk

Zamrik Quant Research

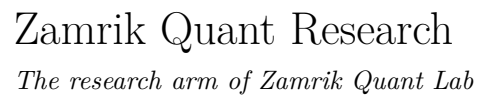
How random processes behave — Itô calculus, diffusions, local time, Fokker–Planck — in theory and in application.

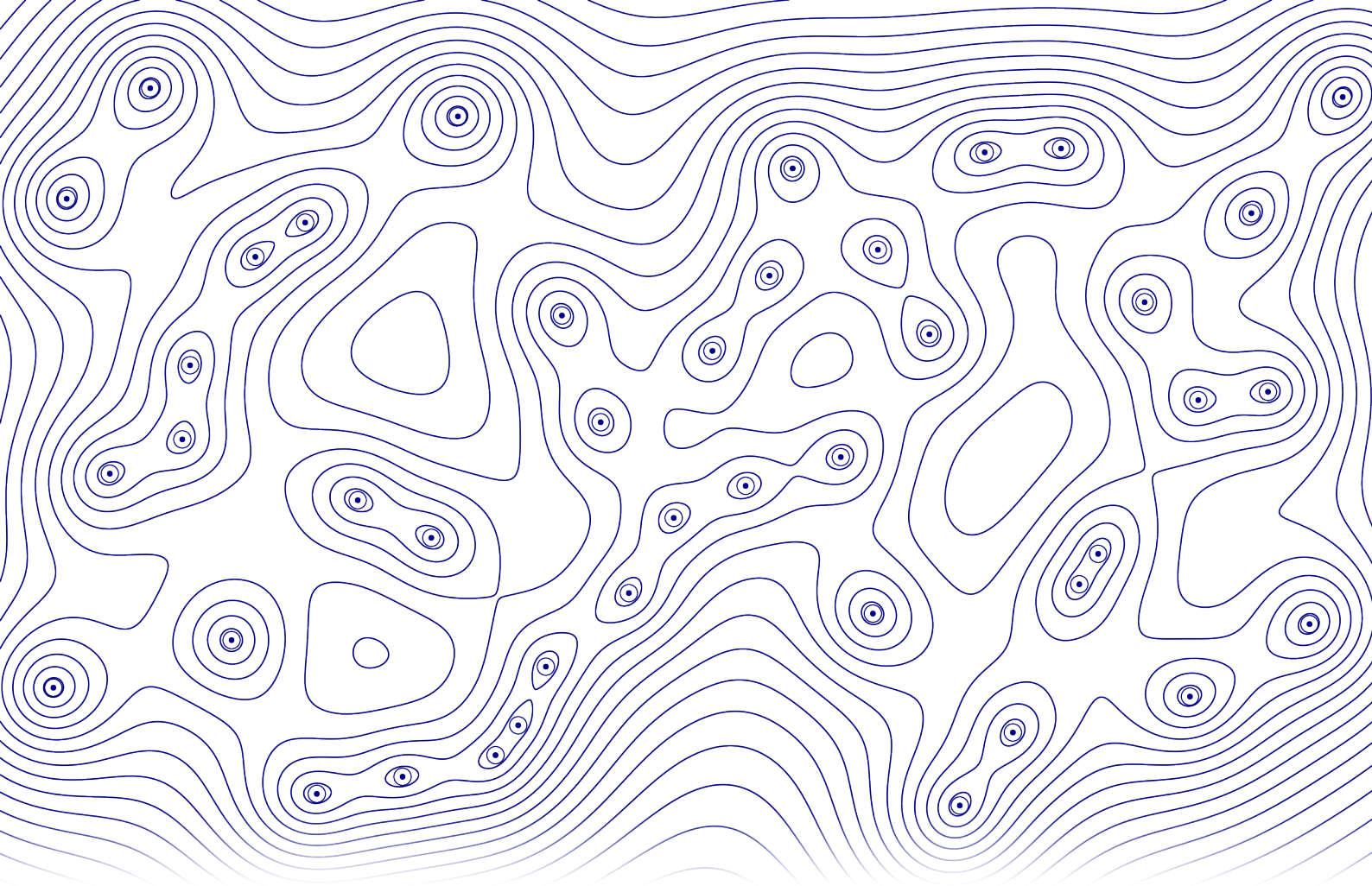
Stochastic analysis is the grammar of every other desk: a price, an order queue, a population or a value function is a functional of a random path, and what can be said about it depends on what can be proved about the path. The Stochastic Analysis desk works on those proofs and on the exact formulas they produce.

Its core questions are the classical ones, asked with care: when a diffusion first reaches a boundary, fixed or moving; how long it spends at each level before it does; and what happens at the boundary itself — reflection, killing, a sticky or skewed point, a reset to the start. The answers come through Itô calculus and martingale methods, local time and Tanaka’s formula, Ray–Knight theorems, time changes, and the Feynman–Kac and Fokker–Planck equations that turn questions about paths into equations for expectations and densities. Wherever the problem allows, the desk derives the law in closed form and then checks it against an independently built simulation.

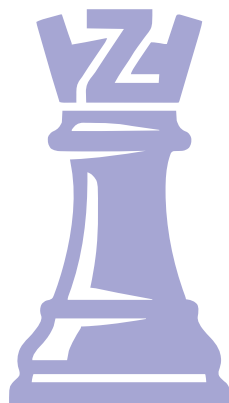
The results travel. The same calculus corrects a barrier that is checked once a day, explains why most stocks lose to the average stock, separates real correlation from the noise of random matrices, sets the acceptance rate of a Metropolis sampler, and models a neuron’s membrane and a day’s rainfall. The desk also studies the numerics that decide whether a simulated path tells the truth: the strong and weak convergence of Euler–Maruyama and Milstein schemes, Brownian-bridge corrections for barriers monitored in discrete time, and the error left when a continuous process is sampled on a grid.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Power function of the Matérn interpolant: $P(\mathbf{x})^2 = \varphi(0) - \varphi_X(\mathbf{x})^\top A^{-1} \varphi_X(\mathbf{x})$, zero at every data point $\mathbf{x}_i$



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