

WORKING PAPER

Optimal Timing of Carbon-Capture Retrofit on a Mature Offshore Gas Platform under Stochastic EU-ETS Prices and a Stochastic Decommissioning Deadline

A real-options analysis with a free-boundary HJB and a closed-form perpetual benchmark

Supervised by Dr. T. Zamrik

07-MAY-2024

Phase portrait of the Brusselator: $\dot{x} = a + x^2y - (b+1)x$, $\dot{y} = bx - x^2y$, $a = 1.12$, $b = 3.05$



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Optimal Timing of Carbon-Capture Retrofit on a Mature Offshore Gas Platform under Stochastic EU-ETS Prices and a Stochastic Decommissioning Deadline

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Abstract. We study when the operator of a mature offshore gas platform should retrofit carbon capture and storage when the price of EU Emissions Trading System (EU ETS) allowances is stochastic and a regulatory decommissioning deadline arrives at a random time. The retrofit is irreversible, and once installed it saves the allowances on the avoided emissions until the deadline. We pose the decision as optimal stopping of a geometric Brownian motion killed at an exponential deadline, show that the decommissioning charge cancels from the decision, derive the free-boundary variational inequality, in which the deadline enters with the term $+\lambda\Phi = -\lambda D$, and solve it in closed form for an open-ended retrofit window and numerically for a finite one. The price model is fitted to EU ETS primary-auction clearing prices on 1,343 auction days from January 2018 to 7 May 2024 (EEX auction results as published by the Infracore Foundation; licence and credit in Section 5.1). The monthly volatility is $\hat{\sigma} = 0.380$; the historical drift, 0.40 per year, exceeds any discount rate and cannot serve as a valuation drift. Investment costs and emissions come from a peer-reviewed study of a North Sea platform; operating costs come from a US DOE/NETL onshore study, used as a stated proxy. With a mean residual life of 12 years the retrofit threshold is 180 €/tCO₂, 2.68 times the zero-NPV price and well above the auction price of 69.16 €/tCO₂ on 7 May 2024; with a 6-year mean life it rises to 239 €/tCO₂. We prove that the threshold rises with the deadline hazard exactly when the annuity on the retrofit capital outweighs the fall in the option premium; at the calibrated costs this holds for every mean residual life below 76 years. The study's range of capital costs moves the 12-year threshold from 71 to 290 €/tCO₂. Each headline number is checked by a grid solution of the variational inequality and by a Monte Carlo simulation of the stopping rule.

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1. Introduction

An offshore gas platform whose combustion plant is covered by the EU ETS pays for each tonne of carbon dioxide it emits. Its operator, when weighing a carbon-capture-and-storage (CCS) retrofit on a mature platform, faces two coupled sources of uncertainty: the path of the allowance price, which sets the value of every tonne the retrofit avoids, and the date at which the platform stops producing, which sets how long the retrofit has to pay for itself. The second is partly a regulatory object: licence terms, decommissioning orders and field economics all move it, and from the operator's side it is a random date.

This paper develops a real-options model of that decision in the tradition of Dixit and Pindyck [1] and McDonald and Siegel [2]. The retrofit is irreversible and costs a lump sum; once installed it turns the avoided emissions into saved allowances at the market price, less the operating costs of the capture train. Two state variables drive the problem: the EU Allowance (EUA) price, modelled as a geometric Brownian motion, and the decommissioning deadline, modelled as the first arrival of a Poisson process with hazard $\lambda \geq 0$, so that the mean residual life is $1/\lambda$ and the median $\ln 2/\lambda$. The operator chooses when to retrofit, within a retrofit window $[0, T]$, $T \leq \infty$.

The contribution is fourfold.

Claim 1 (Theorem 1, Lemma 1). *Under (A1)–(A3) the value function is finite and continuous, and the decommissioning charge D , which falls due at the deadline whether or not the platform was retrofitted, shifts the value by the constant $-\lambda D/(r + \lambda)$ and leaves the retrofit decision unchanged.*

Claim 2 (Theorem 2, Theorem 4). *With an open-ended retrofit window the problem is time-homogeneous and has a closed form of McDonald–Siegel shape in which the deadline acts as an extra discount rate λ ; with a finite window the value is the unique viscosity solution of a variational inequality in which the deadline enters as $+\lambda\Phi = -\lambda D$, and its exercise boundary tends to the closed-form threshold as the remaining window grows.*

Claim 3 (Theorem 3). *The threshold rises with the deadline hazard exactly when the annuity on the retrofit capital, $K/(qc_v + F + \rho K)$, exceeds an option term that depends only on the price dynamics. With no lump-sum outlay the threshold always falls with the hazard; for a capital-heavy retrofit it rises.*

Claim 4 (Proposition 2, Proposition 3). *Fitted to EU ETS auction prices from January 2018 to 7 May 2024, with investment costs from a North Sea platform study and an onshore operating-cost proxy, the threshold is 180 €/tCO₂ at a mean residual life of 12 years and 239 €/tCO₂ at 6 years, against an auction price of 69.16 €/tCO₂ on 7 May 2024. Whether the price reaches the threshold before the deadline depends on a drift the data do not pin down: the probability is 0.79 if the 2018–2024 trend continued and 0.25 under the valuation drift.*



The direction in Claim 3 is the economic content of the paper. A shorter expected life raises the effective discount rate $r + \lambda$, which lowers the option value of waiting and on its own would bring the retrofit forward. It also shortens the period over which the capital K is recovered, which raises the price needed to justify the outlay. Theorem 3 says exactly which effect wins: for the capital-heavy retrofit of the case study, the second wins for all but very long residual lives.

Section 2 fixes the model and proves Claim 1. Section 3 derives the closed form and the direction result. Section 4 characterises the finite-window problem. Section 5 describes the data, the fit and the cost inputs. Section 6 states the numerical scheme and the results, each checked by a second method. Section 7 draws the implications and states the limits. Appendix A gives the algorithm; Appendix B proves Theorem 3.

2. Model

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ satisfying the usual conditions, where $\mathbb{Q}$ is the valuation measure. Allowances are traded and bankable, so the retrofit's revenue is spanned by a traded asset and is valued with the risk-adjusted drift $\mu = r - \delta$, where r is the risk-free rate and $\delta \geq 0$ the convenience yield, the rate-of-return shortfall of Dixit and Pindyck [1, Ch. 5]. The EUA price is a geometric Brownian motion under $\mathbb{Q}$,

$$dP_t = \mu P_t dt + \sigma P_t dW_t, \quad P_0 = p > 0, \quad (2.1)$$

with volatility $\sigma > 0$. Holding an allowance costs nothing to store, so $\delta = 0$, the cost-of-carry case, is the baseline; Section 6 varies δ . The historical drift under the real-world measure is estimated in Section 5 and is used only in Proposition 3. The decommissioning deadline Θ is the first jump of a Poisson process of intensity $\lambda \geq 0$, independent of W ; for $\lambda = 0$ there is no deadline. Write $\rho := r + \lambda$.

Before the retrofit the platform's cash flow is normalised to zero. A retrofit at time $\tau < \Theta$ costs $K \geq 0$ and from then on earns the abatement profit

$$\pi(p) := q(p - c_v) - F \quad (2.2)$$

until Θ , where $q > 0$ is the avoided emissions in tCO₂ per year, $c_v \geq 0$ the variable operating cost of the capture train per tonne and $F \geq 0$ its fixed operating cost per year. At Θ the platform is decommissioned at a charge $D \geq 0$, retrofitted or not; write $\Phi := -D$. The operator may retrofit at any stopping time in the window $[t, T]$ or never ($\tau = \infty$). The value at time t , on $\{\Theta > t\}$, is

$$V(t, p) := \sup_{\tau \in \mathcal{T}_{t, T} \cup \{\infty\}} \mathbb{E}_{t, p} \left[\mathbb{1}_{\{\tau < \Theta\}} \left(\int_{\tau}^{\Theta} e^{-r(s-t)} \pi(P_s) ds - e^{-r(\tau-t)} K \right) + e^{-r(\Theta-t)} \Phi \right]. \quad (2.3)$$

The immediate-retrofit value is



$$G(p) := \mathbb{E}_p \left[\int_0^\Theta e^{-rs} \pi(P_s) ds + e^{-r\Theta} \Phi \right] - K = \frac{qp}{\rho - \mu} - \frac{qc_v + F}{\rho} - \frac{\lambda D}{\rho} - K, \quad (2.4)$$

by Fubini and $\mathbb{E}[e^{-r\Theta}] = \lambda/\rho$, $\mathbb{E} \int_0^\Theta e^{-rs} P_s ds = p/(\rho - \mu)$.

The standing assumptions are:

Assumption 1 (price dynamics). (A1) $\sigma > 0$ and $\mu < \rho$: the expected discounted revenue of a retrofitted platform is finite.

Assumption 2 (deadline). (A2) $\lambda \geq 0$ and $T \in (0, \infty]$; when $\lambda = 0$, (A1) requires $\delta > 0$.

Assumption 3 (costs). (A3) $q > 0$, $c_v \geq 0$, $F \geq 0$, $K \geq 0$, $qc_v + F + \rho K > 0$ and $D \geq 0$.

Lemma 1 (the decommissioning charge cancels). Under (A1)–(A3), $V = \tilde{V} - \lambda D/\rho$, where

$$\tilde{V}(t, p) := \sup_{\tau \in \mathcal{T}_{t, T} \cup \{\infty\}} \mathbb{E}_{t, p} \left[e^{-\rho(\tau-t)} \tilde{G}(P_\tau) \mathbb{1}_{\{\tau < \infty\}} \right], \quad \tilde{G}(p) := \frac{qp}{\rho - \mu} - \frac{qc_v + F}{\rho} - K.$$

In particular the optimal retrofit rule does not depend on D .

Proof. The charge is paid at Θ in both branches, so it contributes $\mathbb{E}[e^{-r(\Theta-t)}] \Phi = -\lambda D/\rho$ whatever τ is. On $\{\tau < \Theta\}$ the strong Markov property of P and the memorylessness of Θ give the conditional value $e^{-r(\tau-t)}(\tilde{G}(P_\tau))$ of the remaining terms, and $\mathbb{Q}(\Theta > \tau \mid \tau) = e^{-\lambda(\tau-t)}$ by independence. Multiplying gives the stated objective. $\square$

Theorem 1 (well-posedness; Claim 1). Under (A1)–(A3), V is finite and continuous on $[0, T) \times (0, \infty)$ and

$$-\frac{\lambda D}{\rho} \leq V(t, p) \leq -\frac{\lambda D}{\rho} + \frac{qp}{\rho - \mu}.$$

Proof. By Lemma 1 it suffices to bound $\tilde{V}$. Never retrofitting gives $\tilde{V} \geq 0$. Since $\tilde{G}(p) \leq qp/(\rho - \mu)$ and $e^{-\rho s} P_s$ is a positive supermartingale under (A1), optional sampling gives $\mathbb{E}[e^{-\rho \tau} \tilde{G}(P_\tau)] \leq qp/(\rho - \mu)$ for every stopping time. Continuity follows from the Lipschitz dependence of P on p and the dominated-convergence argument of Pham [3] (§5.2). $\square$

The continuation region is $\mathcal{C} := \{(t, p) : \tilde{V}(t, p) > \tilde{G}(p)\}$ and the exercise region $\mathcal{S}$ its complement; $\tau^*(t, p) := \inf\{s \geq t : (s, P_s) \in \mathcal{S}\}$ is optimal by the standard Markovian argument.

3. Perpetual benchmark

Take an open-ended window, $T = \infty$. By Lemma 1 the problem is the time-homogeneous stopping problem $\tilde{V}_\infty(p) = \sup_\tau \mathbb{E}_p[e^{-\rho \tau} \tilde{G}(P_\tau)]$: the deadline acts as an extra discount rate. In the continuation region

$$(\mathcal{L} - \rho)\tilde{V}_\infty = 0, \quad \mathcal{L}u(p) = \mu p u'(p) + \frac{1}{2} \sigma^2 p^2 u''(p), \quad (3.1)$$



with general solution $A_1 p^{\beta_1} + A_2 p^{\beta_2}$, where $\beta_{1,2}$ are the roots of

$$\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \rho = 0. \quad (3.2)$$

The quadratic equals $\mu - \rho < 0$ at $\beta = 1$, so under (A1) $\beta_1 > 1 > 0 > \beta_2$, and $\tilde{V}_\infty(0^+) = 0$ forces $A_2 = 0$.

Theorem 2 (perpetual benchmark; Claim 2). Under (A1)–(A3) with $T = \infty$, the continuation region is $(0, p_\infty^*)$ and

$$\tilde{V}_\infty(p) = \begin{cases} (p/p_\infty^*)^{\beta_1} \tilde{G}(p_\infty^*), & 0 < p < p_\infty^*, \\ \tilde{G}(p), & p \geq p_\infty^*, \end{cases} \quad p_\infty^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{\rho - \mu}{\rho} \cdot \frac{qc_v + F + \rho K}{q}.$$

Proof. Value matching $A_1(p^*)^{\beta_1} = \tilde{G}(p^*)$ and smooth pasting $A_1\beta_1(p^*)^{\beta_1-1} = q/(\rho - \mu)$ give $p^* q/(\rho - \mu) \cdot (1 - 1/\beta_1) = (qc_v + F)/\rho + K$, which is the stated threshold. The candidate is C^1 , satisfies $(\mathcal{L} - \rho)\tilde{V}_\infty = 0$ below p^* and $(\mathcal{L} - \rho)\tilde{G} = -(\pi(p) - \rho K) \leq 0$ above it, because $p_\infty^* > (qc_v + F + \rho K)/q$: by the quadratic, $\frac{\beta_1}{\beta_1-1} \cdot \frac{\rho-\mu}{\rho} = \frac{a\beta_1+\mu}{a(\beta_1-1)+\mu} > 1$ with $a = \sigma^2/2$; and $\tilde{V}_\infty \geq \tilde{G}$ below p^* by convexity. The verification theorem [1, Ch. 5] [3, §5.2] completes the proof. $\square$

The threshold is the hurdle factor $\beta_1/(\beta_1 - 1) > 1$ times the zero-NPV price $p_{\text{NPV}} = (\rho - \mu)((qc_v + F)/\rho + K)/q$ at which $\tilde{G} = 0$. At $\lambda = 0$ (with $\delta > 0$) it is the McDonald–Siegel threshold [2] for discount rate r and drift $r - \delta$; for $\lambda > 0$ the deadline raises the discount rate of the post-retrofit stream from r to $r + \lambda$. Figure 1 plots p_∞^* and the hurdle factor against σ at the calibration of Section 5, for $\lambda = 1/12$ and $1/6$.

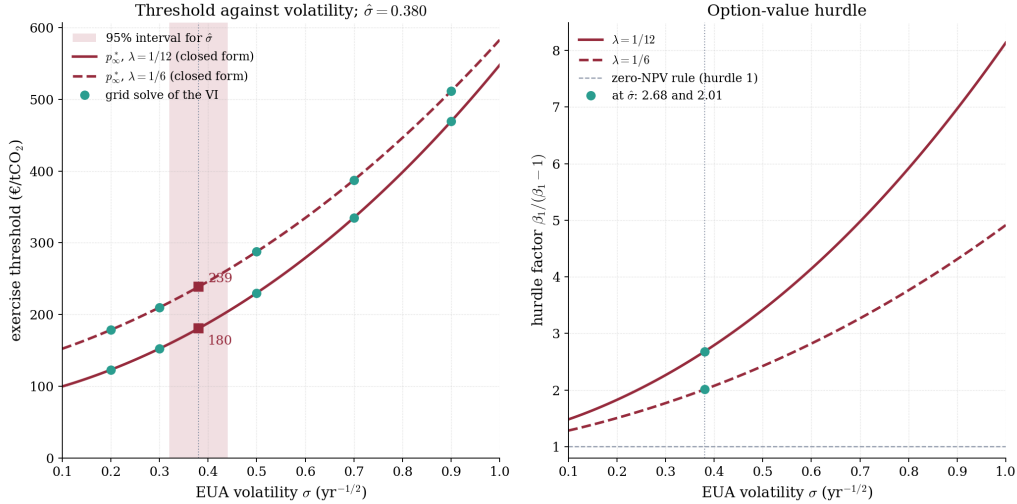


Figure 1: Perpetual exercise threshold $p_\infty^*(\sigma)$ (left) and hurdle factor $\beta_1/(\beta_1 - 1)$ (right) against EUA volatility, for mean residual lives of 12 years ($\lambda = 1/12$, solid) and 6 years ($\lambda = 1/6$, dashed), other parameters at the calibration of Table 1. Teal dots: grid solution of the variational inequality (Section 6), which agrees with Theorem 2 to within 2.2×10^{-5} relative. The shaded band is the 95 % interval of the fitted monthly volatility $\hat{\sigma} = 0.380$.



Theorem 2 also gives the direction of the threshold in the hazard in closed form. Let $a := \sigma^2/2$, $u := a(\beta_1 - 1) + \mu$ and $v := a\beta_1 + \mu$; then $\rho = \beta_1 u$ and $\rho - \mu = (\beta_1 - 1)v$, so $u, v > 0$.

Theorem 3 (direction of the threshold in the hazard; Claim 3). Under (A1)–(A3) with $T = \infty$, $\lambda \mapsto p_\infty^*(\lambda)$ is smooth and

$$\frac{\partial \ln p_\infty^*}{\partial \lambda} = \frac{K}{qc_v + F + \rho K} - \frac{a^2}{uv(u + v - \mu)}.$$

Both terms are positive. Hence (i) with $K = 0$ the threshold strictly decreases in λ ; (ii) it increases at λ if and only if $Kuv(u + v - \mu) > a^2(qc_v + F + \rho K)$.

Proof. Deferred to Appendix B. The first term is the derivative of the log of the capital annuity $qc_v + F + \rho K$; the second collects the hurdle factor and the factor $(\rho - \mu)/\rho$, which move in opposite directions and whose sum is negative. $\square$

The first term is the amortisation effect: a shorter expected life raises the annual charge ρK on the retrofit capital. The second is the option effect: a higher effective discount rate lowers the value of waiting. Theorem 3 decides the race exactly; Section 6 evaluates it at the calibration.

4. Finite-window HJB and the free boundary

Now let $T < \infty$: the retrofit option lapses at T , after which the platform runs unretrofitted until the deadline. By Lemma 1, $\tilde{V}(T, p) = \max\{\tilde{G}(p), 0\}$.

Definition 1 (viscosity solution). A function $u \in C([0, T] \times (0, \infty))$ is a viscosity sub-solution (resp. super-solution) of

$$\max\{\partial_t u + (\mathcal{L} - \rho)u, \tilde{G} - u\} = 0$$

on $[0, T) \times (0, \infty)$ if, at every (t_0, p_0) and every $\varphi \in C^{1,2}$ such that $u - \varphi$ has a local maximum (resp. minimum) at (t_0, p_0) , $\max\{\partial_t \varphi + (\mathcal{L} - \rho)\varphi, \tilde{G} - \varphi\}(t_0, p_0) \geq 0$ (resp. ≤ 0). A viscosity solution is both.

Theorem 4 (HJB characterisation; Claim 2). Under (A1)–(A3), $\tilde{V}$ is the unique viscosity solution of the variational inequality of Definition 1 with $\tilde{V}(T, p) = \max\{\tilde{G}(p), 0\}$ and $0 \leq \tilde{V} \leq qp/(\rho - \mu)$. Equivalently, by Lemma 1, V is the unique viscosity solution of

$$\max\{\partial_t V + (\mathcal{L} - r)V + \lambda(\Phi - V), G - V\} = 0, \quad V(T, p) = \max\{G(p), -\lambda D/\rho\},$$

in which the deadline contributes $+\lambda(\Phi - V)$, that is the source $+\lambda\Phi = -\lambda D$. The exercise region is $\{(t, p) : p \geq p^*(t)\}$ with $p^*(T) = p_{\text{NPV}}$, $p^*(t)$ non-increasing in t , and $p^*(t) \rightarrow p_\infty^*$ of Theorem 2 as $T - t \rightarrow \infty$.

Proof. Sketch. Over $[s, s + h]$ the deadline arrives with probability $\lambda h + o(h)$ and then pays Φ ; otherwise the problem continues. The dynamic-programming principle for the state



$(P, \mathbb{1}_{\{\Theta > s\}})$ [3, Ch. 5] therefore gives the generator term $\lambda(\Phi - V)$, and V is a viscosity sub- and super-solution; the shift of Lemma 1 removes the constant source. Uniqueness follows from the comparison principle for the obstacle problem with linear growth [4] (§III.3), as in the impulse-control setting with Poisson times [5] (§IV.2). Since the coefficients do not depend on t , $\tilde{V}$ depends on $T - t$ only and is non-decreasing in it (a longer window enlarges the set of stopping times), so the boundary is monotone; at T the operator retrofits if and only if $\tilde{G} \geq 0$, and as $T - t \rightarrow \infty$, $\tilde{V}$ increases to $\tilde{V}_\infty$ by monotone convergence. $\square$

5. Data and calibration

5.1 EU ETS allowance prices

The price data are the clearing prices of the EU ETS primary auctions held by the European Energy Exchange (EEX) for the common auction platform and for the German and Polish auctions, as compiled and published by the Infracore Foundation's open-data service [6] under the Creative Commons Attribution 4.0 licence. The licence was read before use; it requires attribution, which this paragraph, Figure 2 and the reference list give. Every auction up to and including 7 May 2024, the date of this paper, was retrieved (2,405 auctions from January 2012); nothing later is used. The file was checked against EEX's own yearly auction reports: each of the 2,405 prices matches an EEX auction of the same date to the cent. For a day with two auctions we take the volume-weighted price; the series is then sampled at the last auction of each month. The calibration window, from 2018-01-08 to 2024-05-07, holds 1,343 auction days and 77 month-ends; the price rose from 7.83 to 69.16 €/tCO₂, with a high of 97.51 on 2022-02-08.

5.2 The price model fitted

The paper's price model is a geometric Brownian motion. With month-end prices P_{t_k} and log returns $R_k = \ln(P_{t_k}/P_{t_{k-1}})$ over the actual intervals $\Delta_k = t_k - t_{k-1}$, the exact Gaussian maximum-likelihood estimates of $d \ln P_t = m dt + \sigma dW_t$ are $\hat{m} = \sum_k R_k / \sum_k \Delta_k$ and $\hat{\sigma}^2 = n^{-1} \sum_k (R_k - \hat{m} \Delta_k)^2 / \Delta_k$, with $\hat{\mu} = \hat{m} + \hat{\sigma}^2/2$. Over the 76 monthly returns,

$$\hat{\sigma} = 0.380 (\pm 0.031), \quad \hat{m} = 0.329 (\pm 0.152), \quad \hat{\mu} = 0.402 (\pm 0.152) \text{ per year}, \quad (5.1)$$

standard errors in brackets. A numerical maximisation of the same likelihood, with no closed form, returns the same estimates to 1×10^{-8} relative.

The fit was checked a second way, by its goodness of fit and by its stability across sampling intervals. The standardised monthly residuals do not reject normality: Jarque–Bera $p = 0.69$, Kolmogorov–Smirnov $p = 0.80$ (skewness 0.10, excess kurtosis 0.44). Auction-day returns are fat-tailed (Jarque–Bera $p < 10^{-100}$), and the volatility estimated from them is higher: Figure 2 (right) shows $\hat{\sigma}$ falling from 0.509 on auction-day returns to 0.466 on weekly and 0.380 on monthly returns, then levelling at 0.356 and 0.361 on two-month and quarterly returns. The first-order autocorrelation of auction-day returns is only -0.031,



so the excess short-horizon variance is spread over several weeks. A single- σ geometric Brownian motion therefore describes the prices at horizons of a month and longer, the horizons of a retrofit decision, and we use the monthly $\hat{\sigma} = 0.380$. Section 6 reports the threshold with the auction-day value 0.509 and with the monthly value 0.435 from the longer window 2013–2024.

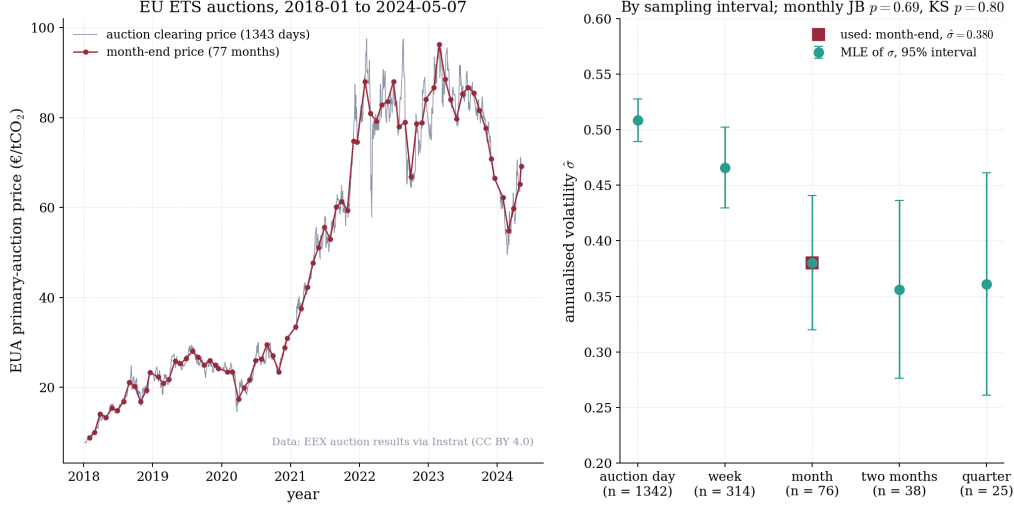


Figure 2: EU ETS primary-auction clearing prices, 2018-01-08 to 2024-05-07 (left; grey: every auction day, red: month-end prices used in the fit), and the maximum-likelihood volatility by sampling interval with 95 % intervals (right). The monthly estimate $\hat{\sigma} = 0.380$ is the one used; monthly residuals pass Jarque–Bera and Kolmogorov–Smirnov tests. Data: EEX auction results, compiled and published by the Instrat Foundation (energy.instrat.pl), CC BY 4.0; aggregation by us.

The drift is a different matter. $\hat{\mu} = 0.40$ per year, with a 95 % interval of 0.10 to 0.70: the price rose nearly ninefold in six years. Used as the drift of the model, with the discount rate of Section 5.3, it violates (A1): $r + \lambda - \hat{\mu} = -0.29$ at $\lambda = 1/12$ and -0.21 at $\lambda = 1/6$, so the expected discounted revenue of a retrofit is infinite and the problem has no solution unless $\lambda > 0.377$ (a mean residual life under 2.7 years); even the lower end of the interval requires $\lambda > 0.078$. A six-year trend in a price set by policy is not an estimate of the growth an operator can bank on, and it is not the drift a valuation needs: as Section 2 explains, a spanned real option is valued with the risk-adjusted drift $r - \delta$. We therefore solve the model under $\mu = r - \delta$ and use $\hat{m}$ only for the real-world question of Proposition 3.

5.3 Costs, emissions and the discount rate

We found no public source for the retrofit cost of a typical mature platform, and use the one peer-reviewed offshore case study we found that reports such costs: Nguyen et al. [7] model a North Sea oil and gas platform with a power demand of about 19 MW and local emissions of about 450 tCO₂ per day, more than 90 % of them from the gas turbines, and assess a steam bottoming cycle combined with monoethanolamine post-combustion



capture on the turbine exhausts. The capture rate is 87–93 % and the platform’s total emissions fall by up to 70 %; the investment is 14–18 M\$ for the steam cycle and the capture plant, excluding site-specific costs which the study puts at 10–96 M\$; the economic model assumes 95 % operation. From these figures:

- Avoided emissions $q = 0.70 \times 450 \times 365 \times 0.95 = 109\,226$ tCO₂ per year. A second reading from the same study, its CO₂-tax saving of 0.02 M\$ per day at 67 \$/t, gives 103 507 tCO₂ per year, 5 % less (Section 6.4).
- Retrofit cost K : the sum of the midpoints, $16 + 53 = 69$ M\$ (range 24–114 M\$), converted at the 2014 ECB reference rate of 1.3285 \$/€ and brought to 2023 prices with the euro-area HICP (ratio of annual averages 1.2339) [8]: $K = 64.09$ M€ (range 22.29–105.88 M€).

The study gives no operating-cost breakdown. For operating costs we use an onshore proxy, the US DOE National Energy Technology Laboratory study *Cost of Capturing CO₂ from Industrial Sources* by Hughes and Zoelle [9], in its iron and steel retrofit case with 90 % capture, the study’s retrofit case with amine post-combustion capture on a low-purity source. Its fixed operating cost, 27.13 M\$ per year against a total plant cost of 878.8 M\$, is 3.09 % of capital per year, which we apply to K : $F = 1.98$ M€ per year. Its variable operating cost excluding fuel, 16.41 M\$ per year over 2.86 MtCO₂ captured, is 5.74 \$/t in December-2018 dollars, or $c_v = 5.78$ €/t after conversion at the 2018 ECB rate (1.1810) and the HICP ratio 2018–2023 (1.1892). The proxy is adjusted in three ways and no more: currency and price level; fixed costs scaled to the offshore capital rather than to the onshore plant; and no fuel cost, because in the offshore design the solvent is regenerated with exhaust heat recovered by the steam cycle, whose cost is already in K , whereas the onshore case buys natural gas for a boiler. No offshore premium is added to operating costs; Section 6.4 doubles c_v and F to bound that omission. Transport and storage operating costs are not included.

The risk-free rate is the euro-area AAA 10-year spot rate of the ECB yield curve on 7 May 2024, $r = 2.477$ % [8]. The ECB permits reuse with the ECB cited as source and modifications stated; the averaging and ratios above are ours. The decommissioning charge needs no value: by Lemma 1 it does not move the decision. The hazard λ is the policy lever, not a calibrated quantity; we report $\lambda = 1/12$ and $1/6$ per year, mean residual lives of 12 and 6 years (medians 8.3 and 4.2 years), and the whole curve in Figure 4.



Table 1: The calibration (money in 2023 euros). Dollar figures are converted at the ECB annual reference rate of their price year and brought to 2023 with the euro-area HICP; the operating costs are an onshore proxy, adjusted as described in Section 5.3.

Parameter	Value	Source
σ	$0.380 \text{ yr}^{-1/2}$	EU ETS auction prices, monthly MLE (Section 5.2) [6]
r	2.477 %	ECB euro-area AAA 10-year spot rate, 7 May 2024 [8]
$\delta, \mu = r - \delta$	0, 0.0248	cost-of-carry valuation drift (Section 2)
q	$109\,226 \text{ tCO}_2/\text{yr}$	avoided emissions, North Sea platform study [7]
K	64.09 M€ (22.29–105.88)	investment incl. site-specific costs [7], 2023 prices [8]
F	1.98 M€/yr	onshore proxy: 3.09 % of K per year [9]
c_v	5.78 €/tCO ₂	onshore proxy: variable O&M without fuel [9]
λ	$1/12, 1/6 \text{ yr}^{-1}$	scenarios: mean residual life 12 and 6 years
D	not needed	cancels from the decision (Lemma 1)
p_0	69.16 €/tCO ₂	last auction price used, 7 May 2024 [6]

6. Numerical scheme and results

6.1 Discretisation

The finite-window problem of Theorem 4 is solved in the log price $x = \ln p$ on a uniform grid $x_0 < \dots < x_N$ spanning $[10^{-4}, 5000]$ €/tCO₂, wide enough that the boundary is never within a factor 5 of the upper edge; the solver refuses to report a boundary that is. In x the generator is $\frac{1}{2}\sigma^2\partial_{xx} + (\mu - \frac{1}{2}\sigma^2)\partial_x$; central differences give a tridiagonal matrix with non-positive off-diagonal entries, hence a monotone scheme, whenever $|\mu - \sigma^2/2| \Delta x \leq \sigma^2$, which holds on every grid used. Time is stepped backward from $\tilde{V}(T) = \max\{\tilde{G}, 0\}$ by implicit Euler, and each step is the linear complementarity problem $(I + \Delta t(\rho - \mathcal{L}_h))V^k \geq V^{k+1}$, $V^k \geq \tilde{G}$, with equality in one of the two at every node, solved exactly by the Brennan–Schwartz algorithm [10]: a forward sweep from the low-price end and a projected back-substitution from the high-price end, where exercise occurs. The open-ended problem of Theorem 2 is the stationary version, solved by one such sweep. Boundary rows are $V = 0$ at x_0 and $V = \tilde{G}$ at x_N . The free boundary is the first exercised node, refined by smooth fit: $\sqrt{V - \tilde{G}}$ is linear in x just below the boundary, and its zero is found from the continuation nodes three to eight below the first exercised one. Algorithm 1 in Appendix



A gives the steps.

Proposition 1 (convergence; Claim 2). Under (A1)–(A3) and the grid condition above, the scheme converges locally uniformly to the viscosity solution $\tilde{V}$ of Theorem 4 as $\Delta t, \Delta x \rightarrow 0$, and its stationary version to $\tilde{V}_\infty$ of Theorem 2.

Proof. The scheme is monotone (an M -matrix with the projection), consistent (central differences and implicit Euler are consistent with $\partial_t + \mathcal{L} - \rho$) and stable (the M -matrix gives a discrete maximum principle, and the projection is non-expansive in the sup norm). The Barles–Souganidis theorem [11], with the comparison principle used in Theorem 4, gives convergence. $\square$

At the baseline the stationary solve converges to the threshold of Theorem 2 as the grid is refined, as Proposition 1 requires: the relative error is 7.0×10^{-5} , 5.9×10^{-5} , 1.5×10^{-5} , 3.2×10^{-6} and 1.6×10^{-7} at $N = 4000, 8000, 16000, 32000$ and 64000 nodes. The same solve reproduces the McDonald–Siegel case of Theorem 2 ($\lambda = 0$, $\delta = 0.02$, threshold 174.79 €/tCO₂) to 3.4×10^{-6} . The grid values respect the bounds of Theorem 1 at every node.

6.2 The thresholds at the calibration

Proposition 2 (calibrated thresholds; Claim 4). At the calibration of Table 1, with $\delta = 0$: - $\lambda = 1/12$ (mean residual life 12 years): $p_\infty^* = 180.5$ €/tCO₂, hurdle factor 2.68 over the zero-NPV price 67.3 €/tCO₂; - $\lambda = 1/6$ (mean residual life 6 years): $p_\infty^* = 238.5$ €/tCO₂, hurdle factor 2.01 over 118.6 €/tCO₂.

At the auction price of 69.16 €/tCO₂ on 7 May 2024 retrofit is not optimal under either hazard, although at $\lambda = 1/12$ the price exceeds the zero-NPV price. Doubling the hazard raises the threshold by 32 %.

Proof. Theorem 2 evaluated at the parameters of Table 1; the checks are in Table 2. $\square$

Table 2 checks each number of Proposition 2 by the grid solve of the variational inequality (Section 6.1), by the time-marching solver run over a 200-year window, and by a Monte Carlo simulation of the stopping rule. The simulation draws 100 000 antithetic price paths under the valuation measure in weekly steps over 100 years, detects the first passage above each candidate threshold with the exact Brownian-bridge crossing probability within each step, and weights each passage by $e^{-(r+\lambda)\tau}$, which carries both discounting and the probability that the deadline has not yet arrived. It uses neither Theorem 2 nor the grid.



Table 2: Proposition 2 and Theorem 2 by three further methods: the stationary grid solve of the variational inequality ($N = 64000$), the time-marching solver of Theorem 4 over a 200-year window ($N = M = 4000$), and the Monte Carlo of the stopping rule (Proposition 1 governs the first two). Relative gaps in brackets; the Monte Carlo value is $\pm$ one standard error.

Quantity	Theorem 2	Grid VI, stationary	Window solver, $T = 200$	Monte Carlo
$p_{\infty}^*, \lambda = 1/12$ (€/tCO ₂)	180.465	180.465 (1.6×10^{-7})	180.452 (7.0×10^{-5})	best of five rules
$p_{\infty}^*, \lambda = 1/6$ (€/tCO ₂)	238.531	238.530 (2.4×10^{-6})	238.566 (1.5×10^{-4})	best of five rules
$\tilde{V}_{\infty}(p_0), \lambda = 1/12$ (M€)	32.1263	32.1263 (1.6×10^{-9})	—	32.072 ± 0.103 (1.7×10^{-3} , 0.5 s.e.)
$\tilde{V}_{\infty}(p_0), \lambda = 1/6$ (M€)	6.7010	6.7010 (6.5×10^{-9})	—	6.695 ± 0.038 (9.6×10^{-4} , 0.2 s.e.)

Five threshold rules were simulated on common paths: 0.8, 0.9, 1, 1.1 and 1.25 times p_{∞}^* . The rule at p_{∞}^* has the highest value in both cases. At $\lambda = 1/12$ it beats the others by 0.78, 0.18, 0.20 and 0.74 M€ (paired standard errors 0.08, 0.06, 0.06, 0.09), and at $\lambda = 1/6$ by 0.42, 0.082, 0.032 and 0.28 M€ (standard errors 0.024, 0.017, 0.017, 0.027). The optimum is flat, as a smooth-pasting optimum must be: a threshold 10 % off costs well under one per cent of the option's value.

Figure 3 shows the boundary of Theorem 4 for a retrofit window of $T = 25$ years. Near the end of the window the boundary falls to the zero-NPV price, 67.3 €/tCO₂ at $\lambda = 1/12$; with a longer remaining window it rises to the perpetual threshold of Theorem 2, within 1 % of it once more than 15.2 years remain at $\lambda = 1/12$ and 7.5 years at $\lambda = 1/6$ (Figure 3, right). The boundaries are ordered in λ at every remaining window, as Theorem 3 predicts for the open-ended limit. At the 25-year start the finite-window threshold is 0.17 % below the closed form at $\lambda = 1/12$; a 10-year window lowers it by 2.7 %.

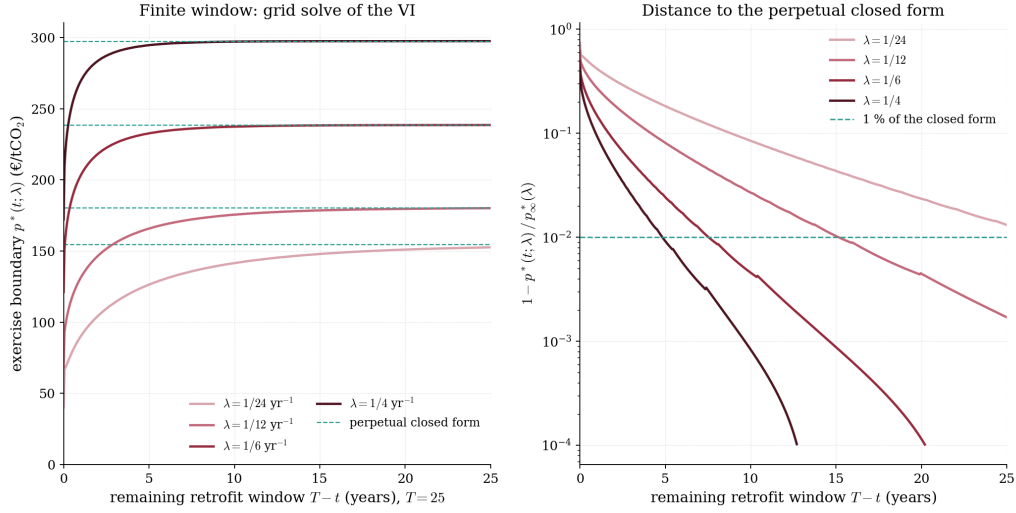


Figure 3: Exercise boundary $p^*(t; \lambda)$ of Theorem 4 for a 25-year retrofit window and four hazards, from the grid solution of the variational inequality (left), against the perpetual thresholds of Theorem 2 (teal, dashed); and the relative distance to the perpetual threshold against the remaining window (right, log scale). The boundary meets the zero-NPV price at the end of the window and the closed form as the window lengthens.

6.3 The deadline hazard

Figure 4 (left) plots $p_\infty^*(\lambda)$ for $\lambda \in [0.01, 0.5]$; the grid solve agrees with Theorem 2 to 2.0×10^{-5} at eight hazards. The threshold rises with the hazard over almost all of this range. The derivative formula of Theorem 3 agrees with a central finite difference of Theorem 2 to 2×10^{-11} ; at $\lambda = 1/12$ the capital term is 6.72 and the option term 3.01, so $\partial \ln p_\infty^* / \partial \lambda = 3.71$; at $\lambda = 1/6$ the terms are 4.31 and 1.34. The sign changes once, at $\lambda_c = 0.0131$: the threshold falls with the hazard only for mean residual lives above 76 years, and rises for every shorter one. Across the capital-cost range the turning point is at mean lives of 44 years (low K) and 92 years (high K). With no lump-sum outlay ($K = 0$) the threshold falls monotonically, from 49.4 to 41.8 €/tCO₂ between $\lambda = 1/12$ and $1/6$, as part (i) of Theorem 3 requires (Figure 4, right).

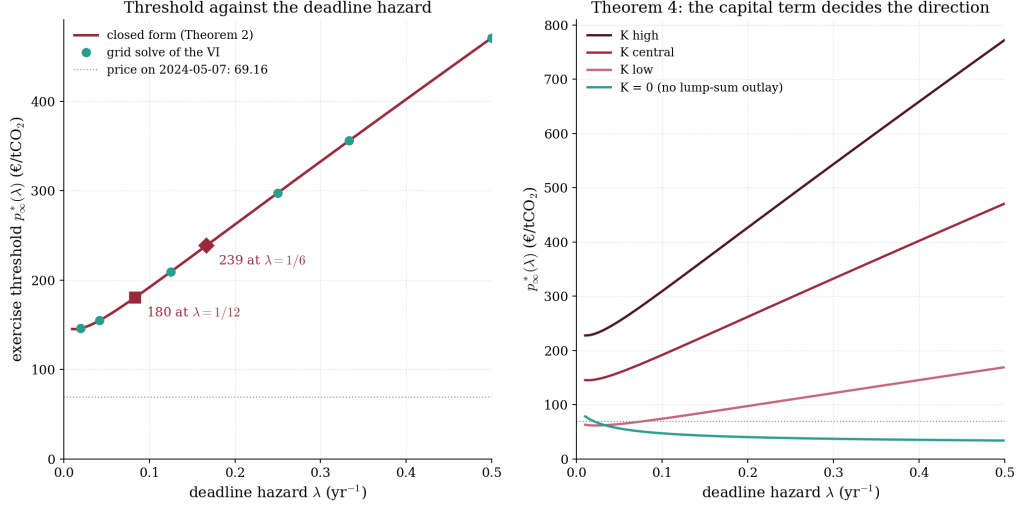


Figure 4: Perpetual threshold against the deadline hazard (left): Theorem 2 (line), the grid solve of the variational inequality (teal dots) and the two hazards of Proposition 2; the dotted line is the auction price on 7 May 2024. Right: the same curve for the low, central and high retrofit cost of Table 1 and for $K = 0$. The capital annuity of Theorem 3 makes the threshold rise with the hazard; without it the threshold falls.

6.4 Sensitivity

Table 3 recomputes Proposition 2 under the alternative inputs named in Section 5. The capital cost dominates: the low end of the study's range puts the 12-year threshold at 70.6 €/tCO₂, close to the May 2024 price, and the high end at 290.4. The volatility comes next. The direction of Theorem 3 holds in every row.



Table 3: Sensitivity of Proposition 2 (thresholds in €/tCO₂, Theorem 2). Each row changes the named inputs and keeps the rest at Table 1; the last column is the rise from $\lambda = 1/12$ to $1/6$, positive in every row as Theorem 3 predicts for capital-heavy retrofits.

Inputs	σ	$p_{\infty}^*, \lambda = 1/12$	$p_{\infty}^*, \lambda = 1/6$	rise
baseline	0.380	180.5	238.5	32 %
σ from monthly returns 2013–2024	0.435	201.8	259.9	29 %
σ from auction-day returns	0.509	234.0	291.4	25 %
K, F at the low end	0.380	70.6	89.6	27 %
K, F at the high end	0.380	290.4	387.5	33 %
$\delta = 0.01$	0.380	185.7	242.9	31 %
$\delta = 0.02$	0.380	191.0	247.3	29 %
$r = 5$ %	0.380	186.4	246.4	32 %
c_v doubled	0.380	192.4	248.6	29 %
F doubled	0.380	217.9	270.2	24 %
q from the study's tax saving	0.380	189.8	251.2	32 %

6.5 Will the price reach the threshold before the deadline?

The thresholds are valuation quantities; whether the price actually reaches them is a question about the real-world drift.

Proposition 3 (reaching the threshold; Claim 4). Let the real-world log price have drift m and volatility σ , and let $p_0 < p^*$. The probability that the price reaches p^* before the deadline is

$$\mathbb{P}(\tau_{p^*} < \Theta) = \mathbb{E}[e^{-\lambda\tau_{p^*}}] = \left(\frac{p_0}{p^*}\right)^{\gamma}, \quad \gamma = \frac{\sqrt{m^2 + 2\sigma^2\lambda} - m}{\sigma^2}.$$

Proof. Θ is independent of the price, so $\mathbb{P}(\Theta > \tau \mid \tau) = e^{-\lambda\tau}$; the Laplace transform of the first passage of a Brownian motion with drift to the level $\ln(p^*/p_0) > 0$ is the stated power [1, Ch. 3]. $\square$

From $p_0 = 69.16$ €/tCO₂ on 7 May 2024, with $\hat{\sigma}$ and the thresholds of Proposition 2, Table 4 gives the probability under four drifts. The Monte Carlo of Section 6.2, run under the historical drift, gives 0.7939 ± 0.0002 at $\lambda = 1/12$ against 0.7942 from Proposition 3.



Table 4: Probability that the EUA price reaches the threshold of Proposition 2 before the deadline (Proposition 3), from 69.16 €/tCO₂ on 7 May 2024 with $\hat{\sigma} = 0.380$.

Real-world drift of $\ln P$	m (yr ⁻¹)	$\lambda = 1/12$	$\lambda = 1/6$
2018–2024 trend, $\hat{m}$	0.329	0.79	0.57
lower end of its 95 % interval	0.032	0.43	0.20
no trend, $m = 0$	0.000	0.36	0.15
valuation drift, $m = r - \sigma^2/2$	-0.048	0.25	0.10

The answer turns on the drift. If the 2018–2024 trend continued, the price would very likely reach the threshold within the platform’s life; with no trend, or under the valuation drift, it more likely would not; and the data cannot tell these apart, since the 95 % interval of the trend runs from 0.03 to 0.63 per year in log terms.

7. Discussion

Four implications follow from the model at the stated inputs. They are properties of the model, not recommendations.

First, at a 12-year mean residual life the zero-NPV price of the retrofit, 67.3 €/tCO₂, was already below the May 2024 auction price, yet the optimal threshold was 180 €/tCO₂. The gap is the value of waiting: with $\hat{\sigma} = 0.38$ the hurdle factor is 2.68. A net-present-value test and an optimal-timing test give opposite answers here.

Second, a shorter expected life raises the threshold for a capital-heavy retrofit, as Theorem 3 makes exact: the annuity on K grows faster than the option premium shrinks. A higher deadline hazard alone therefore makes the retrofit less likely in the model, not more. The decommissioning charge plays no part in this (Lemma 1); what matters is how long the capital has to earn.

Third, the model’s two capital-side inputs act directly on the threshold. A lower lump-sum cost lowers p_∞^* through the capital term of Theorem 2, and at the low end of the cost range the 12-year threshold is 71 €/tCO₂; a convenience yield or a higher discount rate raises it only slightly (Table 3). In the model, anything that lowers K or lengthens the expected life lowers the threshold; shortening the life alone raises it.

Fourth, the decision is far more sensitive to cost than to the price model’s drift, which the valuation does not use; the drift matters for the separate question of when the threshold is reached (Proposition 3), and there the data are not decisive.

The limits are as clear. The geometric Brownian motion is fitted to a period in which the price rose nearly ninefold; regime switches and jumps are absent, and a single σ describes monthly but not daily returns. The costs rest on one platform case study in 2014 prices and an onshore operating-cost proxy without an offshore premium. The case-



study platform also paid a national CO₂ tax, 67 \$/t at the time of the study [7], which the model, counting only the EUA price, leaves out. The valuation drift $r - \delta$ with $\delta = 0$ is a modelling choice, though Table 3 shows the threshold moves little across $\delta \in [0, 0.02]$. The framework extends with little change to abandonment, to repurposing, and to staged capture with several obstacles; related explicit investment rules with time-to-build are in Aïd et al. [12].

A. Algorithm: the variational inequality on a log-price grid

The pseudo-code below is the solver of Section 6.1. It is implemented in `python/ccs_model.py` (`solve_window`, `solve_perpetual`); `python/checks.py` computes every number of the paper from it, from Theorem 2 and from the Monte Carlo of Section 6.2.

Algorithm 1

Implicit time stepping with an exact Brennan-Schwartz LCP solve per step.

```

1  Inputs:
2    sigma, r, delta, lambda      (mu = r - delta, rho = r + lambda)
3    q, c_v, F, K                  (cash-flow parameters; D not needed,
   Lemma 1)
4    T, N, M                      (retrofit window; space and time grid
   sizes)
5    p_min, p_max                 (price-grid ends, p_min > 0)
6
7  Grid:
8    x_i = log(p_min) + i*dx, dx = (log p_max - log p_min)/N, i = 0..N
9    p_i = exp(x_i), dt = T/M
10   G_i = q*p_i/(rho - mu) - (q*c_v + F)/rho - K      (shifted obstacle)
11   m = mu - sigma^2/2, s = sigma^2/(2*dx^2)
12   require |m|*dx <= sigma^2      (monotone scheme)
13
14  Matrix A = I + dt*(rho - L_h), rows i = 1..N-1:
15    a_i = -dt*(s - m/(2*dx))
16    b_i = 1 + dt*(2*s + rho)
17    c_i = -dt*(s + m/(2*dx))
18    row 0: V_0 = 0      row N: V_N = G_N
19
20  Initialise: V_i <- max(G_i, 0)      (t = T)
21
22  For k = M-1, ..., 0:      (backwards in time)
23    d <- V, d_0 <- 0, d_N <- G_N
24    Forward sweep, i = 0..N: w_i = 1/(b_i - a_i*c'_{i-1})
25                                c'_i = c_i*w_i
26                                d'_i = (d_i - a_i*d'_{i-1})*w_i
27    Back substitution with projection, top to bottom:
28                                V_N <- max(d'_N, G_N)
29                                V_i <- max(d'_i - c'_i*V_{i+1}, G_i)
30  Boundary p*(t_k): first node with V_i = G_i, refined by a straight-line

```



```

31     fit of sqrt(V_j - G_j) against x_j, j = i-8..i-3, extrapolated to 0
32
33 Open-ended window (T = infinity): one solve of the same LCP with
34     a_i = -(s - m/(2*dx)), b_i = 2*s + rho, c_i = -(s + m/(2*dx)), d = 0
35
36 Return: V(0, p_i) and p*(t_k), k = 0..M

```

A step costs $O(N)$ operations. The stationary solve with $N = 64\,000$ takes a few hundredths of a second; the 200-year window of Table 2, with $N = 4\,000$ and $M = 4\,000$, a few seconds.

B. Proof of Theorem 3

Write $Q(\beta, \rho) = a\beta^2 + (\mu - a)\beta - \rho$ with $a = \sigma^2/2$, so that $\beta_1(\rho)$ is its larger root and, by the implicit-function theorem,

$$\frac{\partial \beta_1}{\partial \rho} = \frac{1}{\partial_\beta Q(\beta_1, \rho)} = \frac{1}{a(2\beta_1 - 1) + \mu} = \frac{1}{u + v - \mu}, \quad (\text{B.1})$$

where $u = a(\beta_1 - 1) + \mu$ and $v = a\beta_1 + \mu$. From $Q(\beta_1, \rho) = 0$, $\rho = \beta_1 u$ and $\rho - \mu = (\beta_1 - 1)v$; since $\rho > 0$, $\rho - \mu > 0$ and $\beta_1 > 1$, both u and v are positive, and $u + v - \mu = \partial_\beta Q > 0$ at the larger root. Take logarithms in Theorem 2,

$$\ln p_\infty^* = \ln \beta_1 - \ln(\beta_1 - 1) + \ln(\rho - \mu) - \ln \rho + \ln(qc_v + F + \rho K) - \ln q, \quad (\text{B.2})$$

and differentiate in λ , which moves ρ one for one:

$$\frac{\partial \ln p_\infty^*}{\partial \lambda} = -\frac{1}{\beta_1(\beta_1 - 1)} \cdot \frac{1}{u + v - \mu} + \frac{\mu}{\rho(\rho - \mu)} + \frac{K}{qc_v + F + \rho K}. \quad (\text{B.3})$$

With $\rho(\rho - \mu) = \beta_1(\beta_1 - 1)uv$ the first two terms combine to

$$\frac{1}{\beta_1(\beta_1 - 1)} \left[\frac{\mu}{uv} - \frac{1}{u + v - \mu} \right] = \frac{\mu(u + v - \mu) - uv}{\beta_1(\beta_1 - 1)uv(u + v - \mu)} = -\frac{(u - \mu)(v - \mu)}{\beta_1(\beta_1 - 1)uv(u + v - \mu)}, \quad (\text{B.4})$$

and $(u - \mu)(v - \mu) = a^2\beta_1(\beta_1 - 1)$, which gives the stated formula. The option term $a^2/(uv(u + v - \mu))$ is positive, so with $K = 0$ the derivative is negative, which is (i); (ii) is the sign condition rearranged. Smoothness follows from that of β_1 in ρ . $\square$

The proof shows where the earlier intuition about two competing effects is exact: the hurdle factor falls and the factor $(\rho - \mu)/\rho$ rises with the hazard, their net effect is always the negative option term, and only the capital annuity can reverse the sign.



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Zamrik Quant Research

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About the Desk

Optimal Control & Decision Desk

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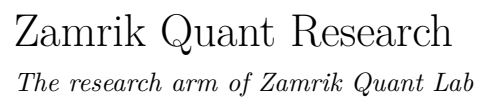
What to do under uncertainty: optimal control and HJB, mean-field games, agent-based models, irreversible decisions, betting, lotteries and insurance.

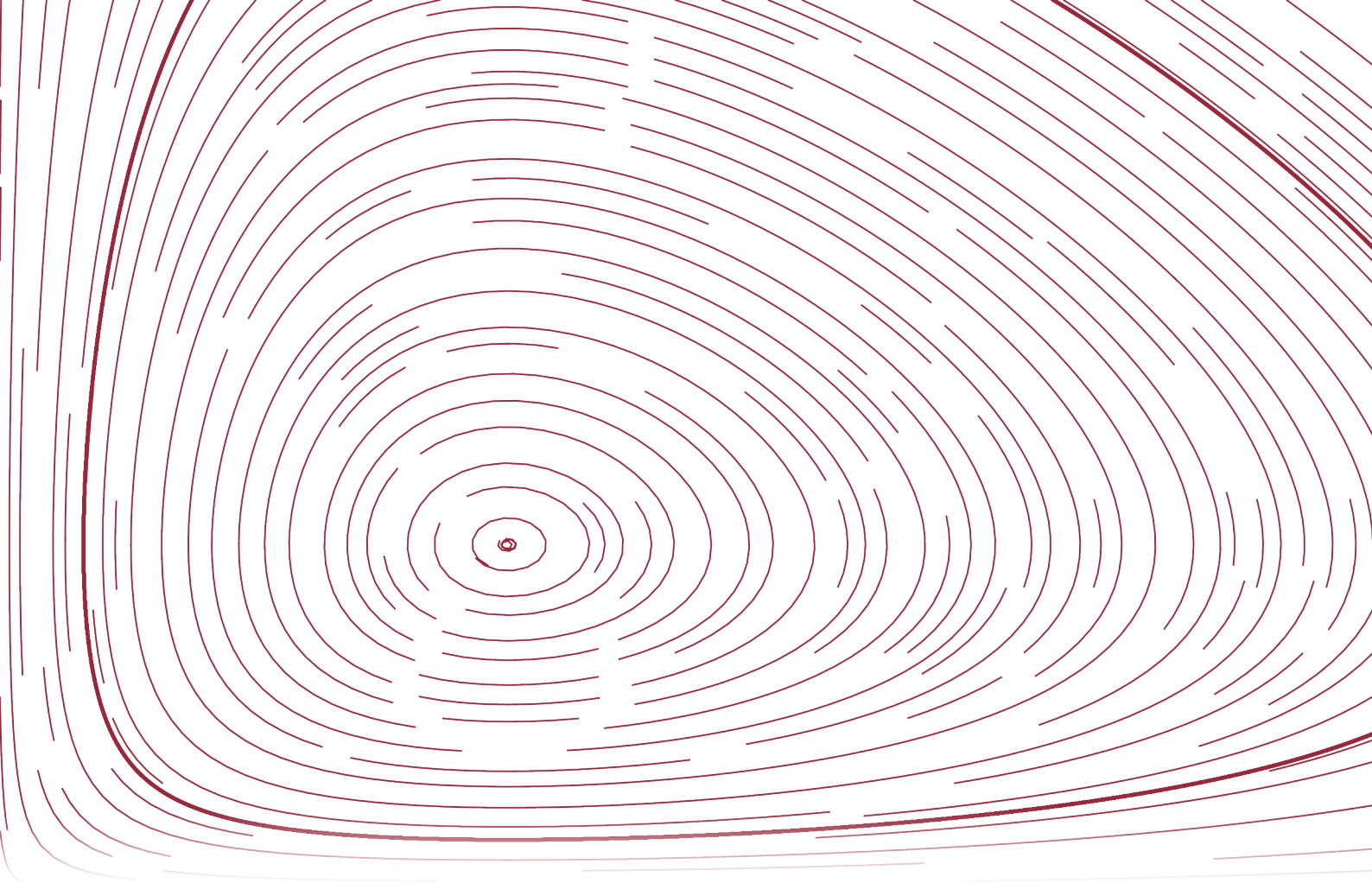
Most of this desk's questions have the same shape: someone must act now, without knowing what comes next, and live with the result. The Optimal Control & Decision desk turns such questions into Hamilton–Jacobi–Bellman equations, optimal stopping and singular control problems, and solves them.

When several parties optimise at once, the problem becomes a game: Dynkin games and wars of attrition between two or three players, and mean field games, in which a large population and its representative member shape each other through coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations. Where the equations give out, agent-based models take over, and the desk says plainly which tool fits which case.

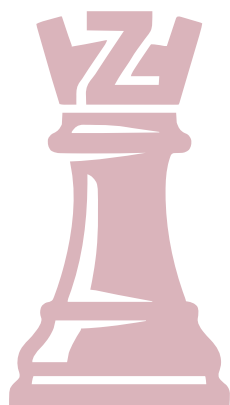
The applications are concrete: when to build an irreversible plant, how a company should pay dividends, how to run a home battery against a time-of-use tariff, when a wind turbine should turn into the wind, how a central bank should answer stagflation, and how much to bet when an edge is small. Its notebooks and essays carry the same reasoning to retirement, market timing, lotteries, insurance and prize competitions, each worked through to a decision a reader could actually make.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Phase portrait of the Lotka–Volterra equations: $\dot{x} = \alpha x - \beta xy$, $\dot{y} = -\gamma y + \delta xy$, $\alpha = 1.11$, $\beta = 0.91$, $\gamma = 1.17$, $\delta = 0.85$



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