



Littlewood's Rule for a Home Battery

Optimal Storage on Ontario's Ultra-Low Overnight Plan, When to Install One, and Why It Does Not Pay Without Solar

Working Paper · Stochastic Optimal Control

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Abstract. Ontario's Ultra-Low Overnight electricity plan charges 3.9 cents a kilowatt-hour overnight and 39.1 cents between 4 and 9 PM on weekdays, and home batteries are sold on that tenfold spread. This paper asks how such a battery should be run when the price is set by the clock, the household's own evening usage is uncertain and stored energy cannot be sold back, and when, if ever, a household should buy one. Running the battery is a revenue-management problem: the morning and evening windows are two fare classes for the same stored energy. The optimal policy fills the battery every night and during the morning mid-peak discharges only down to a reserve set by Littlewood's rule, extended for a second mid-peak window after 9 PM and for the cost of refilling overnight. On 462 days of one house's hourly usage, reconciled to its twelve bills, the reserve for a 13.5 kWh battery is 11.9 kWh in summer, 7.4 in spring and autumn and 11.4 in winter. It earns \$757 a year off the bill, 94% of what a battery that knew each day's usage could earn and 92% when each month's reserve is set without that month, against \$567 for discharging whenever the house draws. Buying the battery is McDonald and Siegel's problem of investing when both the project's value and its cost are random. Its savings over a ten-year warranted life are worth about \$5,500 against an installed cost of \$19,000 to \$24,000, and on the regulator's prices since 2006 and the market's storage prices since 2021 the HJB variational inequality puts the installation boundary at an installed cost of \$430 to \$2,700, depending on the drifts one believes, decades away on present trends.

1. Introduction

Since May 2023 households in Ontario on the Regulated Price Plan have been able to choose an Ultra-Low Overnight (ULO) plan. It sells electricity at 3.9 cents a kilowatt-hour from 11 PM to 7 AM every day and at 39.1 cents from 4 to 9 PM on weekdays, with a mid-peak price of 15.7 cents for the rest of a weekday and 9.8 cents for weekend and holiday days. A battery charged overnight and discharged in the evening buys at 3.9 cents and avoids 39.1: on paper a margin of 35 cents on every kilowatt-hour it moves, and home batteries are marketed on exactly that arithmetic.

The household that owns one faces two decisions of very different kinds. The first is daily and operational: how much of the stored energy to use in the morning mid-peak, when the evening peak, which is worth two and a half times as much, has not happened yet. The second is made once: whether to buy a battery at all, and if not now, at what price. This paper solves both, and evaluates both on fifteen months of one house's hourly smart-meter

readings, reconciled to its twelve bills.

Two features shape the problem. The price is not random: it is fixed by the clock and revised once a year. What is random is the household's own usage, which decides how much of the expensive evening the battery can actually displace. And stored energy cannot be sold: a battery installed without panels only displaces energy the house would otherwise buy, so a kilowatt-hour that the house does not need in the evening is worth no more than the overnight price at which it would have been bought back.

That makes the daily decision a problem of revenue management, not of arbitrage. An airline that sells the same seat at a discount to early bookers and at full fare to late ones must decide how many seats to protect for the late, uncertain, high-fare demand. Littlewood's rule [1] answers it for two fare classes: keep protecting seats for the high fare while the probability that its demand exceeds them is at least the ratio of the low fare to the high one, a critical fractile of the kind Arrow, Harris and Marschak [15] derived for inventory. Belobaba [2] extended the rule heuristically to many fare classes, Brumelle and McGill [3] proved that nested protection levels are optimal for any number of classes, Lee and Hersh [13] let them depend on the time left before departure, and Gallego and van Ryzin [14] priced the same perishable inventory dynamically; the theory is assembled in Talluri and van Ryzin [4].

A battery on the ULO plan is the same problem with the clock in place of the booking curve. The morning mid-peak, which comes first, buys stored energy at the low fare; the evening peak, which comes later and whose size is not known in the morning, buys it at the high fare. Three things are new. There is a second low-fare window after the peak, from 9 to 11 PM, so a kilowatt-hour protected for the evening and not needed by it is not wasted; the fares arrive low, high and low again, a sequence that is not monotone in price, the case Robinson [12] treats for airlines. Every kilowatt-hour used must be bought back overnight, with losses, so the fares are net of a refill cost. And the capacity is restored every night, so the problem repeats daily with the same prices and a fresh draw of demand.

Energy storage is usually valued as an option on a random price. Carmona and Ludkovski [5] value a gas storage facility as an optimal switching problem between injecting, withdrawing and waiting under a stochastic price; Thompson, Davison and Rasmussen [17] solve the same valuation as a real option through its partial differential equation, and Secomandi [16] shows that a capacitated storage asset is traded optimally by price-dependent base-stock targets. Work on electricity storage follows the same route through wholesale prices, measuring what a battery that buys low and sells high would have earned in New York [19] and in PJM [18], and Hoppmann and coauthors [20] review and simulate the economics of household batteries paired with solar panels. On a residential time-of-use plan the price is deterministic and the uncertainty moves to the quantity, which is why the revenue-management view, and not the switching one, gives the optimal policy in closed form.

The installation decision is an investment under uncertainty of the classical kind. Brennan and Schwartz [11] value a resource project whose output price follows a geometric Brownian motion; McDonald and Siegel [6] show that when both the value of a project and its cost

are random, the right to invest is worth waiting for until the value exceeds the cost by a factor that can be large; Dixit and Pindyck [7] collect the theory. Farzin, Huisman and Kort [23] study when to adopt a technology that keeps improving, which is the household's position as battery prices fall, and Fleten, Maribu and Wangensteen [24] apply real options to small-scale, decentralised renewable generation. Here the project's value is the present value of the battery's savings, which moves with the regulator's price spread, and its cost is the installed price of a battery, which has been falling. The household's question, whether and when to install, is exactly theirs.

The paper makes four contributions. First, it states the household's daily problem as two-class revenue management and solves it: the battery should be filled every night (Theorem 1), and on weekdays it should hold back during the morning mid-peak a reserve given by Littlewood's rule extended for the late mid-peak window and for the refill cost (Theorem 3), of which the textbook rule is a special case (Corollary 1). Wear enters as a threshold on each window, and weekends are the first to stop paying (Proposition 1).

Second, it gives the best saving any policy could make, even one that knew each day's usage in advance, in closed form (Theorem 2), and checks it against a linear programme over every hour of the record, so that the reserve policy is measured against a true upper bound and not against another heuristic.

Third, it measures all of this on one house's 462 days of hourly usage, reconciled to its twelve bills. The reserve policy earns \$757 a year off the bill, 94% of the bound and 92% out of sample. Holding a reserve at all is worth \$191 a year; refining the textbook reserve into the extended one is worth only \$11.

Fourth, it states the installation decision as the McDonald and Siegel problem (Theorem 4) and evaluates it on the regulator's price history since 2006 and the market's storage prices since 2021. The battery's savings are worth about a quarter of its installed cost, so even the payback rule refuses it, and the optimal boundary lies far lower still: an installed cost of \$430 to \$2,700 for 13.5 kWh, depending on which drifts are believed.

2. The House and Its Tariff

Definition 1 (The Ultra-Low Overnight plan). The plan divides every day into windows, each with one price per kilowatt-hour: the *overnight* window, 11 PM to 7 AM every day, at c_O ; the *weekend off-peak* window, 7 AM to 11 PM on Saturdays, Sundays and holidays, at c_W ; the *mid-peak* windows, 7 AM to 4 PM and 9 to 11 PM on other days, at c_M ; and the *on-peak* window, 4 to 9 PM on those days, at c_P . Since 1 November 2025 the prices are $c_O = 3.9$, $c_W = 9.8$, $c_M = 15.7$ and $c_P = 39.1$ cents per kilowatt-hour [9]. The holidays are New Year's Day, Family Day, Good Friday, Victoria Day, Canada Day, the Civic Holiday, Labour Day, Thanksgiving, Christmas Day and Boxing Day, a holiday on a weekend being observed on the next weekday.

The Ontario Energy Board sets these prices once a year, each 1 November, and they do not change in between, so within a year the price of every hour is known in advance. Only the energy line of a bill depends on the window. Delivery and regulatory charges are partly

fixed and partly per kilowatt-hour, whatever the hour, and 13% HST is added and the Ontario Electricity Rebate of 23.5% deducted, both on the whole subtotal. A cent saved on the energy line therefore takes $1.13 - 0.235 = 0.895$ cents off the bill, and every dollar figure in this paper is a saving off the bill, after both.

The house is in Ontario, on the ULO plan, and heated by a gas furnace; in winter an attached garage is heated electrically, by a heat pump and two 1,500-watt heaters that are rarely all on at once. Its smart meter records the energy used in every hour, and the utility's Green Button export gives 11,181 hourly readings from 23 June 2025 to 1 October 2026, with the twelve monthly bills from 5 September 2025 to 4 September 2026. The records belong to a private household: the paper reports only totals over the tariff's windows, never the hourly series.

The export labels its times as Eastern Standard Time, but the bills are charged on the local clock, so the convention had to be established from the bills themselves. Every bill's four tier totals are reproduced only when each stamp is read as the local clock time at which its hour starts and a bill covers the days from its start date up to, but not including, its end date. Under that reading the hourly data match the bills (Figure 1). Days with 23 or 25 hours, at the clock changes, and the partial first day are set aside, which leaves 462 complete days: 318 weekdays and 144 weekend or holiday days.

Seasons are calendar months: summer is June to September, winter December to March, and spring and autumn the four months between. The record contains two summers and one winter, so every yearly figure in the paper weights each season by its days in a calendar year, 122, 122 and 121, rather than by its days in the record.

The hourly data against the twelve bills. *Method.* Each hour of the record is assigned to its window by Definition 1, with the holidays, and the hours inside each bill's period are summed by window to rebuild the bill's four tier totals.

Parameters. Twelve bills from 5 September 2025 to 4 September 2026; four tiers each, 48 totals; stamps read as the local clock at the start of each hour.

Result. The rebuilt totals come to 12,850.6 kWh against 12,848.7 kWh billed. The mean absolute difference over the 48 totals is 0.46 kWh, and six of the twelve bills are reproduced to a tenth of a kilowatt-hour on all four tiers.

Error. The largest differences, +5.9 and -4.9 kWh, are both overnight and fall on the two bills that contain a clock change, where an hour is added or removed in the night. The remaining differences are at most 3.1 kWh, all overnight except on one bill, which differs by 1.3 kWh on-peak and 0.4 kWh mid-peak. None is large enough to move a window total by more than a few tenths of a percent.

How much the house draws in each window. *Method.* For every complete day the hourly usage is summed over the windows of Definition 3: morning mid-peak, on-peak and late mid-peak on weekdays, and the whole day from 7 AM to 11 PM on weekend and holiday days. Weekdays are grouped by season.

Parameters. 153 summer weekdays, 82 in spring and autumn and 83 in winter; 144 weekend and holiday days.

Result. In summer the on-peak demand D_P averages 9.82 kWh, with median 9.02 and 90th percentile 14.97; the morning mid-peak D_{M1} averages 11.85 kWh, the late mid-peak D_{M2} 2.99, and a weekend day D_W 24.98. In spring and autumn D_P averages 5.98 kWh (median 5.66, 90th percentile 8.37), D_{M1} 8.71, D_{M2} 2.43 and D_W 17.86. In winter D_P averages 8.92 kWh with median 5.62 and 90th percentile 17.46; D_{M1} averages 11.24, D_{M2} 3.95 and D_W 28.30. The on-peak demand exceeds a 13.5 kWh battery on 18% of summer weekdays and 19% of winter weekdays, and never in spring or autumn. The morning and the evening are correlated, at 0.65 in summer, 0.18 in spring and autumn and 0.15 in winter. Over a calendar year the house uses 12,558 kWh.

Error. Each season's distribution rests on 82 to 153 days, so its upper quantiles, which are what set the reserve, carry sampling error of a kilowatt-hour or so; the out-of-sample test in the Computation section measures what that costs.

Remark 1 (Why winter evenings have a long tail). In winter the median on-peak demand is 5.6 kWh but the mean is 8.9 and the 90th percentile 17.5: most winter evenings are light, and a few are very heavy.

The house itself is heated by gas, so the tail is the garage: its heat pump and heaters run on cold days and in the evening, and switch the on-peak demand from that of a mild day to more than a full battery. The winter nights show the same heating, at more than 2 kWh an hour before dawn (Figure 2). In summer the tail is air conditioning, and it follows the weather through the whole day, which is why the summer morning predicts the summer evening and the winter morning hardly does. The tail matters more than the mean, because Littlewood's rule reads a quantile of the evening's demand, and a heavy tail pushes the reserve up.

3. The Model

Definition 2 (The battery, its losses and its wear). A battery has usable capacity B kWh and can deliver at most $\bar{p}$ kWh in an hour. Its round-trip efficiency is η : to deliver one kilowatt-hour it must be charged with $1/\eta$ from the grid. Each kilowatt-hour it delivers costs w cents of wear. Every kilowatt-hour bought from the grid also carries delivery and regulatory charges of v cents, whatever the window. The *refill cost*

$$s = \frac{c_O + v}{\eta} - v$$

is what it costs, net of the charges the delivered energy avoided, to buy back overnight one kilowatt-hour that the battery delivered.

The base case is a 13.5 kWh battery, the most common size sold for homes, delivering up to 5 kWh in an hour, with $\eta = 0.90$. Its warranty runs for ten years, to 70% of the original capacity, with no limit on the energy it moves for residential use. Under such a warranty the owner pays nothing extra for a kilowatt-hour moved, so the base case sets $w = 0$; the reserve policy moves about 4,570 kWh a year, 339 full cycles, and wear of up to 10 cents is treated as a sensitivity. The charges v are fitted on the twelve bills, as each

bill's delivery and regulatory totals against its days and its kilowatt-hours: 3.04 cents of delivery and 0.48 of regulatory charges per kilowatt-hour, $v = 3.52$. They cancel on the energy the battery merely shifts, but not on its losses, and they raise the refill cost from $c_O/\eta = 4.33$ to $s = 4.72$ cents.

Definition 3 (A day's windows and their demands). On a weekday the house's demand in the morning mid-peak, 7 AM to 4 PM, is D_{M1} ; in the on-peak window, 4 to 9 PM, it is D_P ; and in the late mid-peak, 9 to 11 PM, it is D_{M2} , all in kilowatt-hours. On a weekend or holiday its demand from 7 AM to 11 PM is D_W . Within a season, days are independent draws from that season's joint distribution of these demands and of the hourly loads ℓ_h inside them.

The overnight window needs no demand of its own: the battery never discharges overnight, since the energy it holds was bought at the overnight price, and the house's overnight draw is bought from the grid whether or not there is a battery. The independence of days is a modelling choice; the out-of-sample test, which sets each month's reserve from the other months, checks that nothing important depends on it.

Definition 4 (Policies, the reserve and the day's saving). A *policy* decides in each hour how much to charge from the grid and how much to deliver to the house, using only what has happened so far; it keeps the stored energy between 0 and B , delivers no more in an hour than $\min(\ell_h, \bar{p})$, and exports nothing. The *reserve policy* with reserve R charges to full every night; on a weekday it delivers the house's draw from 7 AM to 4 PM only while more than R is stored, and from 4 to 11 PM without restriction. Its saving on a day on which it delivers d_h in hour h at price c_h is

$$G = \sum_h (c_h - w - s) d_h,$$

in cents before tax, and the value of holding r kWh at 4 PM on a weekday is the expected evening saving

$$V(r) = \mathbb{E} \left[(c_P - w - s) \min(D_P, r) + (c_M - w - s) \min(D_{M2}, (r - D_P)^+) \right].$$

The saving G charges each delivered kilowatt-hour its refill cost s at the moment it is delivered, so energy that stays in the battery is free: it does not need to be bought back. This accounting is what makes the problem separate into days. A year's saving off the bill is 0.895 times the season-weighted sum of G over the days, in dollars. V ignores the limit on hourly power; Remark 2 shows that the limit almost never binds.

Assumption 1 (Standing assumptions). (i) Prices are those of Definition 1 and do not change. (ii) The battery serves only the house and does not lose charge while it stands. (iii) Mid-peak delivery pays: $c_M - w > s$. (iv) The household does not forecast: on a weekday the evening's demands (D_P, D_{M2}) are treated as independent of what happened in the morning, so the reserve of a season is a fixed number.

Prices change every 1 November, which only means that the policy is recomputed once a year. The second assumption is that of a battery installed without panels, which has

no export arrangement; the standing loss of a modern lithium battery over a day is small next to its round-trip loss. Assumption (iii) holds whenever wear is below $c_M - s = 10.98$ cents. Assumption (iv) is a restriction on what the policy uses, not a claim about the data: in summer the morning and evening demands are correlated at 0.65. Under it, Theorem 3 gives the best reserve that does not look at the morning. The simulation on the real days, with the morning and the evening together, finds the same reserve, and Remark 3 bounds what looking at the morning could add.

4. Operating the Battery

Theorem 1 (Fill to the top every night). Under Assumption 1, no policy saves more by charging the battery to less than full overnight: filling it every night is optimal.

Proof. Compare any policy with the one that also fills the battery to the top overnight and otherwise follows it, using the extra energy, x kWh, in the first hours of the day in which the house draws more than the original policy delivers and the power limit allows. Each extra kilowatt-hour cost $(c_O + v)/\eta$ to store. If it is delivered on a weekday it avoids at least $c_M + v - w$, which exceeds its cost by Assumption 1(iii); if it is delivered on a weekend or holiday it avoids $c_W + v - w$, and the filled policy delivers it then only if that exceeds its cost. If it is not delivered, it is still in the battery at 11 PM, and the next night's charge is smaller by exactly the energy it replaces, which recovers its cost, because the battery loses nothing while it stands and the price does not change. Every extra kilowatt-hour therefore earns at least what it cost. $\square$

The argument is the household's version of a simple fact: stored energy keeps. A kilowatt-hour bought overnight and not used is not lost, it is merely bought one night early, so the only question is how to spend the energy, never how much of it to buy. In this house the energy above the reserve is spent almost every weekday: under the reserve policy the morning draws a full battery all the way down to the reserve on 99% of summer weekdays, 92% of those in spring and autumn and every winter weekday.

Theorem 2 (The hindsight bound in closed form). Under Assumption 1, let a day's hourly loads ℓ_h be known in advance and write $\bar{\ell}_h = \min(\ell_h, \bar{p})$. On a weekday the largest saving any policy can make is

$$G^H = (c_P - w - s)p + (c_M - w - s)m,$$

$$p = \min\left(\sum_{h=16}^{20} \bar{\ell}_h, B\right), \quad m = \min\left(\sum_{h \in \mathcal{M}} \bar{\ell}_h, B - p\right),$$

where $\mathcal{M}$ is the mid-peak hours 7 to 15 and 21 to 22; on a weekend or holiday it is $G^H = (c_W - w - s)^+ \min\left(\sum_{h=7}^{22} \bar{\ell}_h, B\right)$. The sum of G^H over the days bounds the saving of every policy, including policies that know the future, charge in any hour and carry energy from one day to the next.

Proof. By Theorem 1 the battery may be taken to be full at 7 AM. Charging during the day

never helps: a kilowatt-hour bought in a window at price c and delivered later in the same window avoids less than it costs, $c + v - w < (c + v)/\eta$, and the on-peak window cannot receive more than the B held at 4 PM. Carrying energy past 11 PM replaces a purchase the next night at its own cost, so it neither gains nor loses. What remains is to choose $d_h \in [0, \bar{\ell}_h]$ with $\sum_h d_h \leq B$ to maximise $\sum_h (c_h - w - s)d_h$, a fractional knapsack whose solution serves the most valuable hours first: the on-peak, then the mid-peak if it pays. The derivation below certifies it by linear-programming duality. $\square$

The bound is not a heuristic of the same kind as the policies it judges. It is the exact optimum of the deterministic problem, so the gap between it and a policy is the most that any information about the day, however good, could add to that policy.

For one day, after Theorem 1 and the two exchange arguments in the proof of Theorem 2, the problem is the linear programme

$$\max_d \sum_{h=7}^{22} a_h d_h \quad \text{subject to} \quad \sum_h d_h \leq B, \quad 0 \leq d_h \leq \bar{\ell}_h, \quad (4.1)$$

with $a_h = c_h - w - s$, so that $a_h = c_P - w - s$ in the on-peak hours, $c_M - w - s$ in the mid-peak hours of a weekday and $c_W - w - s$ on a weekend or holiday. Its dual has a multiplier $\lambda \geq 0$ on the capacity and $\mu_h \geq 0$ on each hour's limit:

$$\min_{\lambda, \mu} \lambda B + \sum_h \mu_h \bar{\ell}_h \quad \text{subject to} \quad \lambda + \mu_h \geq a_h. \quad (4.2)$$

On a weekday take $\lambda = c_P - w - s$ if the on-peak hours alone fill the battery, $\lambda = c_M - w - s$ if the on-peak and mid-peak hours together fill it but the on-peak alone does not, and $\lambda = 0$ otherwise, and set $\mu_h = (a_h - \lambda)^+$. The dual objective is then

$$\lambda B + \sum_h (a_h - \lambda)^+ \bar{\ell}_h = (c_P - w - s)p + (c_M - w - s)m = G^H \quad (4.3)$$

in each of the three cases, as a direct substitution shows, and the greedy primal attains the same value, so both are optimal. The weekend case is the same with a single price.

Two exchanges remove the other freedoms of the general problem. If a policy charges b kWh from the grid at an hour of price c and the energy is delivered later at price c' , the pair is worth $(c' + v - w) - (c + v)/\eta$ per kilowatt-hour delivered. Within a day the only way it can be positive is $c = c_M$ and $c' = c_P$, which would need the battery to have room before 4 PM, and it has room only if it delivered energy in the morning at c_M , for a net of $(c_M + v - w) - (c_M + v)/\eta < 0$ per kilowatt-hour cycled. Across days, a kilowatt-hour held past 11 PM displaces the next night's purchase at the same price. Neither exchange can raise the saving, so the bound for the whole record is the sum of the daily bounds.

Lemma 1 (The value of the energy held at four o'clock). Under Assumption 1, V is concave on $[0, B]$, with right derivative

$$V'(r) = (c_P - c_M) \mathbb{P}(D_P > r) + (c_M - w - s) \mathbb{P}(D_P + D_{M2} > r).$$

Proof. Since $\min(D_{M2}, (r - D_P)^+) = \min(D_P + D_{M2}, r) - \min(D_P, r)$,

$$V(r) = (c_P - c_M) \mathbb{E} \min(D_P, r) + (c_M - w - s) \mathbb{E} \min(D_P + D_{M2}, r).$$

For any demand $X \geq 0$ the right derivative of $\mathbb{E} \min(X, r)$ is $\mathbb{P}(X > r)$, which does not increase with r , so $\mathbb{E} \min(X, r)$ is concave. Both coefficients are positive, $c_P > c_M$ and $c_M - w - s > 0$ by Assumption 1(iii), so V is a positive combination of concave functions. $\square$

The derivative reads as a ledger of where the last kilowatt-hour held at 4 PM ends up. With probability $\mathbb{P}(D_P > r)$ the evening peak needs it and it earns the on-peak price; otherwise, with probability $\mathbb{P}(D_P \leq r < D_P + D_{M2})$, the late mid-peak uses it; and otherwise it stays in the battery and costs nothing to replace. Adding the refill cost s back gives the gross value plotted in Figure 3, which falls from c_P at $r = 0$ towards s for a large reserve.

Theorem 3 (Littlewood's rule for a home battery). Under Assumption 1, let

$$R^* = \max \left\{ r \in [0, B] : (c_P - c_M) \mathbb{P}(D_P > r) \geq (c_M - w - s) \mathbb{P}(D_P + D_{M2} \leq r) \right\}.$$

Among all policies, the reserve policy with reserve R^* is optimal on weekdays. When the demands have continuous distributions and $R^* < B$, the reserve solves

$$(c_P - c_M) \mathbb{P}(D_P > R) = (c_M - w - s) \mathbb{P}(D_P + D_{M2} \leq R).$$

Proof. By Theorem 1 the battery holds B at 7 AM. From 4 PM every delivery pays, since $c_P - w - s > 0$ and $c_M - w - s > 0$, so the evening delivers freely and, by Assumption 1(iv), a weekday's expected saving is $(c_M - w - s)(B - r) + V(r)$, where r is the energy left at 4 PM and $B - r$ cannot exceed what the morning draws. By Lemma 1 this is concave in r , and its right derivative $V'(r) - (c_M - w - s)$ is non-negative exactly when the inequality defining R^* holds, which by the monotonicity in Lemma 1 is an interval $[0, R^*]$. The maximiser over the feasible r is therefore R^* when the morning draws enough to bring the battery down to it, and otherwise the smallest r the morning allows. The reserve policy delivers every kilowatt-hour the morning draws while more than R^* is stored, so it ends the morning at exactly that maximiser on every day, and it never delivers a kilowatt-hour it could have held at a higher expected value [1], [3]. $\square$

The rule weighs the two fares at the margin. The left side is what the last kilowatt-hour held back gains if the evening peak needs it, the extra $c_P - c_M$ over spending it in the morning; the right side is what it loses if the evening, peak and late mid-peak together, does not use it, the morning's net fare $c_M - w - s$ given up. The reserve is the largest amount for which the expected gain is still at least the expected loss.

The rule is the solution of a dynamic programme over the hours of the morning, and the programme shows why a single number is optimal and not merely convenient. Let $J_t(q)$ be the expected saving from hour t to 11 PM with q kWh stored, with the evening value $J_{16} = V$ of Definition 4. For the morning hours $t = 7, \dots, 15$,

$$J_t(q) = \mathbb{E} \left[\max_{0 \leq d \leq \min(\bar{\ell}_t, q)} \{ (c_M - w - s) d + J_{t+1}(q - d) \} \right]. \quad (4.4)$$

The claim is that each J_t is concave with $J'_t(q) \geq c_M - w - s$ for $q \leq R^*$ and $J'_t(q) \leq c_M - w - s$ for $q > R^*$, and that the maximising delivery is

$$d_t^*(q) = \min(\bar{\ell}_t, (q - R^*)^+). \quad (4.5)$$

For $t = 16$ this is Lemma 1 together with the definition of R^* . If it holds for $t + 1$, the bracket's derivative in d is $(c_M - w - s) - J'_{t+1}(q - d)$, which is positive while $q - d > R^*$ and not positive once $q - d \leq R^*$, so the maximiser is d_t^* . Then for $q \leq R^*$ nothing is delivered and $J'_t(q) = \mathbb{E} J'_{t+1}(q) \geq c_M - w - s$; for $q > R^*$ the last kilowatt-hour stored is either delivered this hour, worth $c_M - w - s$, or carried to a state above R^* where it is worth at most that, so $J'_t(q) \leq c_M - w - s$. Concavity passes from J_{t+1} to J_t because the bracket is jointly concave in (q, d) on a convex set.

The protection level is therefore the same in every morning hour, as Brumelle and McGill showed for nested fare classes and Talluri and van Ryzin set out in general [4]: whatever the morning's pattern, deliver down to R^* and no further. On the data, the evening value $V(r)$ simulated hour by hour with the power limit, and maximised against $(c_M - s)r$, gives reserves of 11.9, 7.4 and 11.4 kWh in the three seasons, the same as the rule to within 0.03 kWh.

Corollary 1 (The textbook rule as a special case). Under Assumption 1, (a) if there is no late mid-peak window, $D_{M2} = 0$, the reserve is the quantile of the on-peak demand at which

$$\mathbb{P}(D_P > R^*) = \frac{c_M - w - s}{c_P - w - s};$$

(b) if moreover refilling and wear are free, $w = s = 0$, it is Littlewood's rule, $\mathbb{P}(D_P > R) = c_M/c_P$; and (c) in general R^* is never below the reserve of Littlewood's rule.

Proof. With $D_{M2} = 0$ the condition of Theorem 3 is $(c_P - c_M)\mathbb{P}(D_P > R) = (c_M - w - s)(1 - \mathbb{P}(D_P > R))$, which rearranges to (a); setting $w = s = 0$ gives (b). For (c), $\mathbb{P}(D_P + D_{M2} \leq r) \leq \mathbb{P}(D_P \leq r)$ and $(c_M - w - s)/(c_P - w - s) \leq c_M/c_P$, so at the textbook reserve the inequality defining R^* still holds, and R^* is at least as large. $\square$

In the airline problem a seat sold costs nothing more and a seat left unsold is worth nothing, which is case (b) [1], [2]. A battery differs on both counts: every kilowatt-hour delivered must be bought back, which lowers both fares by s and the fare ratio from $15.7/39.1 = 0.40$ to $10.98/34.38 = 0.32$; and a kilowatt-hour the peak does not need can still be used after 9 PM. Both push the reserve up. The late window also makes the sequence of fares non-monotone, low, high and low again, which in general needs the analysis of Robinson [12]; here it stays simple because the late class is served from whatever the peak leaves. For this house the textbook rule protects 10.0, 6.1 and 9.2 kWh in summer, spring and autumn, and winter, the extended rule 11.9, 7.4 and 11.4.

Proposition 1 (When the battery should rest at weekends). Under Assumption 1(i)-(ii),

a kilowatt-hour delivered in a window of price c pays if and only if $c - w > s$. Hence the battery should be cycled on weekends and holidays only if $w < c_W - s$, it should deliver in the mid-peak windows only if $w < c_M - s$, and in the on-peak window only if $w < c_P - s$. At today's prices and $\eta = 0.9$ the three thresholds are 5.08, 10.98 and 34.38 cents per kilowatt-hour.

Proof. By the accounting of Definition 4 a kilowatt-hour delivered at price c adds $c - w - s$ to the day's saving and one left in the battery adds nothing, so delivery pays exactly when $c - w - s > 0$. On a weekend or holiday every hour from 7 AM to 11 PM has the same price c_W , and the battery is refilled overnight before the next weekday, so the weekend decision does not interact with any weekday reserve. The thresholds follow from $s = 4.72$. $\square$

The three windows drop out in the order of their prices as wear rises. Weekends go first, above 5 cents, which is why a battery whose owner counts its wear at 5 cents a kilowatt-hour or more should simply be left alone from Saturday morning to Sunday night. The mid-peak goes at 11 cents, at which point the battery serves only the peak and the reserve question disappears.

Remark 2 (The power limit, and how rarely it binds). Lemma 1 and Theorem 3 ignore the limit $\bar{p}$ on the energy delivered in an hour. In this house the on-peak draw exceeds 5 kWh in 0.88% of weekday on-peak hours.

The reserve computed with the limit, from the evening simulated hour by hour, differs from the rule by less than 0.03 kWh in every season, and the simulations of the policies keep the limit throughout. Hourly readings hide the minutes, however: an oven and a dryer together can draw more than 5 kW for a quarter of an hour inside an hour whose total is well under 5 kWh, and in those minutes the grid supplies the excess at the on-peak price. The loss is invisible at this resolution, and it is small, since it is bounded by the energy above 5 kW in the busiest quarter-hours of an evening.

Remark 3 (What a forecast of the evening could add). By Theorem 2, no policy, however much it knows about the day ahead, can save more than the hindsight bound. For this house the gap between the bound and the reserve policy is \$48.8 a year.

That figure is the most that any forecast could be worth to this household, and in practice a forecast would recover only part of it. The natural one is the morning itself: in summer the morning mid-peak predicts the evening peak, with a correlation of 0.65, because a hot morning is followed by a hot evening, and a reserve that rose on hot mornings, or with the day's forecast temperature, would capture some of the gap. In spring, autumn and winter the correlation is below 0.2 and there is little to capture. A household deciding whether to buy a forecasting service for its battery therefore has a firm upper limit on what it can pay.

5. When to Install

Definition 5 (The savings, the installed cost and the right to wait). Let V_t be the value at time t of the savings of a battery installed then: the present value, at the household's

discount rate r , of its yearly savings over a ten-year life in which its capacity fades linearly to 70%, at the prices of time t . Let K_t be the installed cost of the battery at time t . Both are geometric Brownian motions,

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_V dW_t^V, \quad \frac{dK_t}{K_t} = \mu_K dt + \sigma_K dW_t^K, \quad d\langle W^V, W^K \rangle_t = \rho dt,$$

and $X_t = V_t/K_t$. Write $\delta_V = r - \mu_V$ and $\delta_K = r - \mu_K$. The household chooses a stopping time τ at which to install, to maximise $\mathbb{E}[e^{-r\tau}(V_\tau - K_\tau)]$; the value of this right is $F(V, K)$.

The savings move almost in proportion to the general level of the prices: scaling all four prices by a common factor scales the fares of Theorem 3 nearly in proportion, the volumetric charges v aside, and leaves the reserve nearly where it was. So V_t moves with the price spread, and its drift and volatility are taken from the spread's. The OEB's on-peak minus off-peak spread, at its 36 price-setting dates since 2006 other than the emergency flat prices of 2020 to 2022, has an expected growth rate of $\mu_V = 4.2\%$ a year (its logarithm grew 2.1% a year) and a volatility of $\sigma_V = 20\%$ [9]. The median quoted price of home storage, 2021 to 2024, has $\mu_K = -5.2\%$ (its logarithm fell 5.8% a year) and $\sigma_K = 10\%$ [10]. Learning curves point the same way, at a faster pace for the cells than for an installed home battery: Schmidt and coauthors [21] project storage costs from experience rates, and Ziegler and Trancik [22] measure three decades of decline in the price of lithium-ion cells. Neither drift is well determined: their standard errors are 4.6 and 5.4 percentage points, which is why the boundary is reported under four beliefs about them. With eight half-years of costs there is no usable estimate of ρ , and the base case sets it to zero.

Theorem 4 (The installation boundary). Under Definition 5, suppose $\delta_V > 0$ and $\delta_K > 0$. Let $\sigma^2 = \sigma_V^2 + \sigma_K^2 - 2\rho\sigma_V\sigma_K$ and let β be the larger root of

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (\delta_K - \delta_V)\beta - \delta_K = 0.$$

Then $\beta > 1$, and it is optimal to install the first time the savings reach

$$X_t \geq x^* = \frac{\beta}{\beta - 1},$$

that is, the first time the installed cost falls to $K^* = V_t/x^*$. The value of the right to install is

$$F(V, K) = K(x^* - 1)\left(\frac{V}{x^*K}\right)^\beta \quad \text{for } V < x^*K, \quad F(V, K) = V - K \quad \text{otherwise.}$$

Proof. The quadratic is negative at $\beta = 0$, where it equals $-\delta_K$, and at $\beta = 1$, where it equals $-\delta_V$, and it is convex, so its larger root exceeds 1. That F is the value function and the stated time is optimal is the solution of the optimal stopping problem of McDonald and Siegel [6], [7]; the derivation below reduces the variational inequality to one variable, solves it, and checks the conditions of the verification theorem. $\square$

The boundary is a ratio, not a price: a battery is worth installing when its savings are

worth x^* times its cost, whatever the level of either. Since $x^* > 1$, the cost must fall below the value of the savings by a margin, and the margin is set by how fast the two are expected to move apart and how uncertain they are. When the spread is expected to rise almost as fast as the discount rate, δ_V is small, β approaches 1 and the margin becomes very large, because waiting then costs almost nothing.

The value function satisfies the HJB variational inequality of optimal stopping [8], [25],

$$\max \{ \mathcal{L}F - rF, V - K - F \} = 0, \quad (5.1)$$

where

$$\mathcal{L} = \frac{1}{2}\sigma_V^2 V^2 \partial_{VV} + \rho\sigma_V\sigma_K V K \partial_{VK} + \frac{1}{2}\sigma_K^2 K^2 \partial_{KK} + \mu_V V \partial_V + \mu_K K \partial_K. \quad (5.2)$$

Doubling both V and K doubles the payoff at every stopping time, so F is homogeneous of degree one, $F(V, K) = Kf(x)$ with $x = V/K$. Then $F_V = f'$, $F_K = f - xf'$, $F_{VV} = f''/K$, $F_{VK} = -xf''/K$ and $F_{KK} = x^2 f''/K$, and in the region where waiting is optimal the inequality becomes the ordinary differential equation

$$\frac{1}{2}\sigma^2 x^2 f''(x) + (\delta_K - \delta_V) x f'(x) - \delta_K f(x) = 0. \quad (5.3)$$

Its solutions are powers x^β with β a root of the quadratic of Theorem 4; the right to install is worthless as $x \rightarrow 0$, which rules out the negative root, so $f(x) = Ax^\beta$. Value matching and smooth pasting at the boundary,

$$Ax^{*\beta} = x^* - 1, \quad \beta Ax^{*\beta-1} = 1, \quad (5.4)$$

give $x^* = \beta/(\beta - 1)$ and $A = (x^* - 1)/x^{*\beta}$.

It remains to check the two inequalities. Below the boundary, $f(x) - (x - 1)$ is convex, since $\beta > 1$, and vanishes with its derivative at x^* , so $f(x) > x - 1$: waiting is worth more than installing. Above it, $F = V - K$ and $\mathcal{L}F - rF = -\delta_V V + \delta_K K = K(\delta_K - \delta_V x)$, which is negative for $x > \delta_K/\delta_V$. The quadratic at its root gives $\frac{1}{2}\sigma^2 \beta(\beta - 1) = \delta_V \beta - \delta_K(\beta - 1) > 0$, so $x^* = \beta/(\beta - 1) > \delta_K/\delta_V$ and the inequality holds throughout the region where installing is optimal. With these, Itô's formula applied to $e^{-rt}F(V_t, K_t)$ shows that no stopping time does better, the verification theorem for optimal stopping [8].

Corollary 2 (Why the payback rule installs too early). The net-present-value rule, install when $V \geq K$, and the payback rule, install when the installed cost is at most the first year's saving times the battery's life, both install at some $X_t < x^*$, and each gives away $F(V, K) - (V - K) > 0$ when it does.

Proof. Since $\beta > 1$, $x^* > 1$, and by the derivation $F(V, K) > V - K$ for every $V < x^* K$. The first year's saving times the life exceeds V , which discounts the savings and lets them fade, so the payback rule accepts every cost the net-present-value rule accepts and some it refuses. $\square$

Both rules compare the battery with not buying it. The optimal rule compares it with buying it later, when it may be cheaper and the spread may be wider, and that alternative is worth more than installing until the savings are worth x^* times the cost. For this house both simple rules already refuse the battery at today's quotes, so the corollary matters for what it says about the future: a falling installed cost will reach the point where the battery pays back long before it reaches the point where it should be bought.

Remark 4 (A battery that fades, and a spread set by a regulator). The savings value V_t is a present value over a finite, fading life, and the price spread behind it moves in annual steps set by a regulator rather than continuously.

Neither undoes the model. Installing later buys a new battery with a full life, so V_t depends on time only through the prices, and the homogeneity of the derivation holds. The spread, by contrast, is a policy variable: it fell when the Fair Hydro Plan lowered prices in July 2017 and rose by more than half in November 2019, when that plan's pricing ended and the Ontario Electricity Rebate rose to 31.8% [9]. A geometric Brownian motion is a summary of the drift and dispersion of those steps, as Brennan and Schwartz used one for an output price [11]. The rebate moves too, and it moved against the 2019 price change, so the spread a bill actually shows is steadier than the price spread: the volatility used here is, if anything, too high, which raises x^* but cannot turn today's refusal into an acceptance. The cost series is a national median of quoted prices in the United States; only its drift and volatility are used, and the level is the Ontario quote.

6. Computation

Algorithm 1 — From hourly usage to the reserve and the year's saving

Inputs: hourly usage readings; the bills' tier totals; the prices and holidays of Definition 1; B , η , $\bar{p}$, w ; the bills' delivery and regulatory totals. *Output:* the reserve of each season, the yearly saving of the reserve policy, of the naive policy and of hindsight, and the out-of-sample saving.

1. Read each stamp as the local clock time at which its hour starts; keep the complete days.
2. Assign every hour to its window by Definition 1 and rebuild each bill's tier totals; stop if they do not match.
3. Fit each bill's delivery and regulatory totals on its days and kilowatt-hours; the per-kilowatt-hour coefficients sum to v . Set $s = (c_O + v)/\eta - v$.
4. For every day form D_{M1} , D_P , D_{M2} or D_W , and group weekdays by season.
5. For each season, take R^* as the largest r on a grid of 0.01 kWh with $(c_P - c_M) \hat{\mathbb{P}}(D_P > r) \geq (c_M - w - s) \hat{\mathbb{P}}(D_P + D_{M2} \leq r)$, using the season's empirical frequencies.
6. Simulate every day hour by hour from a full battery at 7 AM: deliver $\min(\ell_h, \bar{p}, q - \text{floor})$, with the floor R^* in weekday hours before 4 PM and zero otherwise, and on weekends only if $c_W - w > s$; the day's saving is $\sum_h (c_h - w - s) d_h$. The naive policy is the same with $R^* = 0$.

7. Compute the hindsight bound of each day by Theorem 2, and check its sum against the linear programme over all hours.
8. Annualise: each season's mean daily saving times its days in a calendar year, summed, times 0.895, in dollars.
9. Out of sample: for each calendar month, repeat step 5 without that month and simulate the month with the result.

Cost. One pass over the 11,181 hours per policy. The linear programme of step 7 has 33,267 variables and is solved by HiGHS in under a second, and the whole computation, out-of-sample test and sensitivities included, takes about two seconds. Step 2 is the step that matters: without it the windows would be guessed from the file's label, and on that label the summer windows are an hour off.

Algorithm 2 — From price histories to the installation boundary

Inputs: the OEB's TOU prices since 2006; the half-yearly median price of home storage; the yearly saving of the reserve policy at each capacity from Algorithm 1; the life, the fade, the discount rate r and today's installed cost K_0 . *Output:* β , the threshold x^* , the boundary cost K^* , the value of the right to install and the expected time to the boundary, under each belief about the drifts.

1. Form the spread S_i , on-peak minus off-peak, at each date t_i on which the prices changed, leaving out the emergency flat prices.
2. Estimate the log drift as $\ln(S_n/S_1)/(t_n - t_1)$ and σ_V^2 as the time-weighted variance of the log changes about it; set μ_V to the log drift plus $\frac{1}{2}\sigma_V^2$. Do the same for the storage price, to get μ_K and σ_K .
3. Compute $V_0 = \sum_{k=0}^9 A(B_k) e^{-(r-\mu_V)(k+1/2)}$, where $B_k = B(1 - 0.3(k + \frac{1}{2})/10)$ is the faded capacity in year k and $A(B_k)$ the yearly saving at that capacity, interpolated from Algorithm 1.
4. Solve the quadratic of Theorem 4 for β ; set $x^* = \beta/(\beta - 1)$ and $K^* = V_0/x^*$.
5. With $x_0 = V_0/K_0$, the value of the right to install is $K_0(x^* - 1)(x_0/x^*)^\beta$.
6. With $m = (\mu_V - \frac{1}{2}\sigma_V^2) - (\mu_K - \frac{1}{2}\sigma_K^2)$, the drift of $\ln X$: if $m > 0$ the expected time to the boundary is $\ln(x^*/x_0)/m$; otherwise the probability of ever reaching it is $(x_0/x^*)^{2|m|/\sigma^2}$.
7. Check step 4 independently: for a range of candidate boundaries, solve the value of installing at each by finite differences, and take the best.

Cost. Steps 1 to 6 are closed forms and take milliseconds; step 7 solves 600 tridiagonal systems of 4,000 unknowns, in about a second. The step that carries the uncertainty is step 2: with 19 years of spreads and 4 years of costs, the drifts are estimated far less precisely than the volatilities.

The rule against the evening programme and the best reserve on the data.

Method. Algorithm 1, step 5, gives R^* in each season. Two checks use the data differently. The first simulates every weekday of the season with a constant reserve R on a grid of 0.1 kWh, mornings and evenings together, with the power limit, so that the real dependence

between the morning and the evening is kept; it reports the best R . The second is the evening programme of the derivation: $V(r)$ simulated hour by hour with the power limit, and the r that maximises $V(r) - (c_M - s)r$.

Parameters. $B = 13.5$ kWh, $\eta = 0.90$, $\bar{p} = 5$ kWh, $w = 0$, $s = 4.72$ cents.

Result. The rule gives 11.91 kWh in summer, 7.43 in spring and autumn and 11.38 in winter. The best constant reserve on the data is 12.0, 7.6 and 11.4 kWh, and the evening programme 11.9, 7.4 and 11.4. Using the rule's reserve instead of the best one on the data costs less than 0.06 cents a weekday in every season. The textbook rule's reserves, 10.0, 6.1 and 9.2 kWh, cost 6.5, 4.5 and 4.3 cents a weekday, \$11.4 a year; holding no reserve at all costs \$190.6 a year.

Error. The grids are 0.1 kWh, and the saving is flat near its maximum (Figure 3), so a reserve within a few tenths of a kilowatt-hour of the best costs almost nothing. The agreement with the best reserve on the data, which keeps the summer correlation of 0.65 between morning and evening, shows that Assumption 1(iv) does not move the reserve.

Out of sample: each month's reserve set without it. *Method.* Algorithm 1, step 9: for each of the 17 calendar months of the record, from June 2025 to October 2026, the seasonal reserves are recomputed from the other months and the month is simulated with them; the year is then assembled from the simulated days as before.

Parameters. As in the base case.

Result. The out-of-sample saving is \$740.7 a year, against \$757.1 in sample, and 91.9% of the hindsight bound against 93.9%. The reserves set without a month range from 11.4 to 13.0 kWh in summer, 6.8 to 7.7 in spring and autumn, and 9.2 to 13.5 in winter.

Error. The loss from not knowing the season's distribution in advance is \$16.4 a year, 2.2% of the saving. Of it, \$12.8 is winter, \$2.5 summer and \$1.0 spring and autumn: the record holds a single winter, and removing one of its four months moves the winter reserve by as much as 2.2 kWh. A household with several winters of data would lose less.

The closed form against a linear programme over every hour. *Method.* The linear programme of the derivation, written over the whole record rather than day by day: in every hour a charge $b_t \in [0, \bar{p}]$ bought at the hour's price plus v , a delivery $d_t \in [0, \min(\ell_t, \bar{p})]$, and the stored energy $q_t = q_{t-1} + \eta b_t - d_t \in [0, B]$ carried from hour to hour, so that the programme may charge in any hour and carry energy across days. It is solved with HiGHS and compared with the sum over days of the closed form of Theorem 2.

Parameters. The 11,089 hours of the 462 complete days; base case, $w = 0$.

Result. The programme's optimum is 116,776.3 cents before tax, and the sum of the closed forms is 116,776.3 cents: equal to the tenth of a cent. Annualised by season, the hindsight bound is \$805.9 a year off the bill.

Error. None beyond the solver's tolerance. The equality confirms both exchange arguments of Theorem 2: with every freedom allowed, the optimum never charges outside the overnight window and never gains by carrying energy from one day to the next.

What the battery earns, against hindsight and the naive policy. *Method.* Algorithm 1, steps 6 to 8, over all 462 days, with each season weighted by its days in a calendar year.

Parameters. Base case: $B = 13.5$ kWh, $\eta = 0.90$, $\bar{p} = 5$, $w = 0$, $s = 4.72$ cents, the reserves of the rule.

Result. The reserve policy saves \$757.1 a year off the bill, 93.9% of the hindsight bound of \$805.9. Discharging whenever the house draws saves \$566.5. Weekends and holidays contribute \$69.4 of the reserve policy's saving. The policy moves 4,571 kWh a year through the battery, 339 full cycles, and the morning draws a full battery down to the reserve on 99% of summer weekdays, 92% of those in spring and autumn and all winter weekdays, so the energy that Theorem 1 says to buy is almost always spent.

Error. The figures are those of one house over 462 days. The season weighting removes the double count of summer in the record; a plain average over the days would overstate the year by about \$20.

Wear, losses and the battery's size. *Method.* Algorithm 1 rerun with the reserve recomputed for each setting.

Parameters. Wear w of 0, 2.5, 5, 7.5 and 10 cents per kilowatt-hour; round-trip efficiency 0.85, 0.90 and 0.95; usable capacity from 2.5 to 20 kWh. Otherwise the base case.

Result. The reserve policy saves \$757.1, \$656.1, \$557.8, \$493.3 and \$430.9 a year at the five levels of wear (Table 1). The naive policy falls faster, from \$566.5 to \$197.5, because it spends the most energy at the least valuable times. The weekend term disappears between 5 and 7.5 cents, as Proposition 1 places it at 5.08. Round-trip efficiency matters little: \$737.3 at 0.85 and \$774.8 at 0.95. Capacity matters more, with diminishing returns: \$205.2 a year at 2.5 kWh, \$402.5 at 5, \$647.8 at 10, \$757.1 at 13.5 and \$918.7 at 20, so the saving per kilowatt-hour of capacity falls from \$82 to \$46 (Figure 4).

Error. Each figure carries the sampling error of the record, the same for all settings, so the comparisons between them are sharper than the levels.

A fading battery's savings and their present value. *Method.* Algorithm 2, step 3: the yearly saving of the reserve policy at each year's faded capacity, interpolated from the capacity runs of Algorithm 1, discounted at r .

Parameters. Ten years; capacity falling linearly from 13.5 kWh to 70% of it; $r = 5\%$; prices held at today's level; installed cost $K_0 = \$21,500$, the middle of the 2026 quotes.

Result. The saving is \$751 in the first year and \$634 in the tenth, \$6,935 in all. Discounted, it is worth $V_0 = \$5,499$, 26% of the installed cost: the net present value of installing today is $-\$16,001$, and the simple payback, the installed cost over the first year's saving, is 28.4 years against a ten-year warranty. The prices would have to be 3.9 times today's for the net present value to be zero. The province's rebate of up to \$5,000 for home batteries applies only with new solar panels; even if it applied, the net present value would be $-\$11,001$.

Error. The fade is the warranty's floor, so a battery that kept more of its capacity would

save a little more; at the warranted 70% the last year saves 84% of the first, and no plausible fade changes the sign.

The boundary on the regulator’s prices and the market’s. *Method.* Algorithm 2 for four beliefs about the drifts: as estimated; the spread flat in expectation, $\mu_V = 0$; the installed cost flat, $\mu_K = 0$; and both flat. Step 7 checks the boundary by finite differences for the first two.

Parameters. $\sigma_V = 20.4\%$ from 36 TOU price settings, 2006 to 2025 [9]; $\sigma_K = 10.1\%$ from eight half-yearly medians, 2021 to 2024 [10]; $\mu_V = 4.2\%$ and $\mu_K = -5.2\%$ as estimated; $\rho = 0$; $r = 5\%$; $K_0 = \$21,500$.

Result. With the drifts as estimated, $\beta = 1.07$ and $x^* = 15.3$: install when the cost falls to $K^* = \$435$ for 13.5 kWh, \$32 per kilowatt-hour. With the spread flat, $\beta = 1.54$, $x^* = 2.85$ and $K^* = \$1,933$; with the cost flat, $K^* = \$717$; with both flat, $K^* = \$2,719$ (Table 2). The expected time to the boundary is 50, 65 and 131 years in the first three cases, and with both drifts flat the boundary is reached with probability 0.29 at all. The right to install is worth \$4,736 with the drifts as estimated and \$966 with the spread flat.

Error. The finite-difference search finds boundaries of 15.23 and 2.85, against 15.29 and 2.85 from the formula, the difference being the search’s grid, and the value of the right to install agrees to the cent. The real uncertainty is in the drifts, whose standard errors are 4.6 and 5.4 points: across all four beliefs the boundary stays between \$430 and \$2,700, an eighth to a fiftieth of today’s quotes.

Source. `compute_numbers.py`, Algorithm 1 on the 462 days, base case except for the wear; dollars a year off the bill, each season weighted by its days in a calendar year.

Columns. Wear per kilowatt-hour delivered; the hindsight bound of Theorem 2; the reserve policy of Theorem 3; discharging whenever the house draws; the reserve policy as a share of the bound; whether the battery is cycled on weekends (Proposition 1).

Table 1: A year of savings off the bill by policy and wear, for a 13.5 kWh battery at today’s prices.

wear (cents/kWh)	hindsight (\$)	reserve policy (\$)	naive policy (\$)	reserve / hindsight	weekends
0	805.9	757.1	566.5	93.9%	yes
2.5	696.8	656.1	457.5	94.2%	yes
5	587.7	557.8	348.4	94.9%	yes
7.5	511.8	493.3	272.4	96.4%	no
10	436.9	430.9	197.5	98.6%	no

The reserve policy comes closer to the bound as wear rises, because wear removes first the weekend and then the margin on the mid-peak, which are the parts of the problem where foresight helps. The naive policy gives up a quarter of the reserve policy’s saving at no wear and more than half at 10 cents.

Source. `compute_numbers.py`, Algorithm 2; $\sigma_V = 20.4\%$, $\sigma_K = 10.1\%$, $\rho = 0$, $r = 5\%$, $K_0 = \$21,500$, a 13.5 kWh battery; costs and values in Canadian dollars.

Columns. The belief about the drifts; μ_V and μ_K ; β and the threshold x^* of Theorem 4; V_0 , the value of the savings today; the boundary cost K^* ; the value of the right to install; the expected years to the boundary.

Table 2: The installation boundary for a 13.5 kWh battery under four beliefs about the drifts, in Canadian dollars.

drifts	μ_V	μ_K	β	x^*	V_0	K^*	right to install	years
as estimated	4.2%	-5.2%	1.07	15.3	6,657	435	4,736	50
spread flat	0	-5.2%	1.54	2.85	5,499	1,933	966	65
cost flat	4.2%	0	1.12	9.28	6,657	717	3,939	131
both flat	0	0	1.98	2.02	5,499	2,719	367	p = 0.29

With both drifts flat the logarithm of V/K drifts down, at $-\frac{1}{2}\sigma_V^2 + \frac{1}{2}\sigma_K^2$, so the boundary is reached at all only with probability 0.29, which is the entry in the last column. A rising spread raises the value of the savings, but it raises the threshold faster, because it makes waiting cheap: with the drifts as estimated the boundary is lowest even though the savings are worth the most. In every row the boundary is a small fraction of today's \$21,500.

Script. `fig_recon.py`, which plots the 48 tier totals of `compute_numbers.py`: each bill's four totals as billed and as rebuilt from the hourly data under Definition 1.

Shows. That the hourly data, read on the local clock at the start of each hour, reproduce the bills.

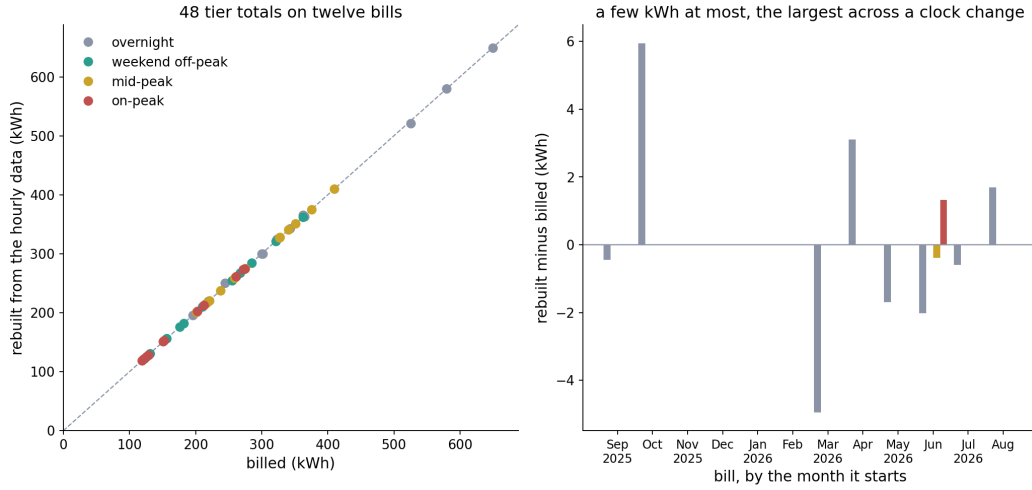


Figure 1: Left: each of the 48 tier totals on the twelve bills, billed against rebuilt from the hourly readings, by tier, on the line of equality; right: the difference between rebuilt and billed for every bill and tier, in kilowatt-hours, with the bills labelled by the month they start.

On the left every point lies on the line of equality, from about 120 kWh on-peak in a mild month to 650 kWh overnight in winter. On the right the differences are all below 6 kWh; the two largest sit on the bills of October 2025 and March 2026, which contain the clock changes, and all but one bill's differences are in the overnight tier.

Script. `fig_windows.py`, which plots the season means of the hourly usage, by hour of the day, from `compute_numbers.py`, with the 10th to 90th percentiles of summer weekdays.

Shows. Where the house's demand falls against the tariff's windows, and how the seasons differ.

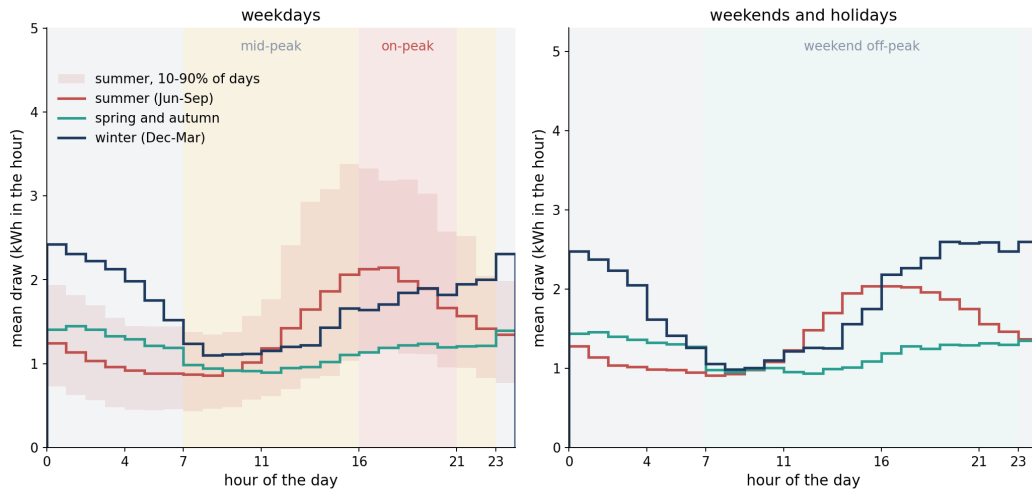


Figure 2: The house's mean usage in each hour of the day by season, on weekdays on the left and on weekend and holiday days on the right, with the overnight, mid-peak, on-peak and weekend off-peak windows shaded and the 10th to 90th percentile range of summer weekdays drawn as a band.

In summer the weekday draw climbs through the afternoon to more than 2 kWh an hour at 5 PM and stays high through the peak, which is air conditioning; the band shows that a hot summer evening draws more than 3 kWh an hour. In winter the draw is highest at night and rises again through the evening, which is the garage's heating, while in spring and autumn the house draws close to 1 kWh an hour all day. The weekends follow the same seasons without the weekday pattern of the morning.

Script. `fig_reserve.py`, which plots the marginal value of Lemma 1 plus the refill cost for each season, computed on the season's empirical distribution, and the mean weekday saving of a battery run with a constant reserve, simulated on the data.

Shows. Theorem 3 at work: where the evening's marginal value crosses the morning's price, and that this point is the top of the simulated saving.

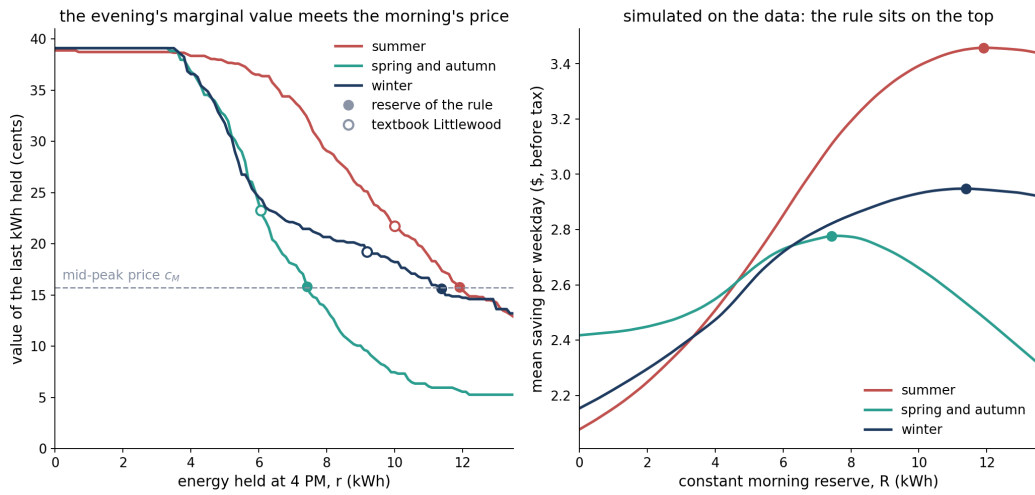


Figure 3: Left: the value of the last kilowatt-hour held at 4 PM against the energy held, by season, with the mid-peak price as a dashed line, the reserves of the extended rule as filled dots and the textbook Littlewood reserves as open dots; right: the mean saving per weekday against a constant morning reserve, simulated on the data, with the rule's reserves marked.

On the left the marginal value starts at the on-peak price, 39.1 cents, while the evening is certain to need the energy, and falls as the reserve grows; the extended rule stops where it meets the mid-peak price. The textbook reserves, the open dots, sit higher on the curves, where a kilowatt-hour held is still worth 19 to 23 cents against the 15.7 the morning pays, so they protect too little. On the right the simulated saving peaks at the rule's reserve in every season, \$3.46 a summer weekday, \$2.95 in winter and \$2.78 in spring and autumn, before tax.

Script. `fig_savings.py`, which plots the yearly savings of Algorithm 1 against the wear and against the battery's capacity, from `compute_numbers.py`.

Shows. How wear removes the battery's uses in order (Proposition 1), and the diminishing return to capacity.

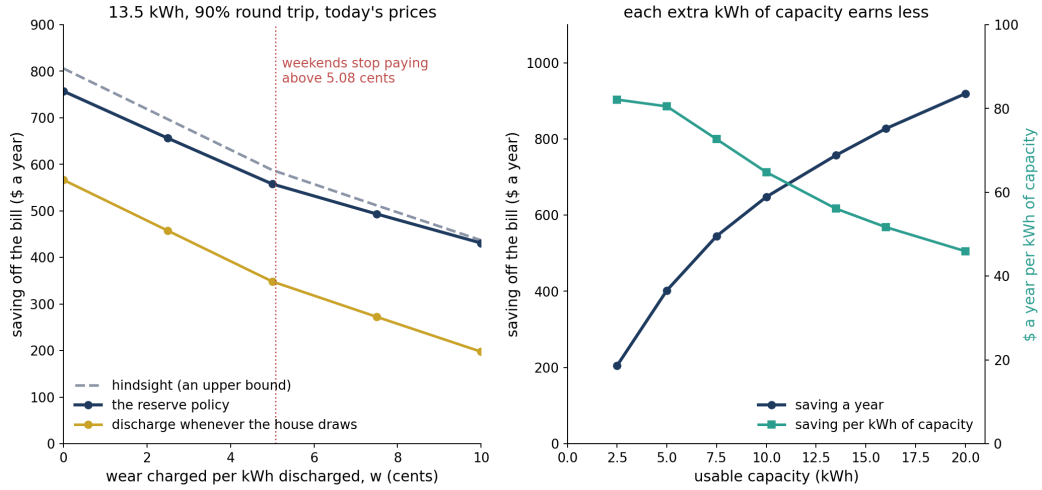


Figure 4: Left: the yearly saving off the bill against the wear charged per kilowatt-hour delivered, for the hindsight bound, the reserve policy and discharging whenever the house draws, with the weekend threshold marked; right: the reserve policy's yearly saving against usable capacity, and the saving per kilowatt-hour of capacity on a second axis.

On the left the reserve policy stays within 6% of the bound at every wear, while the naive policy runs \$190 to \$230 below the reserve policy, and the bound bends at 5.08 cents, where weekends stop paying. On the right the first 5 kWh of capacity earn \$403 a year and the next 15 kWh only \$516 more, because the evenings that need more than 10 kWh are a minority.

Script. `fig_prices.py`, which plots the price spreads of the OEB's tables and the median storage prices from `sources.json`, as read by `compute_numbers.py`, with the fitted trend.

Shows. The two series behind Definition 5: the spread that drives the savings and the price that sets the cost.

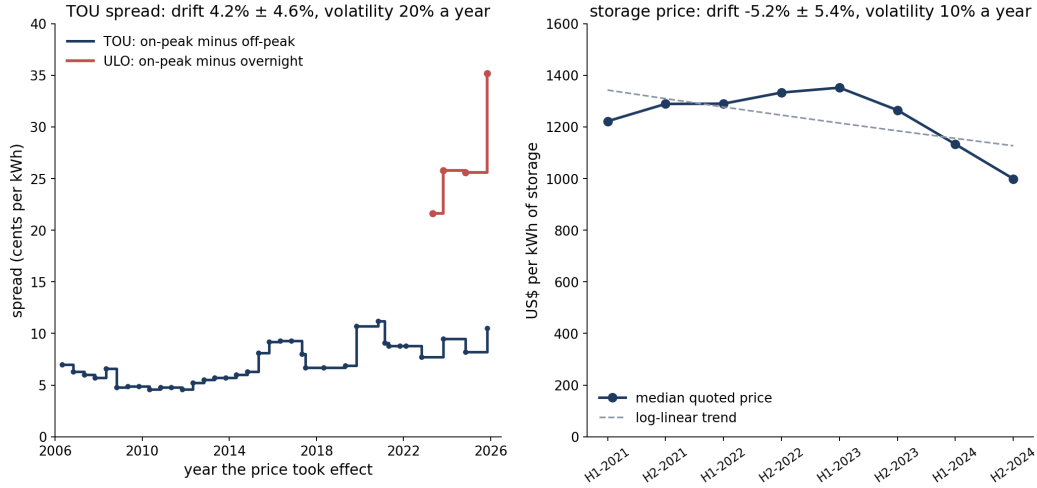


Figure 5: Left: the on-peak minus off-peak spread of the time-of-use plan since 2006 and the on-peak minus overnight spread of the ultra-low overnight plan since 2023, in cents per kilowatt-hour, at each date the prices changed; right: the median quoted price of home storage per kilowatt-hour by half-year from 2021 to 2024, with its log-linear trend.

The TOU spread more than doubled from 2012 to 2020 in steps, fell back after 2020 and rose again in November 2025; the ULO spread is three times as wide and jumped from 25.6 to 35.2 cents in November 2025. The storage price rose until the first half of 2023 and then fell by a quarter in eighteen months, which is what gives its trend a negative drift with a large standard error.

Script. `fig_entry.py`, which evaluates Theorem 4 over a range of drifts of the spread for four drifts of the installed cost, and the value of the right to install against the installed cost for two of the beliefs, from `compute_numbers.py`.

Shows. Where the installation boundary lies, how it moves with the drifts, and the smooth pasting that defines it.

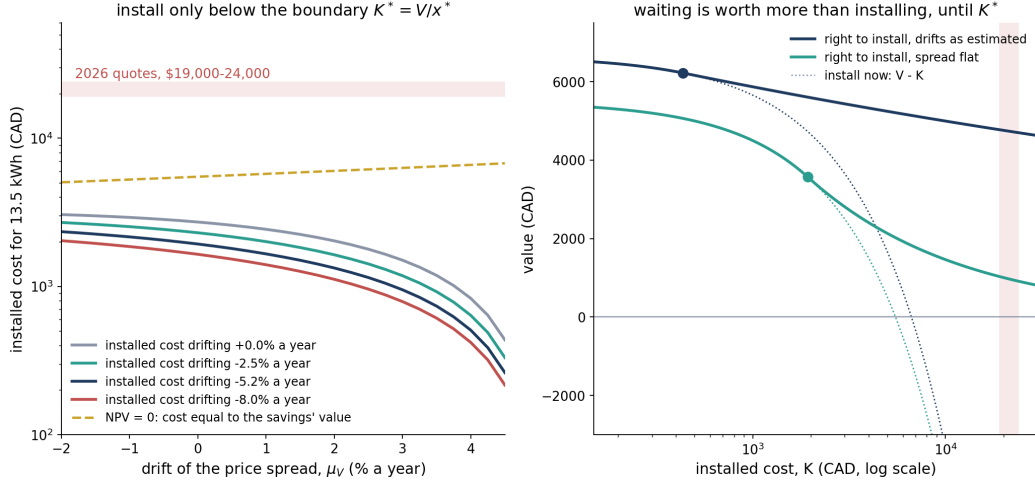


Figure 6: Left: the installed cost below which installing a 13.5 kWh battery is optimal, against the drift of the price spread, for four drifts of the installed cost, on a logarithmic scale, with the cost at which the net present value is zero and the band of 2026 quotes; right: the value of the right to install against the installed cost, for the drifts as estimated and with the spread flat, with the value of installing at once as dotted lines and the boundaries marked.

On the left the boundary lies between a few hundred and three thousand dollars for every belief, an order of magnitude below the quotes and below the cost at which the net present value turns positive; it falls steeply as the spread's drift approaches the discount rate. On the right the right to install exceeds the value of installing at every cost above the boundary and meets it tangentially there; at today's quotes, the band on the right, installing is worth $-\$15,000$ to $-\$16,000$, while waiting is worth $\$966$ to $\$4,736$.

Example 1 (The numbers for this house). A 13.5 kWh battery delivering up to 5 kW with 90% round trip, installed for $\$21,500$, the middle of the 2026 quotes, on this house's usage at today's prices, discounted at 5%, with the spread expected to stay where it is.

Parameters. $B = 13.5$, $\bar{p} = 5$, $\eta = 0.90$, $w = 0$, $s = 4.72$ cents; $K_0 = \$21,500$; $r = 5\%$; $\mu_V = 0$, $\sigma_V = 20.4\%$, $\mu_K = -5.2\%$, $\sigma_K = 10.1\%$, $\rho = 0$.

Computation. Theorem 1 says to fill the battery every night and Theorem 3 to hold back 11.9 kWh during the morning mid-peak in summer, 7.4 in spring and autumn and 11.4 in winter: in summer and winter the morning may use only the top 1.6 to 2.1 kWh. Run that way the battery saves $\$757$ a year off the bill, $\$751$ in its first year as it starts to fade and $\$634$ in its tenth. At 5% those savings are worth $V_0 = \$5,499$, so $x_0 = V_0/K_0 = 0.256$ and the net present value of installing today is $-\$16,001$. The quadratic of Theorem 4 with $\delta_V = 0.05$ and $\delta_K = 0.102$ has $\beta = 1.542$, so $x^* = 2.85$ and

$$K^* = \frac{V_0}{x^*} = \frac{5,499}{2.85} = \$1,933, \quad (6.1)$$

or $\$143$ per kilowatt-hour of capacity against $\$1,593$ in the quotes. The installed cost must fall by a factor of 11 before installing is optimal, and with $\ln X$ drifting up at 3.7% a year

the boundary is expected in 65 years.

Shows. The battery is operated well by a single number per season, and bought by none of the rules: it fails the payback rule by a factor of 2.9, the net-present-value rule by a factor of 3.9, and the optimal rule by a factor of 11.

7. Discussion

Remark 5 (What a household on this plan can do). A household that already has a battery should set the reserve its controller keeps back by season, to the values of Theorem 3, and leave the battery idle at weekends if it counts wear above 5 cents a kilowatt-hour. A household without one should not buy one at today's quotes to use on this plan alone.

Most home batteries let their owners set a reserve that the battery will not discharge below, and a seasonal reserve of about 12 kWh in summer and winter and 7.5 in spring and autumn is all Theorem 3 asks for; it captures 94% of what the battery could earn with perfect knowledge of each day. The household can capture the same spread without any battery by moving loads: every kilowatt-hour of laundry, dishwashing or water heating moved from the peak to the night saves $(c_P - c_O) \times 0.895 = 31.5$ cents, with no losses and no equipment, and households do shift use when the peak is priced: Faruqui and Sergici [26] survey fifteen experiments in which they did. The province's battery rebate applies only to batteries sized to new solar panels, and a battery charged by its own panels is a different investment, outside this paper; on the grid alone the battery's savings are worth a quarter of its price.

Remark 6 (What one house and fifteen months cannot show). The measurements are those of a single house over 462 days, with one winter, read hour by hour; the drifts behind the installation boundary are estimated from 19 years of regulated prices and four years of quoted battery prices.

Another house would have other reserves: Theorem 3 applies to any household, but its numbers depend on the tail of that household's evening, and a house heated electrically, or with an electric car charged in the evening, would protect more and save more. One winter is a thin basis for the winter reserve, as the out-of-sample test shows, and the hourly readings hide the minutes in which the power limit binds. The installation boundary rests on drifts that the data cannot pin down, which is why it is reported under four beliefs, and on no correlation between the spread and the installed cost, which cannot be estimated from eight half-years. The paper also leaves out what a battery may be worth beyond the bill: power during an outage, which some households value highly, and any future payment for exporting to the grid or for serving it at times of stress. A household that values those enough could rationally install where this paper says not to; the paper measures the part of the case that a bill can show.

8. Conclusion

A home battery on Ontario's Ultra-Low Overnight plan solves a revenue-management problem every weekday. Its stored energy can be sold to the house twice: cheaply in the

morning, which comes first, and dearly in the evening, whose size is uncertain. The answer is Littlewood's: fill the battery every night, and in the morning spend it only down to a reserve at which the chance that the evening needs the last kilowatt-hour, weighed by the extra the evening pays, balances the morning fare given up. On a battery the rule has two amendments, a second cheap window after 9 PM and the cost of buying each kilowatt-hour back overnight, and both raise the reserve.

For the house studied here the reserve is about 12 kWh of 13.5 in summer and winter and 7.4 in spring and autumn, and it earns \$757 a year off the bill, 94% of the best any policy could do with perfect knowledge of each day, and 92% out of sample. Holding a reserve is worth \$191 a year; the amendments to the textbook rule are worth \$11.

The installation question has a sharper answer. The battery's savings over its warranted life are worth about \$5,500 against an installed cost of \$19,000 to \$24,000, so it fails even the payback rule. The optimal rule, from McDonald and Siegel's variational inequality with a random spread and a falling cost, is stricter still: install only when the savings are worth x^* times the cost, which for this house means an installed cost between \$430 and \$2,700, decades away on present trends. A battery is worth running well, and on this plan alone, not yet worth buying.

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