

TECHNICAL NOTE

Lee's Moment Formula in the Heston Model

Explosion Times, Wing Slopes, and How Slowly the Smile Reaches Them

Supervised by Dr. T. Zamrik

15-DEC-2025



Sample paths of stochastic logistic growth: $dX_t = r X_t(1 - X_t/K) dt + \sigma X_t dW_t$, $r = 7.77$, $K = 1$, $\sigma = 0.28$, $X_0 = 0.03$



TN-2026-59564509
zamrik.com



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Suggested Citation

Derivatives & Volatility Desk, Zamrik Quant Research (2025). 'Lee's Moment Formula in the Heston Model: Explosion Times, Wing Slopes, and How Slowly the Smile Reaches Them'. *Zamrik Quant Research Technical Note TN-2026-59564509*. Available at <https://zamrik.com/research-items/tn-2026-59564509/>.

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Lee's Moment Formula in the Heston Model

Explosion Times, Wing Slopes, and How Slowly the Smile Reaches Them

Supervised by Dr. T. Zamrik¹

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Abstract. Lee's moment formula fixes the slopes of the implied total variance at extreme strikes through the largest finite moments of the price. In the Heston model these follow from the moment-explosion time, a closed form with three branches that we write in one convention, so the wing slopes β_R and β_L are explicit at every maturity. As the maturity falls the critical moments grow like $1/T$ and the ratio of the two slopes tends to $\arccos \rho / (\pi - \arccos \rho)$; as it grows they tend to the roots of a quadratic at the rate T^{-2} , and the slopes to closed forms in the ratio $(1 - \rho)/(1 + \rho)$. For an equity-like set ($v_0 = \theta = 0.04$, $\kappa = 1.5$, $\xi = 0.5$, $\rho = -0.7$, $T = 1$), $\beta_R = 0.024222$ and $\beta_L = 0.085741$. Damped Fourier prices down to 4.9×10^{-70} , agreeing with 34-digit arithmetic to 8.5×10^{-14} , show how slowly the smile gets there: at a strike 2981 times the forward the call wing is still 9.16 per cent above its slope, and at $1/2981$ of the forward the put wing is 16.2 per cent above. Lee's slope bounds the wings; it describes no quoted strike.

TN-2026-59564509 · Derivatives & Volatility Desk · Technical Note · Published
15-DEC-2025 · Reading time: about 17 min

1. Introduction

Far from the money, the implied volatility of a model with finite moments cannot grow without limit. Lee [1] proved that the implied total variance $V(x) = \sigma_{\text{imp}}^2(x)T$, as a function of the log-moneyness $x = \log(K/F)$, grows at most linearly, with a slope in $[0, 2]$ on each side fixed by how many moments of the price are finite. Practitioners use the bound to shape parametric smiles: raw SVI, studied by Gatheral and Jacquier [11], is linear in both wings with slopes $b(1 \pm \rho)$, and a smile is often extended beyond its last quote along a straight line in V . Both uses assume that the slope is nearly reached where it is applied. For the Heston model we compute the slopes exactly and measure how far the smile is from them at strikes up to e^8 times the forward.

In the Heston model a moment of the price is finite only up to some maturity: for $u > 1$ or $u < 0$, $\mathbb{E}[S_T^u]$ is finite for $T < T^*(u)$ and infinite after. Andersen and Piterbarg [2] identified $T^*(u)$ with the blow-up time of a Riccati equation; Keller-Ressel [7] studied explosion times and critical moments for affine stochastic volatility models, including long

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maturities; and Benaim and Friz [4] showed how a known moment generating function turns Lee's upper limit into a limit. The critical moments are therefore the roots of $T^*(u) = T$, and through Lee's formula they are all the slopes need.

Lee's theorem and the explosion time are known; we carry out the step between them with every branch and check it. We write $T^*(u)$ in one convention and show that its logarithmic branch needs $\kappa < \rho\xi$ (Proposition 1). We prove that the critical moments grow like $1/T$ at short maturity, so that the slope ratio tends to $\arccos \rho / (\pi - \arccos \rho)$ (Proposition 2, Corollary 3), and that at long maturity they reach the roots of a quadratic at the rate T^{-2} , with slopes in the ratio $(1 - \rho)/(1 + \rho)$ (Proposition 3). Then we measure how far the smile is from its slopes. Friz, Gerhold, Gulisashvili and Sturm [8] refined the wing expansion beyond Lee's term; we report only the measured rate.

2. The Model and the Wings

Definition 1 (The Heston model). Under a pricing measure $\mathbb{P}$ the price S_t and its variance v_t satisfy [5]

$$dS_t = S_t \sqrt{v_t} dW_t^1, \quad dv_t = \kappa(\theta - v_t) dt + \xi \sqrt{v_t} dW_t^2, \quad d\langle W^1, W^2 \rangle_t = \rho dt,$$

with $S_0 = 1$, initial variance v_0 , mean reversion κ , long-run variance θ , vol-of-vol ξ and correlation ρ . Rates are zero, so the forward is $F = 1$, and $X_T = \log S_T$ is the log-price at maturity T .

Zero rates cost nothing: the wings depend only on moments of S_T/F .

Assumption 1 (Standing hypotheses). The parameters of Definition 1 satisfy $v_0, \kappa, \theta, \xi > 0$, $-1 < \rho < 1$ and $\kappa > \rho\xi$; expectations are under $\mathbb{P}$.

Under the share measure $S_T d\mathbb{P}$ the variance mean-reverts at rate $\kappa - \rho\xi$, so the last condition says it still does; it holds whenever $\rho \leq 0$. Proposition 1 does not need it. No Feller condition is needed.

Definition 2 (The wings). For a strike K , $x = \log(K/F)$ is the log-moneyness and $V(x) = \sigma_{\text{imp}}(x)^2 T$ the implied total variance, $\sigma_{\text{imp}}(x)$ being the Black volatility of the out-of-the-money option at K (the call for $x \geq 0$, the put for $x < 0$). The critical exponents are

$$\tilde{p} = \sup\{p \geq 0 : \mathbb{E}[S_T^{1+p}] < \infty\}, \quad \tilde{q} = \sup\{q \geq 0 : \mathbb{E}[S_T^{-q}] < \infty\},$$

and the wing slopes are $\beta_R = \limsup_{x \rightarrow +\infty} V(x)/x$ and $\beta_L = \limsup_{x \rightarrow -\infty} V(x)/|x|$.

A slope $\beta > 0$ means the implied volatility grows like $\sqrt{\beta|x|/T}$.

Theorem 1 (Lee's moment formula [1]). Let $S_T > 0$ with $\mathbb{E}[S_T] = 1$, and let $\psi(\beta) = (2 - \beta)^2 / (8\beta)$, a decreasing bijection of $(0, 2]$ onto $[0, \infty)$, with $\psi^{-1}(\infty) = 0$. Then $\beta_R, \beta_L \in [0, 2]$ and

$$\beta_R = \psi^{-1}(\tilde{p}) = 2 - 4(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p}) = 2(\sqrt{\tilde{p} + 1} - \sqrt{\tilde{p}})^2, \quad \beta_L = \psi^{-1}(\tilde{q}).$$



Proof. Let $c(x) = \mathbb{E}[(S_T - e^x)^+]$ and $c_B(x, V)$ Black's call, increasing in V . (a) For $p > 0$, $\mathbb{E}[S_T^{1+p}] = p(1+p) \int_{\mathbb{R}} e^{px} c(x) dx$, from $S^{1+p} = p(1+p) \int_0^\infty K^{p-1} (S - K)^+ dK$. (b) $(S - K)^+ \leq S^{1+p} K^{-p}$, so $c(x) \leq \mathbb{E}[S_T^{1+p}] e^{-px}$. (c) With $a = x/\sqrt{V} - \sqrt{V}/2$, $b = a + \sqrt{V}$ and R the Mills ratio, $c_B(x, V) = \varphi(a)[R(a) - R(b)]$; for $V = \beta x$, $0 < \beta < 2$, this gives $\log c_B(x, \beta x) = -\psi(\beta)x + O(\log x)$. First, $V(x) < 2x$ eventually, as $c_B(x, 2x) \rightarrow \frac{1}{2}$ while $c(x) \rightarrow 0$. If $p < \tilde{p}$ and $V(x_n) \geq \beta x_n$ along $x_n \rightarrow \infty$ with $\beta \in (\psi^{-1}(p), 2)$, then (b) and (c) force $\psi(\beta) \geq p$, a contradiction; so $\beta_R \leq \psi^{-1}(\tilde{p})$. If $\beta_R < \beta < 2$, then $c(x) \leq c_B(x, \beta x)$ for large x , and (a) gives $\mathbb{E}[S_T^{1+p}] < \infty$ for $p < \psi(\beta)$; so $\tilde{p} \geq \psi(\beta)$ and $\psi^{-1}(\tilde{p}) \leq \beta_R$. The left wing is the right wing of $1/S_T$ under $dQ = S_T dP$, since $\mathbb{E}_{\mathbb{Q}}[S_T^{-1-q}] = \mathbb{E}[S_T^{-q}]$ and the put at x is e^x times the $\mathbb{Q}$ -call on $1/S_T$ at $-x$, with the same implied variance. $\square$

The proof follows [1]. We evaluate $\psi^{-1}(r) = 2/(\sqrt{r+1} + \sqrt{r})^2$, which subtracts nothing; for large r , $\psi^{-1}(r) = 1/(2r) + O(r^{-2})$.

Remark 1 (When the upper limit is a limit). Benaïm and Friz [3] showed that Lee's upper limits are limits when the tail of $\log S_T$ varies regularly, and gave in [4] criteria on the moment generating function that cover the Heston model. There $V(x)/|x|$ converges, so its distance from β at a given strike has a rate, which Table 1 measures.

3. When the Moments Explode

Let $M(t, x, v) = \mathbb{E}[e^{uX_T} \mid X_t = x, v_t = v]$ for real u , where finite. By the Markov property it solves

$$\partial_t M - \frac{1}{2}v \partial_x M + \kappa(\theta - v) \partial_v M + \frac{1}{2}v \partial_{xx} M + \rho\xi v \partial_{xv} M + \frac{1}{2}\xi^2 v \partial_{vv} M = 0, \quad M(T, x, v) = e^{ux}. \quad (3.1)$$

With $M = \exp(ux + A(\tau) + B(\tau)v)$, $\tau = T - t$, the terms free of v and the coefficients of v vanish separately:

$$\frac{dB}{d\tau} = \frac{1}{2}\xi^2 B^2 + (\rho\xi u - \kappa)B + \frac{1}{2}(u^2 - u), \quad \frac{dA}{d\tau} = \kappa\theta B, \quad A(0) = B(0) = 0, \quad (3.2)$$

so $\mathbb{E}[S_T^u] = \exp(A(T) + B(T)v_0)$ while B stays finite. We write $b = \rho\xi u - \kappa$ and $\Delta = b^2 - \xi^2 u(u-1)$, the discriminant of the right-hand side as a quadratic in B . For $u > 1$ or $u < 0$ its constant term is positive and B rises from zero; the same equations at complex u give the characteristic function used for prices.

Proposition 1 (The explosion time). Let $v_0, \kappa, \theta, \xi > 0$, $-1 < \rho < 1$, and $u > 1$ or $u < 0$. Then $\mathbb{E}[S_T^u] < \infty$ exactly when $T < T^*(u)$, where

$$T^*(u) = \begin{cases} \infty, & \Delta \geq 0, \ b \leq 0, \\ \frac{1}{\sqrt{\Delta}} \log \frac{b + \sqrt{\Delta}}{b - \sqrt{\Delta}}, & \Delta \geq 0, \ b > 0, \\ \frac{2}{\sqrt{-\Delta}} \left(\frac{\pi}{2} - \arctan \frac{b}{\sqrt{-\Delta}} \right), & \Delta < 0, \end{cases}$$



the middle line read as $2/b$ when $\Delta = 0$. Under Assumption 1 the middle branch never occurs, and $T^*(u) = \infty$ exactly on $[u_-^\infty, 0) \cup (1, u_+^\infty]$, where $u_-^\infty < 0 < 1 < u_+^\infty$ are the roots of Δ in u .

Proof. The moment is finite exactly while B exists [2]. With $q(B) = \frac{1}{2}\xi^2 B^2 + bB + c$, $c = \frac{1}{2}(u^2 - u) > 0$, B increases from 0, stays bounded if q has a positive root, and otherwise explodes at $T^* = \int_0^\infty dB/q(B)$. The roots $(-b \pm \sqrt{\Delta})/\xi^2$ have positive product: for $\Delta \geq 0$ both are positive if $b \leq 0$ ($b = 0$ forces $\Delta < 0$), giving $T^* = \infty$, and both negative if $b > 0$, where partial fractions give the logarithm; for $\Delta < 0$, $q = \frac{1}{2}\xi^2[(B + b/\xi^2)^2 - \Delta/\xi^4]$ gives the arctangent. Under Assumption 1, Δ is concave in u with $\Delta(0) = \kappa^2$ and $\Delta(1) = (\rho\xi - \kappa)^2$ positive, so $\Delta \geq 0$ exactly on an interval $[u_-^\infty, u_+^\infty] \supset [0, 1]$. Where $b = 0$ with $u \notin [0, 1]$, $\Delta < 0$; as $b(0) = -\kappa$ and the zero $\kappa/(\rho\xi)$ of b exceeds 1 when $\rho > 0$, $b < 0$ on the interval. $\square$

Andersen and Piterbarg [2] state the result in their own parameterisation. We compute with the equivalent forms $2 \operatorname{artanh}(\sqrt{\Delta}/b)/\sqrt{\Delta}$ and $2 \operatorname{atan2}(\sqrt{-\Delta}, b)/\sqrt{-\Delta}$, which avoid cancellation as $\Delta \rightarrow 0$.

Corollary 1 (The critical moments). Under Assumption 1, for every $T > 0$ there are $u_+^*(T) > u_+^\infty$ and $u_-^*(T) < u_-^\infty$ with $T^*(u_\pm^*(T)) = T$, and $\mathbb{E}[S_T^u] < \infty$ exactly when $u_-^*(T) < u < u_+^*(T)$. $T \mapsto u_+^*(T)$ is continuous and strictly decreasing and $T \mapsto u_-^*(T)$ continuous and strictly increasing, with $u_\pm^*(T) \rightarrow \pm\infty$ as $T \rightarrow 0$.

Proof. On (u_+^∞, ∞) , $T^* = 2 \operatorname{atan2}(\sqrt{-\Delta}, b)/\sqrt{-\Delta}$ is real-analytic, tends to ∞ as $u \downarrow u_+^\infty$ (the angle tends to π , as $b < 0$) and to 0 as $u \rightarrow \infty$. For $1 < u < u'$ and $T < T^*(u')$, Jensen's inequality gives $\mathbb{E}[S_T^u] \leq \mathbb{E}[S_T^{u'}]^{u/u'} < \infty$, so T^* is non-increasing; were it constant on some $[u, u']$ it would, being analytic, be constant throughout, against its limits. So T^* decreases strictly from ∞ to 0 and $u_+^*(T)$ is its inverse at T . The left side is the same for $u' < u < 0$. Orders in $[u_-^\infty, u_+^\infty]$ have finite moments at every maturity, and at $u_\pm^*(T)$, where $T^* = T$, the moment is infinite by Proposition 1. $\square$

Keller-Ressel [7] studies explosion times and critical moments for affine models in general; here they are roots of a closed form (Figure 1).

Corollary 2 (The wing slopes of the Heston smile). Under Assumption 1, for every $T > 0$,

$$\beta_R(T) = 2 \left(\sqrt{u_+^*(T)} - \sqrt{u_+^*(T) - 1} \right)^2, \quad \beta_L(T) = 2 \left(\sqrt{1 - u_-^*(T)} - \sqrt{-u_-^*(T)} \right)^2,$$

both in $(0, 2)$ and strictly increasing in T , and $V(x)/|x|$ tends to them as $x \rightarrow \pm\infty$.

Proof. By Corollary 1, $\tilde{p} = u_+^*(T) - 1$ and $\tilde{q} = -u_-^*(T)$; apply Theorem 1. ψ^{-1} decreases and $\tilde{p}, \tilde{q}$ decrease in T , so the slopes increase. The limits are Remark 1. $\square$

For the equity-like set at $T = 1$ the slopes are $\beta_R = 0.024222$ and $\beta_L = 0.085741$ (Example 1), small against Lee's bound of 2.



Proposition 2 (Short maturity: critical moments of order $1/T$). Under Assumption 1, as $T \rightarrow 0$,

$$u_{\pm}^*(T) T = c_{\pm} + O(T), \quad c_+ = \frac{2 \arccos \rho}{\xi \sqrt{1 - \rho^2}}, \quad c_- = -\frac{2(\pi - \arccos \rho)}{\xi \sqrt{1 - \rho^2}}.$$

Proof. For large $|u|$,

$$-\Delta = \xi^2(1 - \rho^2)u^2 - (\xi^2 - 2\rho\xi\kappa)u - \kappa^2 > 0, \quad T^* = \frac{2 \operatorname{atan2}(\sqrt{-\Delta}, b)}{\sqrt{-\Delta}}.$$

As functions of $1/u$ near 0, the quotients

$$\frac{\sqrt{-\Delta}}{|u|} = \xi \sqrt{1 - \rho^2} + O(1/u), \quad \frac{b}{|u|} = \pm \rho \xi + O(1/u) \quad (u \rightarrow \pm\infty)$$

are analytic, and $\operatorname{atan2}$ ignores a positive factor, so

$$|u| T^*(u) = \frac{2 \operatorname{atan2}(\sqrt{1 - \rho^2}, \pm \rho)}{\xi \sqrt{1 - \rho^2}} + O(1/u).$$

Since $\operatorname{atan2}(\sqrt{1 - \rho^2}, \rho) = \arccos \rho$ and $\operatorname{atan2}(\sqrt{1 - \rho^2}, -\rho) = \pi - \arccos \rho$, $u T^*(u) = c_{\pm} + O(1/u)$. As $u_{\pm}^*(T) \rightarrow \pm\infty$ and $T^*(u_{\pm}^*(T)) = T$, $u_{\pm}^*(T) T = c_{\pm} + O(T)$. $\square$

For $\rho = -0.7$, $\xi = 0.5$: $c_+ = 13.141314$ and $c_- = -4.455125$ (Table 2).

Corollary 3 (Short maturity: the implied-variance slopes and their ratio). Under Assumption 1, as $T \rightarrow 0$,

$$\frac{\beta_R(T)}{T} \rightarrow \frac{\xi \sqrt{1 - \rho^2}}{4 \arccos \rho}, \quad \frac{\beta_L(T)}{T} \rightarrow \frac{\xi \sqrt{1 - \rho^2}}{4(\pi - \arccos \rho)}, \quad \frac{\beta_L(T)}{\beta_R(T)} \rightarrow \frac{\arccos \rho}{\pi - \arccos \rho}.$$

Proof. $r \psi^{-1}(r) \rightarrow \frac{1}{2}$ as $r \rightarrow \infty$, and by Proposition 2, $\tilde{p}T \rightarrow c_+$ and $\tilde{q}T \rightarrow |c_-|$. So $\beta_R/T = \tilde{p} \psi^{-1}(\tilde{p})/(\tilde{p}T) \rightarrow 1/(2c_+)$, likewise $\beta_L/T \rightarrow 1/(2|c_-|)$, and the ratio tends to $c_+/|c_-|$. $\square$

As $V/T = \sigma_{\text{imp}}^2$, β/T is the wing slope of the variance smile ($|x| \rightarrow \infty$ first). It stays finite as $T \rightarrow 0$, and the ratio depends on ρ alone: 1 at $\rho = 0$, 2.9497 at $\rho = -0.7$.

Proposition 3 (Long maturity: the critical moments tend to the roots of a quadratic). Under Assumption 1 let $u_-^{\infty} < 0 < 1 < u_+^{\infty}$ be the roots of

$$\Delta(u) = \xi^2(\rho^2 - 1)u^2 + (\xi^2 - 2\rho\xi\kappa)u + \kappa^2,$$

let $b_{\pm} = \rho\xi u_{\pm}^{\infty} - \kappa < 0$ and $D = \sqrt{(\xi^2 - 2\rho\xi\kappa)^2 + 4\xi^2(1 - \rho^2)\kappa^2}$, and let $q = (2\kappa - \rho\xi)/(\xi\sqrt{1 - \rho^2})$. As $T \rightarrow \infty$,

$$u_{\pm}^*(T) = u_{\pm}^{\infty} \pm \frac{4\pi^2}{D(T + 2/|b_{\pm}|)^2} (1 + O(T^{-2})),$$



and the wing slopes tend to

$$\beta_R(\infty) = 2\sqrt{\frac{1+\rho}{1-\rho}}(\sqrt{q^2+1}-q), \quad \beta_L(\infty) = 2\sqrt{\frac{1-\rho}{1+\rho}}(\sqrt{q^2+1}-q),$$

so that $\beta_L(\infty)/\beta_R(\infty) = (1-\rho)/(1+\rho)$.

Proof. For u just beyond u_+^∞ let $s = \sqrt{-\Delta(u)}$; as $b < 0$, $T^*(u) = 2(\pi - \arctan(s/|b|))/s = 2\pi/s - 2/|b| + O(s^2)$. At $u = u_+^*(T)$ this gives $s = 2\pi(1 + O(T^{-3}))/ (T + 2/|b|)$, and $s^2 = D(u - u_+^\infty)(1 + O(u - u_+^\infty))$, D being $|d\Delta/du|$ at either root, while $b = b_+ + O(T^{-2})$; the rate follows, and likewise at u_-^∞ . For the slopes put $u_+^\infty = \cosh^2 a$, $u_-^\infty = -\sinh^2 c$. At a root with $b < 0$, $\kappa/\xi = \rho u + \sqrt{u(u-1)}$, so $\rho \cosh 2a + \sinh 2a = -\rho \cosh 2c + \sinh 2c = 2\kappa/\xi - \rho$. With $\rho = \tanh g$ this reads $\sinh(2a+g) = \sinh(2c-g) = q$, so $2a+g = 2c-g = \operatorname{arsinh} q$, and $a > 0$ is $\kappa > \rho\xi$. By Corollary 2, $\beta_R(\infty) = 2(\cosh a - \sinh a)^2 = 2e^{-2a}$ and $\beta_L(\infty) = 2e^{-2c}$; with $e^g = \sqrt{(1+\rho)/(1-\rho)}$ and $e^{-\operatorname{arsinh} q} = \sqrt{q^2+1}-q$ the formulas follow, and the ratio is e^{-2g} . $\square$

Keller-Ressel [7] treats the long-maturity limit for affine models in general. For the equity-like set $u_+^\infty = 11.703875$, $u_-^\infty = -1.507796$, $D = 1.684488$, $b_+ = -5.5964$, $b_- = -0.9723$, and $\beta_R(\infty) = 0.044650$, $\beta_L(\infty) = 0.253015$, in the ratio 5.6667, independent of κ and ξ .

4. Computing the Wings

For real α with $\mathbb{E}[S_T^{1+\alpha}] < \infty$, Carr and Madan [9] invert the transform of $e^{\alpha x}c(x)$. Exchanging integrals,

$$\begin{aligned} \int_{\mathbb{R}} e^{(\alpha+iv)x} \mathbb{E}[(S_T - e^x)^+] dx &= \mathbb{E}\left[\int_{-\infty}^{X_T} e^{(\alpha+iv)x} (e^{X_T} - e^x) dx\right] \\ &= \frac{\mathbb{E}[S_T^{\alpha+1+iv}]}{(\alpha+iv)(\alpha+1+iv)}, \quad \alpha > 0, \end{aligned} \quad (4.1)$$

and the put, with inner integral over (X_T, ∞) of $e^{(\alpha+iv)x}(e^x - e^{X_T})$, gives the same for $\alpha < -1$. Hence

$$\text{price}(x) = \frac{e^{-\alpha x}}{\pi} \int_0^\infty \operatorname{Re}\left[e^{-ivx} \frac{\mathbb{E}[S_T^{\alpha+1+iv}]}{(\alpha+iv)(\alpha+1+iv)}\right] dv, \quad (4.2)$$

the call for $\alpha \in (0, u_+^*(T) - 1)$, the put for $\alpha \in (u_-^*(T) - 1, -1)$: the strips of Corollary 1. The moment function at complex order is evaluated in the little-trap form discussed by Lord and Kahl [6], with principal branches, and checked against the integrated complex Riccati equation.

For each strike we take the α minimising $\log \mathbb{E}[S_T^{1+\alpha}] - \alpha x - \log(\alpha(\alpha+1))$, the integrand at $v = 0$; there its phase is stationary, so it does not oscillate before it decays. Over 74 strikes the integral of the modulus exceeds the modulus of the integral by at most 1.78: no digits are lost. A formula that forms the price from terms of order one cannot do this. With $\alpha = -\frac{1}{2}$, between the poles, the integral returns $c(x) - 1$; adding 1 back gives a



relative error of 1.1×10^{-7} at $x = 1$, returns 0 for 3.3×10^{-34} at $x = 4$, and 7.8×10^{-16} for 6.4×10^{-68} at $x = 8$.

With $k = |x|$, $s = \sqrt{V}$, $a = k/s - s/2$ and $b = k/s + s/2$, the out-of-the-money Black call is

$$c_B(k, s) = N(-a) - e^k N(-b) = \varphi(a) [R(a) - R(b)],$$

$$R(z) = \frac{1 - N(z)}{\varphi(z)} = \sqrt{\frac{\pi}{2}} \operatorname{erfcx}(z/\sqrt{2}), \quad (4.3)$$

and the put at $x < 0$ is $e^x c_B(k, s)$. We solve

$$-\frac{1}{2}a^2 - \frac{1}{2} \log(2\pi) + \log[R(a) - R(b)] + \min(x, 0) = \log(\text{price}) \quad (4.4)$$

for s by Brent's method. The exponential smallness sits in $-a^2/2$, computed and never exponentiated. The only subtraction, $R(a) - R(b)$, cancels by a factor near $(k + V/2)/V$, at most 38.4 here, since $V(x)/|x| \geq 0.0264$: fewer than two digits. Jäckel [10] inverts Black's formula to full accuracy for all inputs; only the far wings are needed here, where the logarithmic form is well conditioned, $\partial_s \log c_B$ being of order k^2/s^3 . Against a 50-digit inversion of the reference prices the implied variances agree to 2.4×10^{-15} at all 74 strikes.

Algorithm 1 — Critical moments, slopes and the wing smile

Inputs: $v_0, \kappa, \theta, \xi, \rho$ with $\kappa > \rho\xi$; a maturity T ; log-strikes x . *Output:* $u_{\pm}^*(T)$, $\beta_R(T)$, $\beta_L(T)$ and $V(x)$ at each strike.

1. Compute the roots $u_-^\infty < 0 < 1 < u_+^\infty$ of $\Delta(u)$ (Proposition 3).
2. Solve $T^*(u) = T$ by Brent's method on (u_+^∞, ∞) and on $(-\infty, u_-^\infty)$, with T^* from Proposition 1 in its $\operatorname{atan2}$ and artanh forms: $u_{\pm}^*(T)$.
3. Set $\beta_R = 2/(\sqrt{u_+^*} + \sqrt{u_+^* - 1})^2$ and $\beta_L = 2/(\sqrt{1 - u_-^*} + \sqrt{-u_-^*})^2$.
4. For each x choose α in $(0, u_+^* - 1)$ if $x \geq 0$, in $(u_-^* - 1, -1)$ if $x < 0$, minimising $\log \mathbb{E}[S_T^{1+\alpha}] - \alpha x - \log(\alpha(\alpha + 1))$.
5. Evaluate the damped integral by 32-point Gauss-Legendre rules on panels of width $\min(\frac{1}{4}, 1/(4\sigma_\alpha))$, σ_α^2 the second derivative of $\log \mathbb{E}[S_T^{1+\alpha}]$, until the integrand falls below 10^{-22} of its value at zero; keep the price as a logarithm.
6. Solve the logarithmic Black equation for s by Brent's method; $V(x) = s^2$.

Cost. Steps 1 to 3 take about $60 \mu\text{s}$ per maturity; a price and its inversion 0.05 s at $x = 8$, against about 4 s for the 34-digit reference. The verification, 134 checks, runs in 33 s on 32 processes.

5. Verification

Method. Proposition 1 against the Riccati equation integrated directly, in the angle $B = \tan \phi$: $d\phi/d\tau = \frac{1}{2}\xi^2 \sin^2 \phi + b \sin \phi \cos \phi + \frac{1}{2}(u^2 - u) \cos^2 \phi$, $\phi(0) = 0$, smooth through the blow-up, integrated by an eighth-order Runge-Kutta method (relative tolerance 10^{-13})



until $\phi = \pi/2$ or $\tau = 2000$. Critical moments were also found as roots of the integrated equation.

Parameters. The equity-like set at 22 orders in $[-50, 200]$; a set with $\kappa < \rho\xi$ ($\kappa = 0.1$, $\xi = 1$, $\rho = 0.9$) at 8 orders, the only way to reach the logarithmic branch.

Result. The 23 finite explosion times agree with the closed form to a relative 1.8×10^{-12} at worst, across all three branches, e.g. $T^*(-2) = 4.922772$ and $T^*(-4) = 1.446423$. At the 7 orders with $T^* = \infty$ the angle settles at the arctangent of the smaller root, never reaching $\pi/2$. The roots of the integrated equation at $T = 1$ agree with u_+^* and u_-^* to 6.5×10^{-13} and 4.7×10^{-14} .

Error. The worst relative error is of the size of the integrator's tolerance. A scan of 8000 orders finds no logarithmic branch for the equity-like or the positive-correlation set, as Proposition 1 requires, and 226 for the third set.

Method. Each price by Algorithm 1 in double precision and by the same integral in mpmath at 34 digits (tanh-sinh on panels of width 2, truncated at 10^{-38} of the integrand at zero); the reference was rerun at 44 digits on panels of width 1 at $x = \pm 8$. A second damping, with the integrand at $v = 0$ a hundred times larger, and put-call parity across the two strips test the formula itself.

Parameters. The equity-like set at $T = 1$, $x = \pm 0.25, \pm 0.5, \dots, \pm 8.25$, and the set of Example 2 at $x = \pm 1, \pm 2, \pm 4, \pm 8$: 74 prices from 4.9×10^{-70} to 1.6×10^{-2} .

Result. The worst relative gap between double precision and the 34-digit price is 8.5×10^{-14} , at $x = 7.25$; the two references agree to 7×10^{-34} in log-price. The second damping agrees to 1.4×10^{-13} , parity holds to 1.1×10^{-16} , and the little-trap moment function matches the complex Riccati equation to 7.5×10^{-13} at 40 damped points.

Error. Every price carries about thirteen correct digits, the smallest included.

Script. `fig_tstar.py`: Proposition 1 on a grid of orders with the integrated-Riccati test points, and Corollary 1 on 301 maturities from 0.01 to 1000.

Shows. Proposition 1 and Corollary 1 for the equity-like set.

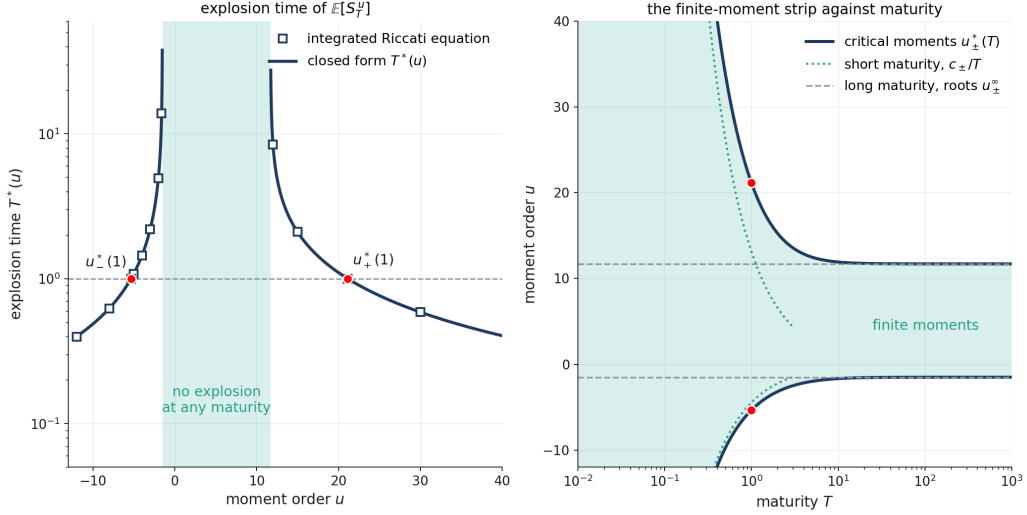


Figure 1: Left, the explosion time of the moment of order u against u on a logarithmic scale for the equity-like set, infinite on the shaded band, with integrated-Riccati points and the two critical moments at maturity one; right, the strip of finite moments against maturity with its short-maturity and long-maturity limits.

T^* is infinite on the band $[-1.5078, 11.7039]$ and meets $T = 1$ at $u_-^* = -5.3422$ and $u_+^* = 21.1454$. The strip follows $c_{\pm}/T$ at short maturity and at $T = 10$ is within 0.22 of the band.

Method. $V(x)/|x|$ from Algorithm 1 against β_R or β_L of Corollary 2; the observed order between $|x|$ and $2|x|$ is $\log_2$ of the ratio of successive gaps.

Parameters. The equity-like set at $T = 1$; $|x| = 0.25, 0.5, 1, 2, 4, 8$.

Result. Table 1. $V(x)/|x|$ decreases at every step on both wings and stays above its slope, as Theorem 1 and Corollary 2 require, at all 66 strikes of the grid.

Error. At $x = 8$, a strike 2981 times the forward, the call wing is still 9.16 per cent above $\beta_R = 0.024222$; at $x = -8$ the put wing is 16.2 per cent above $\beta_L = 0.085741$. The observed orders fall from 1.30 to 0.67 and from 0.80 to 0.57: each doubling of the distance gains less.

Source. Algorithm 1 at $T = 1$ for the equity-like set; prices checked against 34-digit arithmetic.

Columns. $|x|$; for each wing $V(x)/|x|$, its excess over $\beta_R = 0.024222$ or $\beta_L = 0.085741$, and the observed order against the previous row.



$ x $	call $V(x)/x$	above β_R	order	put $V(x)/ x $	above β_L	order
0.25	0.07510	210.1%	—	0.22735	165.2%	—
0.5	0.04486	85.2%	1.30	0.16693	94.7%	0.80
1	0.03462	42.9%	0.99	0.13520	57.7%	0.71
2	0.03008	24.2%	0.83	0.11726	36.8%	0.65
4	0.02775	14.6%	0.73	0.10646	24.2%	0.61
8	0.02644	9.16%	0.67	0.09966	16.2%	0.57

Script. `fig_wings.py`: $V(x)$ at the 64 grid strikes with $|x| \leq 8$, and the slopes of Corollary 2.

Shows. Theorem 1 and Corollary 2 at $T = 1$.

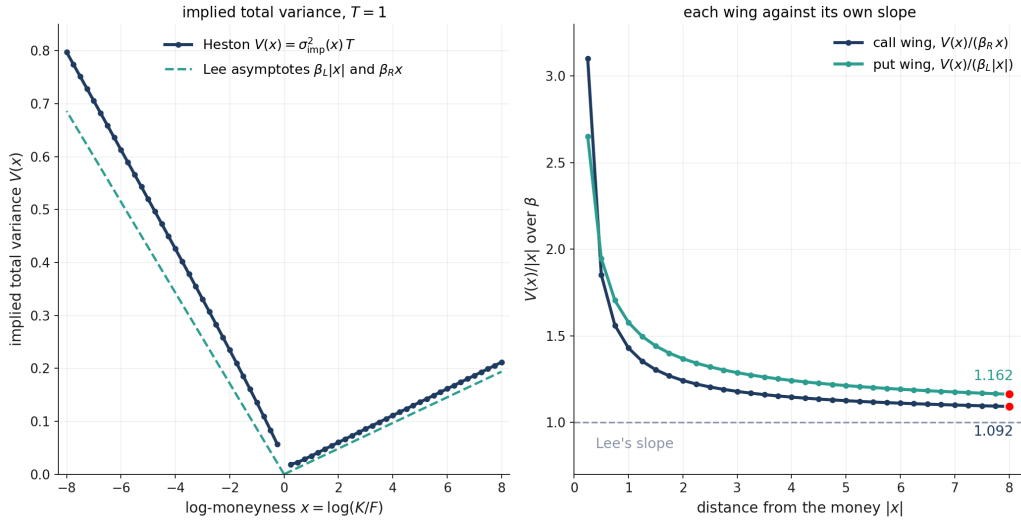


Figure 2: Left, the implied total variance of the Heston smile at maturity one on both wings against the two Lee asymptotes through the origin; right, each wing divided by its own Lee line, approaching one slowly from above.

The call wing runs nearly parallel to its asymptote: its local slope dV/dx is 4.3 per cent above β_R at $x = 1$, where V itself is 42.9 per cent above $\beta_R x$. On the put wing the local slope is still 18.6 per cent above β_L at $x = -1$ and 7.1 per cent at $x = -8$. At $|x| = 8$ the ratios are 1.092 and 1.162.

Remark 2 (The refined expansion). Friz, Gerhold, Gulisashvili and Sturm [8] extend Lee's leading term in the Heston model to a refined expansion at large strikes. We have not computed its coefficients, so the next term is not checked here. What we measured is the rate of Table 1, and its observed orders are still falling at $|x| = 8$: the asymptotic rate cannot be read off strikes up to e^8 times the forward, and a single power law fitted over that range would mislead.

Method. $u_{\pm}^*(T)$ by Algorithm 1 for T from 10^{-3} to 10^3 , against Propositions 2 and 3 and



Corollary 3, with observed orders per decade; at $T = 10^{-3}$ and $T = 100$ the roots were recomputed on the integrated Riccati equation.

Parameters. The equity-like set; $T = 1, 0.1, 0.01, 0.001$ and $T = 1, 10, 100, 1000$.

Result. Table 2. At $T = 10^{-3}$ the orders are 1.0006 and 1.0005; $\beta_R/T = 0.038029$ against 0.038048, $\beta_L/T = 0.112198$ against 0.112230, and $\beta_L/\beta_R = 2.950328$ against $\arccos \rho / (\pi - \arccos \rho) = 2.949707$, order 0.9999. At $T = 1000$, $u_{\pm}^* - u_{\pm}^{\infty} = \pm 2.34 \times 10^{-5}$, orders 1.997 and 1.984; the rate of Proposition 3 matches both to a relative 1.8×10^{-6} , an error falling with order 1.99 and 1.93, while without the shift $2/|b_{\pm}|$ it would be 0.07 and 0.41 per cent.

Error. The integrated-equation roots agree to 1.1×10^{-10} (relative) at $T = 10^{-3}$ and 1.8×10^{-14} (absolute) at $T = 100$. The closed forms for $\beta(\infty)$ match Lee's map at the roots of Δ to 4.4×10^{-16} on 22 parameter sets, and the slopes at $T = 1000$ to 2.1×10^{-6} and 1.2×10^{-5} .

Source. Algorithm 1, steps 1 to 3, for the equity-like set; limits from Propositions 2 and 3.

Columns. Left: short maturities and $u_{\pm}^*(T)T$. Right: long maturities and $u_{\pm}^*(T)$. Last row: the limits.

T	$u_{+}^*(T)T$	$u_{-}^*(T)T$	T	$u_{+}^*(T)$	$u_{-}^*(T)$
1	21.145373	-5.342205	1	21.145373	-5.342205
0.1	13.861067	-4.536058	10	11.919096	-1.669034
0.01	13.212389	-4.463130	100	11.706201	-1.510046
0.001	13.148412	-4.455924	1000	11.703898	-1.507820
limit	13.141314	-4.455125	limit	11.703875	-1.507796

6. How the Slopes Depend on the Parameters

Script. `fig_maps.py`: Corollary 2 at $T = 1$ on 73 correlations in $[-0.9, 0.9]$ and 46 vols-of-vol in $[0.1, 1]$, other parameters as in the equity-like set.

Shows. Corollary 2 across the two parameters that shape the wings.

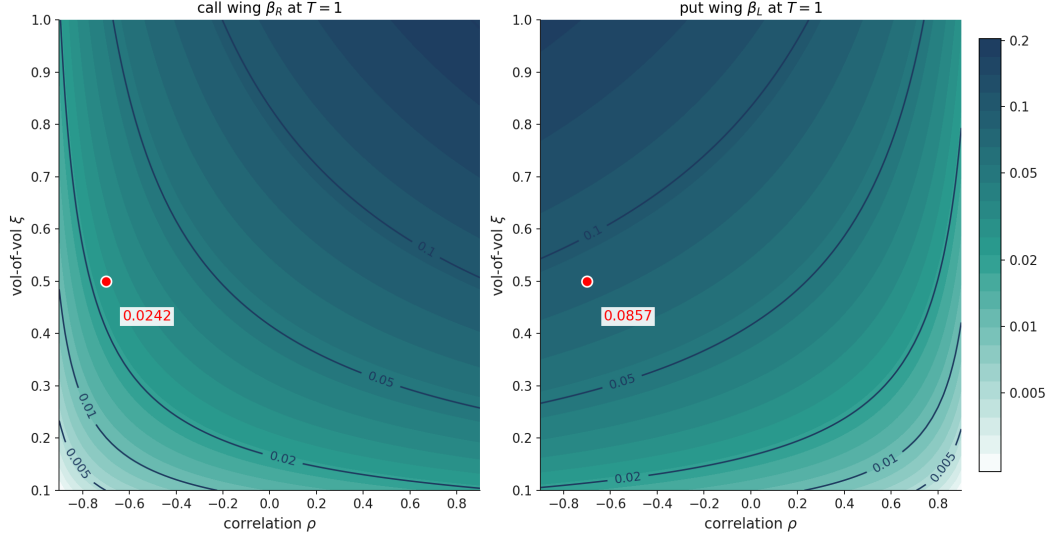


Figure 3: Lee's call-wing slope beta R on the left and put-wing slope beta L on the right at maturity one, as filled contour maps over correlation and vol-of-vol on a common logarithmic colour scale, with the equity-like set marked.

β_R ranges over $[0.0022, 0.2035]$ and β_L over $[0.0023, 0.1786]$. In every row β_R rises and β_L falls with ρ ; in every column both rise with ξ . At $\rho = 0$ they are equal (0.1198 at $\xi = 1$), as Δ and b are then symmetric under $u \mapsto 1 - u$.

Method. Corollary 2 on the maturity grid, against Corollary 3 and Proposition 3 [7].

Parameters. The equity-like set; $T = 0.25, 0.5, 1, 2, 5$.

Result. β_R rises from 0.008416 to 0.04175 and β_L from 0.02612 to 0.2030, towards 0.044650 and 0.253015; the ratio β_L/β_R rises from 3.104 to 4.862, between its limits 2.9497 and 5.6667. The variance slopes β/T fall, from 0.0337 to 0.00835 (call) and 0.1045 to 0.0406 (put).

Error. At $T = 5$ the slopes are 6.5 and 19.8 per cent below their limits, although the critical moments are within 0.78 and 0.48 of $u_{\pm}^{\infty}$. Lee's map explains it: $|\mathrm{d} \log \psi^{-1}(r)/\mathrm{d}r| = 1/\sqrt{r(r+1)}$ is 0.083 at $\tilde{p} = 11.48$ and 0.410 at $\tilde{q} = 1.99$, and times the distances gives 6.5 and 19.7 per cent.

Example 1 (An equity-like smile, wing by wing). Take $v_0 = \theta = 0.04$, $\kappa = 1.5$, $\xi = 0.5$, $\rho = -0.7$, $T = 1$. The band of Proposition 1 is $[-1.5078, 11.7039]$; $u_+^* = 21.145373$ and $u_-^* = -5.342205$, so $\beta_R = 0.024222$ and $\beta_L = 0.085741$: the put wing is 3.540 times as steep.

In volatility, Lee's asymptote at $x = 8$ is $\sqrt{8\beta_R} = 44.0$ per cent against the model's 46.0; at $x = -8$, 82.8 against 89.3; at $x = \pm 1$, 15.6 and 29.3 against 18.6 and 36.8. The calls at $x = 8$ cost 6.4×10^{-68} and the put at $x = -8$ 2.6×10^{-22} : far beyond any quote, still off the asymptotes.

Example 2 (Positive correlation). Take $\rho = +0.7$ in the equity-like set ($\kappa > \rho\xi = 0.35$ still). At $T = 1$, $u_+^* = 6.0452$ and $u_-^* = -18.4485$, so $\beta_R = 0.090352$ and $\beta_L = 0.026392$:



the call wing is 3.42 times as steep. Corollary 3 gives $\beta_L/\beta_R \rightarrow 0.3390$ as $T \rightarrow 0$ (0.33896 at $T = 10^{-3}$); Proposition 3 gives $\beta_R(\infty) = 0.31931$ and $\beta_L(\infty) = 0.05635$, ratio 0.17647.

$V(x)/|x|$ is 0.1457 at $x = 1$ against 0.0385 at $x = -1$, and 0.1059 at $x = 8$ against 0.0290 at $x = -8$. The sets are not mirror images: flipping ρ also changes $\kappa - \rho\xi$, so $u_+^* = 6.0452$ is not $1 - u_-^* = 6.3422$ of Example 1.

Remark 3 (Extrapolating a quoted smile). Raw SVI [11] has total-variance wing slopes $b(1 \pm \rho)$, and a common extrapolation continues a smile beyond its last quote at Lee's slope. In the Heston model at $T = 1$ no traded strike is near that slope: $V(x)/x$ is 42.9 per cent above β_R at $x = 1$ and 9.16 per cent at $x = 8$. The local slope dV/dx is close early on the call wing (4.3 and 3.3 per cent above β_R at $x = 1$ and 8) but not on the put wing (18.6 and 7.1 per cent above β_L).

So a wing continued at slope β is nearly right on the call side and too flat on the put side, and a cap of β on $V(x)/|x|$ is violated by the model's own smile at every computed strike. At long maturity the Heston slopes stand in the ratio $(1 - \rho)/(1 + \rho)$, that of the two raw SVI slopes with the same ρ .

7. Conclusion

In the Heston model Lee's wing slopes follow from the critical moments, the roots of the closed-form explosion time $T^*(u) = T$, whose logarithmic branch needs $\kappa < \rho\xi$; for an equity-like set at one year $\beta_R = 0.024222$ and $\beta_L = 0.085741$. At short maturity the critical moments grow like $c_{\pm}/T$ and the variance-smile slopes have ratio $\arccos \rho/(\pi - \arccos \rho)$; at long maturity they reach the roots of a quadratic at rate T^{-2} , with slopes in the ratio $(1 - \rho)/(1 + \rho)$. Fourier prices down to 4.9×10^{-70} , accurate to 8.5×10^{-14} , show the smile approaching slowly: 9.16 per cent above β_R at a strike 2981 times the forward and 16.2 per cent above β_L at $1/2981$ of it. Lee's slope bounds the wings; it describes no quoted strike.



References

- [1] Lee, R. W. (2004). The moment formula for implied volatility at extreme strikes. *Mathematical Finance*. <https://doi.org/10.1111/j.0960-1627.2004.00200.x>
- [2] Andersen, L. B. G., & Piterbarg, V. V. (2006). Moment explosions in stochastic volatility models. *Finance and Stochastics*.
<https://doi.org/10.1007/s00780-006-0011-7>
- [3] Benaim, S., & Friz, P. (2009). Regular variation and smile asymptotics. *Mathematical Finance*. <https://doi.org/10.1111/j.1467-9965.2008.00354.x>
- [4] Benaim, S., & Friz, P. (2008). Smile asymptotics II: models with known moment generating functions. *Journal of Applied Probability*.
<https://doi.org/10.1239/jap/1208358948>
- [5] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*.
<https://doi.org/10.1093/rfs/6.2.327>
- [6] Lord, R., & Kahl, C. (2010). Complex logarithms in Heston-like models. *Mathematical Finance*. <https://doi.org/10.1111/j.1467-9965.2010.00416.x>
- [7] Keller-Ressel, M. (2010). Moment explosions and long-term behavior of affine stochastic volatility models. *Mathematical Finance*.
<https://doi.org/10.1111/j.1467-9965.2010.00423.x>
- [8] Friz, P., Gerhold, S., Gulisashvili, A., & Sturm, S. (2011). On refined volatility smile expansion in the Heston model. *Quantitative Finance*.
<https://doi.org/10.1080/14697688.2010.541486>
- [9] Carr, P., & Madan, D. (1999). Option valuation using the fast Fourier transform. *The Journal of Computational Finance*. <https://doi.org/10.21314/JCF.1999.043>
- [10] Jäckel, P. (2015). Let's be rational. *Wilmott*. <https://doi.org/10.1002/wilm.10395>
- [11] Gatheral, J., & Jacquier, A. (2013). Arbitrage-free SVI volatility surfaces. *Quantitative Finance*. <https://doi.org/10.1080/14697688.2013.819986>



Zamrik Quant Research

The research arm of Zamrik Quant Lab

About the Desk

Derivatives & Volatility Desk

Zamrik Quant Research

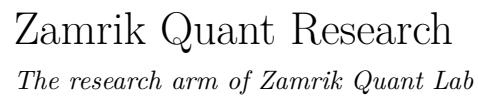
Options, volatility and rates: pricing, hedging and the free boundary.

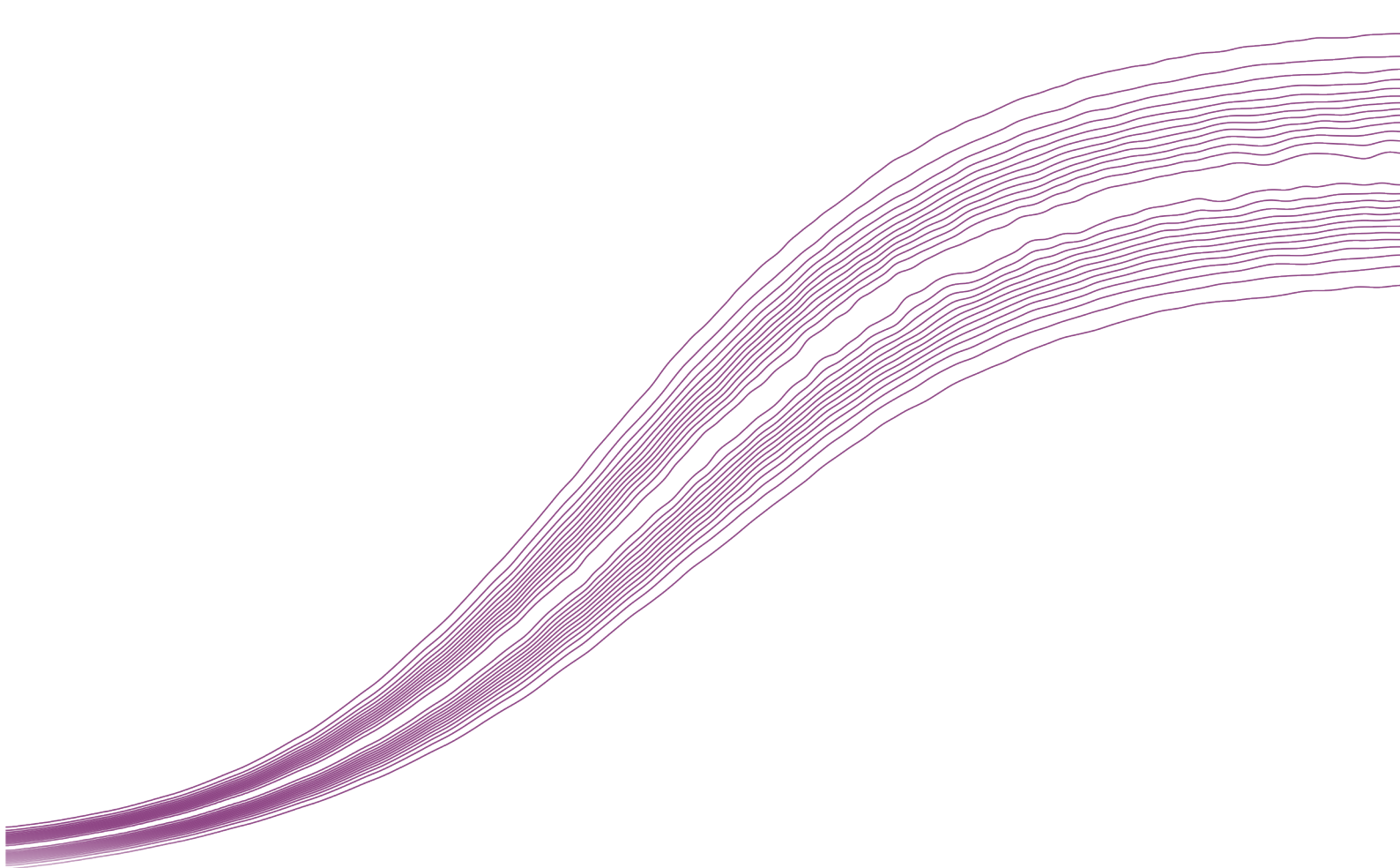
A derivative is a promise whose value depends on a path nobody controls. The Derivatives & Volatility desk works out what such promises are worth, how to hedge them, and when their holder should act.

The foundations come first: arbitrage, risk-neutral measures and the entropy view of them, and the benchmark approach, which prices with the growth-optimal portfolio instead. From there the desk takes on contracts whose exercise is itself a decision — the Russian option on a running maximum, and the Israeli option that either side may cancel — solved two independent ways, by Hamilton–Jacobi–Bellman equations and by reflected backward SDEs, checked against each other, and extended to ambiguity about the drift.

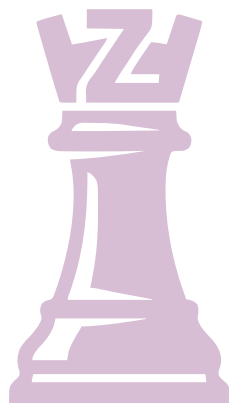
Volatility is modelled as a random process in its own right, including Wishart stochastic volatility for several assets at once, linked to the order flow that drives it. Rates are treated in the Heath–Jarrow–Morton framework, through bond prices and the derivatives written on forward rates. The desk also asks how much risk a position should carry at all, through the Kelly criterion and the growth-optimal portfolio, and its essays explain products that surprise their owners, such as leveraged funds whose long-run return falls short of their multiple.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Fokker–Planck density of stochastic logistic growth: $\partial_t p = -\partial_x [r x(1 - x/K) p] + \frac{1}{2} \sigma^2 \partial_{xx} (x^2 p)$



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