

TECHNICAL NOTE

The Distribution of Growing Perpetuities and the Gordon Growth Formula

Supervised by Dr. T. Zamrik

22-JUL-2023

Matérn RBF interpolant: $s(\mathbf{x}) = \sum_{i=1}^n \lambda_i \varphi(\|\mathbf{x} - \mathbf{x}_i\|)$, $A\lambda = f$, $\varphi(r) = (1 + \sqrt{5}\varepsilon r + \frac{5}{3}\varepsilon^2 r^2) e^{-\sqrt{5}\varepsilon r}$, $n = 39$



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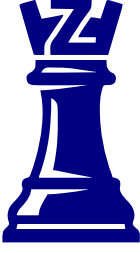
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The Distribution of Growing Perpetuities and the Gordon Growth Formula

Supervised by Dr. T. Zamrik¹

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Abstract. A cash flow growing at expected rate g with volatility σ and discounted at $r > g$ for ever is worth $S_0/(r - g)$ by the Gordon growth formula. That number is the mean of the present value, not its typical size. The present value is an exponential functional of a drifted Brownian motion, and Dufresne's identity gives its law exactly: $2S_0/(\sigma^2\gamma_\nu)$, with γ_ν a Gamma variable of shape $\nu = 1 + 2(r - g)/\sigma^2$. This one number fixes the shape. The mean is the Gordon value at every volatility; the median, as a share of it, is 0.689 at $r - g = 3$ per cent and $\sigma = 20$ per cent, and 0.495 at $\sigma = 30$ per cent. The variance is finite only while $\sigma^2 < 2(r - g)$. At those two settings 70.0 and 77.4 per cent of outcomes fall below the Gordon value, a share that exceeds one half for every ν and tends to one as volatility grows. A ten-year model's terminal value has a median of 0.584 of its mean, and a simulation free of the identity reproduces its quantiles within 0.43 per cent.

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1. Introduction

Every discounted-cash-flow valuation ends the same way. Beyond an explicit horizon the cash flow is assumed to grow at a constant rate g , it is discounted at $r > g$, and the rest of its life is priced at $S_0/(r - g)$, with S_0 the current rate of the flow: the formula of Gordon and Shapiro [5]. It prices terminal values, which carry 74 per cent of the expected value in the ten-year model of this note, and indexed liabilities.

The formula is exact only for the expected flow, $\int_0^\infty e^{-rt} \mathbb{E}[S_t] dt$, so its single number is a mean. Whether a mean is a fair summary depends on the law of the present value, which the formula does not show. This note writes that law down for geometric growth and measures how far the typical outcome sits from the quoted number.

With geometric growth the present value is an integral over an infinite horizon of the exponential of a drifted Brownian motion: an exponential functional of Brownian motion, with an exact law. Dufresne [1] found it in 1990, studying perpetuities in risk theory and pension funding: the integral is a constant over a Gamma variable. Yor [2] studied the

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same functionals at a finite horizon and recovered the identity, and Geman and Yor [4] tied them to Bessel processes, Asian options and perpetuities.

In valuation, where the integral is the terminal value or the liability, the identity is seldom quoted. Its Gamma law has one shape parameter, $\nu = 1 + 2(r - g)/\sigma^2$, the discount spread measured against half the variance of growth, and ν decides what the Gordon formula leaves out: where the median sits, which moments exist, and how much of the law lies below the quoted value.

Neither the identity nor the route of its proof, time reversal and the stationary law of a diffusion, is new here; Matsumoto and Yor [3] survey both. The note adds a reading, an organisation and measurements.

The reading is that the Gordon value is the mean of an inverse-gamma law and nothing more. The organisation is that every quantity depends on (r, g, σ) only through ν : the median and quantiles as shares of the Gordon value, the loss of the variance at $\sigma^2 = 2(r - g)$, and a probability of falling below the Gordon value that exceeds one half at every ν , proved by reflection about the mode. The measurements are a table over nine settings; a simulation of the functional that does not use the identity and confirms it; a simulator that closes the tail beyond a horizon exactly; and the terminal value of a ten-year model, whose median is not the product of its two factors' medians.

2. The Perpetuity

Definition 1 (The growing cash flow and its perpetuity). Let W be a standard Brownian motion on a filtered probability space satisfying the usual conditions [6]. A cash flow paid continuously at the rate

$$S_t = S_0 \exp((g - \frac{1}{2}\sigma^2)t + \sigma W_t), \quad t \geq 0,$$

has initial rate S_0 , expected growth rate g and volatility σ : $\mathbb{E}[S_t] = S_0 e^{gt}$. At a constant discount rate r its present value, the perpetuity, and its shape are

$$P = \int_0^\infty e^{-rt} S_t dt, \quad \nu = 1 + \frac{2(r - g)}{\sigma^2}.$$

The Gordon value is $S_0/(r - g)$, and $P(r - g)/S_0$ is the present value as a share of it.

The shape compares the spread $r - g$ with half the variance: ν is large when growth is nearly certain and close to one when volatility swamps the spread. At $r - g = 0.03$ it is 7, 2.5 and 5/3 at $\sigma = 0.10, 0.20$ and 0.30 .

Assumption 1 (Standing hypotheses). $S_0 > 0$, $\sigma > 0$ and $r > g$, so that $\nu > 1$; the rates r and g and the volatility σ are constants.

The condition $r > g$ is that of Gordon's formula. Theorem 1 needs only $\nu > 0$, that is $r > g - \frac{1}{2}\sigma^2$: the perpetuity is then finite almost surely, but for $\nu \leq 1$ its mean is infinite and there is no Gordon value.



Definition 2 (The gamma and inverse-gamma laws). For $\nu > 0$, γ_ν denotes a random variable with density

$$f_\nu(x) = \frac{x^{\nu-1}e^{-x}}{\Gamma(\nu)}, \quad x > 0,$$

the Gamma law of shape ν and unit scale, with median $m(\nu)$ and quantile function Q_ν . Its reciprocal $1/\gamma_\nu$ is inverse-gamma, with density $x^{-\nu-1}e^{-1/x}/\Gamma(\nu)$, and

$$\mathbb{E}[\gamma_\nu^{-k}] = \frac{\Gamma(\nu-k)}{\Gamma(\nu)} \quad (k < \nu), \quad \mathbb{E}[\gamma_\nu^{-k}] = \infty \quad (k \geq \nu);$$

in particular $\mathbb{E}[1/\gamma_\nu] = 1/(\nu-1)$ for $\nu > 1$.

The moments are the Gamma integral $\int_0^\infty x^{\nu-k-1}e^{-x}dx$, finite exactly when $\nu > k$. The inverse-gamma tail is a power law, $\mathbb{P}(1/\gamma_\nu > x) \sim x^{-\nu}/\Gamma(\nu+1)$, so the shape is also the tail exponent.

Lemma 1 (The present value is a rescaled exponential functional). Under Assumption 1, P/S_0 has the law of $(4/\sigma^2)A_\infty$, where

$$A_\infty = \int_0^\infty e^{2(\beta_u - \nu u)} du$$

and β is a standard Brownian motion; A_∞ is finite almost surely.

Proof. By Definition 1, $e^{-rt}S_t/S_0 = \exp(\sigma W_t - (r - g + \frac{1}{2}\sigma^2)t)$. By Brownian scaling [6], $\beta_u = \frac{1}{2}\sigma W_{4u/\sigma^2}$ is a standard Brownian motion with $\sigma W_t = 2\beta_{\sigma^2 t/4}$. Substituting $u = \sigma^2 t/4$, $dt = (4/\sigma^2)du$, the exponent becomes $2\beta_u - (r - g + \frac{1}{2}\sigma^2)4u/\sigma^2 = 2(\beta_u - \nu u)$, so $P/S_0 = (4/\sigma^2)A_\infty$ path by path. As $\beta_u/u \rightarrow 0$ almost surely [6] and $\nu > 0$, the integrand is eventually below $e^{-\nu u}$, and $A_\infty < \infty$. $\square$

The clock $u = \sigma^2 t/4$ measures time in units of variance. At $\sigma = 0.20$ one unit is a century, over which the exponent falls by $2\nu = 5$ on average, so present values level off slowly (Figure 1).

3. The Law

Theorem 1 (Dufresne's identity). For $\nu > 0$, A_∞ has the law of $1/(2\gamma_\nu)$. Under Assumption 1, therefore,

$$P \stackrel{d}{=} \frac{2S_0}{\sigma^2\gamma_\nu}.$$

Proof. Put $X_s = \beta_s - \nu s$ and $A_t = \int_0^t e^{2X_s} ds$. (i) Fix t . The process $\hat{X}_s = X_t - X_{t-s}$, $0 \leq s \leq t$, has the law of $(X_s)_{0 \leq s \leq t}$, and $\hat{X}_t = X_t$; substituting $s \mapsto t-s$, $A_t = \int_0^t e^{2(\hat{X}_t - \hat{X}_s)} ds$, so A_t has the law of $Z_t = \int_0^t e^{2(X_t - X_s)} ds$. (ii) $Z_t = e^{2X_t} \int_0^t e^{-2X_s} ds$; Itô's formula, $de^{2X_t} = e^{2X_t}(2d\beta_t + 2(1-\nu)dt)$, and the product rule give

$$dZ_t = (1 + 2(1-\nu)Z_t)dt + 2Z_t d\beta_t, \quad Z_0 = 0,$$



with generator $\mathcal{L}f(z) = 2z^2 f''(z) + (1 + 2(1 - \nu)z)f'(z)$. (iii) A stationary density with zero flux solves $(2z^2 p)' = (1 + 2(1 - \nu)z)p$. For $p(z) = z^{-\nu-1}e^{-1/(2z)}$,

$$(2z^2 p)' = (2(1 - \nu)z^{-\nu} + z^{-\nu-1})e^{-1/(2z)} = (1 + 2(1 - \nu)z)p,$$

and p , integrable for $\nu > 0$, is proportional to the density of $1/(2\gamma_\nu)$ by Definition 2. (iv) A_t increases to $A_\infty \in (0, \infty)$ almost surely, so by (i) Z_t converges in law to the law μ of A_∞ . For f smooth with compact support in $(0, \infty)$, Itô's formula gives $\mathbb{E}f(Z_t) = \int_0^t \mathbb{E}\mathcal{L}f(Z_s) ds$, since $f(0) = 0$ and the stochastic integral has a bounded integrand; as $\mathcal{L}f$ is bounded and continuous, dividing by t and letting $t \rightarrow \infty$ gives $\int \mathcal{L}f d\mu = 0$. Integrated once, the stationary equation holds for μ with a constant flux c , whose solutions are $(K + cs(z))p(z)$ with the scale function $s(z) = \int_1^z y^{\nu-1}e^{1/(2y)} dy$. As $s(0+) = -\infty$ and $s(\infty) = \infty$, positivity forces $c = 0$: μ has density proportional to p . Lemma 1 gives the law of P . $\square$

The identity is Dufresne's [1]; Yor [2] gave another proof, and Matsumoto and Yor [3] survey the subject. Time reversal turns the perpetuity into the stationary law of a diffusion.

The mean needs neither the identity nor the scaling. The integrand is non-negative, so Tonelli's theorem exchanges expectation and integral, and by Definition 1

$$\mathbb{E}[P] = \int_0^\infty e^{-rt} \mathbb{E}[S_t] dt = S_0 \int_0^\infty e^{-(r-g)t} dt = \frac{S_0}{r-g}. \quad (3.1)$$

The volatility drops out in the first step, where $\mathbb{E}[e^{\sigma W_t}] = e^{\sigma^2 t/2}$ cancels the $-\frac{1}{2}\sigma^2 t$ in the exponent. Theorem 1 agrees: $\mathbb{E}[2S_0/(\sigma^2 \gamma_\nu)] = 2S_0/(\sigma^2(\nu - 1)) = S_0/(r - g)$. The same step gives a second moment without the identity: writing A_∞^2 as twice the integral over $s < t$ of $e^{2X_s + 2X_t}$, with $X_u = \beta_u - \nu u$, independent increments give

$$\mathbb{E}[A_\infty^2] = 2 \int_0^\infty e^{(8-4\nu)s} ds \int_0^\infty e^{(2-2\nu)v} dv = \frac{1}{4(\nu-1)(\nu-2)} \quad (\nu > 2), \quad (3.2)$$

which the computation section uses for an error bar. The mean cannot see the volatility; the rest of the law depends on it.

Corollary 1 (The Gordon value is the mean). Under Assumption 1, $\mathbb{E}[P] = S_0/(r - g)$ for every $\sigma > 0$, and the present value as a share of the Gordon value has the law

$$\frac{P(r - g)}{S_0} \stackrel{d}{=} \frac{\nu - 1}{\gamma_\nu}.$$

Its median is $(\nu - 1)/m(\nu)$ and its q -quantile $(\nu - 1)/Q_\nu(1 - q)$. Its moment of order k is finite exactly when $k < \nu$; the variance is finite exactly when $\sigma^2 < 2(r - g)$, and then $\text{Var}(P) = (S_0/(r - g))^2/(\nu - 2)$. Each of these depends on (r, g, σ) only through ν .

Proof. By Theorem 1 and $\sigma^2(\nu - 1) = 2(r - g)$, the share has the law of $2(r - g)/(\sigma^2 \gamma_\nu) = (\nu - 1)/\gamma_\nu$. As $x \mapsto (\nu - 1)/x$ decreases, the q -quantile of the share is $\nu - 1$ over the $(1 - q)$ -quantile of γ_ν . Definition 2 gives the moments: the mean is $(\nu - 1)/(\nu - 1) = 1$, the order- k moment is finite exactly for $k < \nu$, and $\mathbb{E}[(\nu - 1)^2 \gamma_\nu^{-2}] = (\nu - 1)/(\nu - 2)$, so the share has variance $1/(\nu - 2)$; and $\nu > 2$ exactly when $\sigma^2 < 2(r - g)$. $\square$



At $r - g = 0.03$ the median share is 0.900, 0.689 and 0.495 at $\sigma = 0.10, 0.20$ and 0.30 , and the variance is lost at $\sigma^* = \sqrt{2(r - g)} = 0.245$. A power tail of exponent ν holds the mean at one.

Corollary 2 (Most outcomes fall below the Gordon value). Under Assumption 1,

$$\mathbb{P}\left(P < \frac{S_0}{r - g}\right) = \mathbb{P}(\gamma_\nu > \nu - 1) > \frac{1}{2}$$

for every $\nu > 1$. With $r - g$ fixed, the probability tends to $\frac{1}{2}$ as $\sigma \rightarrow 0$ and to 1 as $\sigma \rightarrow \infty$.

Proof. By Corollary 1, $(\nu - 1)/\gamma_\nu < 1$ exactly when $\gamma_\nu > \nu - 1$. Put $a = \nu - 1 > 0$, the mode of f_ν . For $0 < x < a$ and $u = x/a$,

$$\log \frac{f_\nu(a + x)}{f_\nu(a - x)} = a \log \frac{1 + u}{1 - u} - 2x = 2a(\operatorname{artanh} u - u) > 0,$$

so $\mathbb{P}(a < \gamma_\nu < 2a) > \mathbb{P}(0 < \gamma_\nu < a)$; hence $\mathbb{P}(\gamma_\nu > a) > \mathbb{P}(\gamma_\nu < a)$, and the first exceeds one half. As $\sigma \rightarrow \infty$, $\nu \downarrow 1$ and $\mathbb{P}(\gamma_\nu \leq a) \leq \int_0^a x^{\nu-1} dx / \Gamma(\nu) = a^\nu / \Gamma(\nu + 1) \rightarrow 0$. As $\sigma \rightarrow 0$, $\nu \rightarrow \infty$; γ_ν is a sum of $\lfloor \nu \rfloor$ independent unit exponentials and an independent Gamma variable of shape below one, so $(\gamma_\nu - \nu)/\sqrt{\nu}$ tends in law to a standard normal and $\mathbb{P}(\gamma_\nu > \nu - 1) \rightarrow \frac{1}{2}$. $\square$

The corollary does not claim monotonicity. On 6001 values of ν from $1 + 10^{-6}$ to 50 the probability decreases strictly in ν , from 0.999999 to 0.538, so it rises with volatility at a fixed spread.

Source. Corollaries 1 and 2, evaluated with SciPy's Gamma quantile and survival functions; every entry agrees with SciPy's separate inverse-gamma distribution to 10^{-10} .

Columns. The spread $r - g$, the volatility σ and the shape ν ; the median and the 10 and 90 per cent quantiles of the share $P(r - g)/S_0$; the probability of falling below the Gordon value; whether the variance is finite, $\sigma^2 < 2(r - g)$.

$r - g$	σ	ν	median	10 per cent	90 per cent	below Gordon	variance
0.02	0.10	5	0.856	0.500	1.644	0.629	finite
0.02	0.20	2	0.596	0.257	1.880	0.736	infinite
0.02	0.30	1.444	0.394	0.146	1.656	0.812	infinite
0.03	0.10	7	0.900	0.570	1.541	0.606	finite
0.03	0.20	2.5	0.689	0.325	1.863	0.700	finite
0.03	0.30	1.667	0.495	0.197	1.815	0.774	infinite
0.05	0.10	11	0.937	0.649	1.424	0.583	finite
0.05	0.20	3.5	0.788	0.416	1.765	0.660	finite
0.05	0.30	2.111	0.621	0.274	1.883	0.726	finite



Proposition 1 (The terminal value). Under Assumption 1, fix a horizon $T > 0$ and split $P = \int_0^T e^{-rt} S_t dt + V_T$. The terminal value is $V_T = e^{-rT} S_T P'$, where

$$P' = \int_0^\infty e^{-rs} \frac{S_{T+s}}{S_T} ds \stackrel{d}{=} \frac{2}{\sigma^2 \gamma_\nu}$$

is independent of $\mathcal{F}_T$. Hence $\mathbb{E}[V_T] = S_0 e^{-(r-g)T} / (r-g)$, and $V_T / \mathbb{E}[V_T] \stackrel{d}{=} LQ$ with $L = \exp(\sigma\sqrt{T}\xi - \frac{1}{2}\sigma^2 T)$, ξ standard normal, and $Q = (\nu-1)/\gamma_\nu$ independent of L . Its median is the root x of

$$\mathbb{E}\left[\Phi\left(\frac{\log(x/Q) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right] = \frac{1}{2},$$

with Φ the standard normal distribution function.

Proof. Substituting $t = T + s$, $V_T = e^{-rT} S_T P'$. The ratio $S_{T+s}/S_T = \exp((g - \frac{1}{2}\sigma^2)s + \sigma(W_{T+s} - W_T))$ is built from the increments of W after T , a Brownian motion independent of $\mathcal{F}_T$ by the Markov property, so P' is independent of S_T and has the law of P/S_0 , given by Theorem 1. Then $\mathbb{E}[V_T] = e^{-rT} S_0 e^{gT} / (r-g)$, the factor $e^{-rT} S_T / \mathbb{E}[e^{-rT} S_T] = \exp(\sigma W_T - \frac{1}{2}\sigma^2 T)$ has the law of L , and $(r-g)P'$ that of Q by Corollary 1. Given Q , $\mathbb{P}(LQ \leq x \mid Q) = \Phi((\log(x/Q) + \frac{1}{2}\sigma^2 T)/(\sigma\sqrt{T}))$. $\square$

At $T = 10$, $r = 0.08$, $g = 0.05$ and $\sigma = 0.20$ ($\nu = 2.5$) the terminal value carries $e^{-(r-g)T} = 0.741$ of the expected value. Its median is 0.584 of its mean, with 10 and 90 per cent quantiles at 0.188 and 2.060. Multiplying the two factors' medians, 0.819 and 0.689, would give 0.564. Geman and Yor [4] treat the perpetuity with the finite-horizon functional.

4. Computation

Algorithm 1 — The perpetuity by simulation, with the tail beyond the horizon closed exactly

Inputs: the spread $r-g$, the volatility σ , a horizon T , a step Δ with $N = T/\Delta$, the number of paths n , a seed. *Output:* for each path the explicit present value V_0 and the terminal value V_T , with $P/S_0 = V_0 + V_T$.

1. Set $D_0 = 1$, where $D_t = e^{-rt} S_t / S_0$, and $\nu = 1 + 2(r-g)/\sigma^2$.
2. For $k = 1, \dots, N$ set $\log D_{k\Delta} = \log D_{(k-1)\Delta} - (r-g + \sigma^2/2)\Delta + \sigma\sqrt{\Delta}\xi_k$, with ξ_k independent standard normals: exact at the grid points.
3. Integrate by the trapezoid rule: $V_0 = \Delta(D_0/2 + D_\Delta + \dots + D_{(N-1)\Delta} + D_T/2)$.
4. Draw γ from the Gamma law of shape ν , independent of the path, and set $V_T = 2D_T/(\sigma^2\gamma)$.
5. Return V_0 and V_T ; the share of the Gordon value is $(r-g)(V_0 + V_T)$.

Cost. $N+1$ random draws per path; at $T = 10$ and $\Delta = 0.02$ the 4.8 million paths of the nine settings of Table 1 take under a minute on one core.



Step 4 is Proposition 1: a tail cut at $T = 10$ would lose on average 74 per cent of the Gordon value at $r - g = 0.03$. The algorithm's numbers rely on Theorem 1, which the next check verifies without it.

Method. $A_U = \int_0^U e^{2(\beta_u - \nu u)} du$ simulated directly, with exact Gaussian increments of β on a grid of step h and the trapezoid rule, and the same paths integrated again at step $2h$; nothing uses Theorem 1. The horizon U makes the tail mean, $e^{2(1-\nu)U}/(2(\nu-1))$ by Tonelli, at most 10^{-9} , so by Markov's inequality the truncation moves a path by more than 10^{-4} with probability below 10^{-5} . The 10, 25, 50, 75 and 90 per cent quantiles are compared with those of $1/(2\gamma_\nu)$, with standard errors from twenty batches, and the whole sample by the Kolmogorov–Smirnov distance.

Parameters. $\nu = 1.5, 2.5, 7$ with $U = 21, 7, 2$; $h = 0.002$; 200 000 paths each; seeds 101 to 103.

Result. The medians are 0.42179, 0.23021 and 0.07487 against 0.42266, 0.22981 and 0.07497. Over the fifteen quantiles the largest relative error is 0.43 per cent and the largest deviation 2.62 standard errors; the Kolmogorov–Smirnov distances are 0.0018, 0.0017 and 0.0029, with p-values 0.54, 0.59 and 0.063.

Error. Halving the step from 0.004 moves every quantile by at most 0.042 per cent, 0.22 standard errors, while each path changes by 0.27 to 0.54 per cent in root mean square: a small random error per path, not a bias. The sample means at $\nu = 2.5$ and 7 lie 2.79 and -1.85 standard errors from the Tonelli mean $1/(2(\nu-1))$, with the standard error from the Tonelli second moment; at $\nu = 1.5$ the variance is infinite and the mean, 0.974, carries no valid error bar. Checking Lemma 1, $\sigma^2 P/(4S_0)$ from Algorithm 1 at $\sigma = 0.20$ and the direct A_U at $\nu = 2.5$ differ by a two-sample Kolmogorov–Smirnov distance of 0.0023, p-value 0.68.

Method. Algorithm 1, scored on the share $P(r-g)/S_0$ against 1. Where the variance is finite the standard error comes from the exact variance $1/(\nu-2)$: for $\nu \leq 4$ the fourth moment is infinite and the sample standard deviation is unreliable, usually too small. Where the variance is infinite the mean is reported, not scored, and the half moment $(\nu-1)^{1/2} \Gamma(\nu-\frac{1}{2})/\Gamma(\nu)$, whose estimator has finite variance, is checked.

Parameters. $r - g = 0.03$; $\sigma = 0.10, 0.20, 0.30$ ($\nu = 7, 2.5, 5/3$); $T = 10$, $\Delta = 0.02$; 400 000, 10^6 and 10^6 paths; seeds 204 to 206.

Result. The mean share is 0.99961 ± 0.00071 at $\sigma = 0.10$ and 0.99627 ± 0.00141 at $\sigma = 0.20$. At $\sigma = 0.30$ the running mean is 0.976, 0.957, 0.989 and 1.004 after 10^3 , 10^4 , 10^5 and 10^6 paths, and 64.6 per cent of thousand-path batch means fall below one, against 56.3 and 52.8 per cent at $\sigma = 0.20$ and 0.10; the batch means at $\sigma = 0.30$ run from 0.796 to 12.9.

Error. The scored means lie -0.55 and -2.64 standard errors from one; the sample standard error would put the second at -2.98 . The half moments, 0.97921, 0.92069 and 0.83900 against 0.97941, 0.92132 and 0.83908, lie -0.62 , -1.62 and -0.16 standard errors off. Small simulations of a skewed law usually under-report its mean.

Method. Algorithm 1: the share of paths below the Gordon value against $\mathbb{P}(\gamma_\nu > \nu - 1)$



from Corollary 2, with the binomial standard error, and the median share against Corollary 1.

Parameters. The nine settings of Table 1; $T = 10$, $\Delta = 0.02$; 400 000 paths, 10^6 at $r - g = 0.03$ with $\sigma = 0.20$ and 0.30 ; seeds 201 to 209.

Result. At $r - g = 0.03$ the shares below are 0.6072, 0.7001 and 0.7736 at $\sigma = 0.10$, 0.20 and 0.30, against 0.6063, 0.7000 and 0.7738; over the nine settings they run from 0.5835 to 0.8116.

Error. The largest deviation is 1.29 standard errors, at $r - g = 0.02$ and $\sigma = 0.30$, the largest absolute difference 0.00088; the simulated medians agree with Corollary 1 within 0.0013.

Method. Algorithm 1 at the horizon T , keeping the explicit part and V_T apart, against Proposition 1, whose integral equation is solved by quadrature in both orders (they agree to 10^{-9}). Quantiles carry batch standard errors; the mean, the standard error of the exact variance, $\text{Var}(V_T)/\mathbb{E}[V_T]^2 = e^{\sigma^2 T}(\nu - 1)/(\nu - 2) - 1 = 3.475$.

Parameters. $S_0 = 1$, $r = 0.08$, $g = 0.05$, $\sigma = 0.20$, $T = 10$, $\Delta = 0.02$, 10^6 paths, seed 205: the paths of the mean check at $\sigma = 0.20$.

Result. The terminal value's mean is 24.571 ± 0.046 against 24.694; its median 14.417 ± 0.018 against 14.420, which is 0.584 of the mean; its 10 and 90 per cent quantiles 4.661 and 50.660 against 4.651 and 50.859. Of the paths, 71.49 per cent end below the mean, against 71.49. The terminal value carries 0.7399 of the total expected value against $e^{-(r-g)T} = 0.7408$.

Error. The mean, median and two quantiles lie -2.68 , -0.15 , $+1.48$ and -2.30 standard errors from Proposition 1, the share of value -2.97 , the explicit ten years -0.32 . The mean, the share of value and the 90 per cent quantile lean low together: they read one sample's upper tail three ways.

Script. `fig_paths.py`: six of 400 paths at $r = 0.08$, $g = 0.05$, $\sigma = 0.20$, exact steps of 0.02 years to 600 years, chosen at the 5, 25, 45, 65, 85 and 97 per cent quantiles of their present values.

Shows. Left, the flows on a log scale with their mean $S_0 e^{gt}$; right, their running present values over a century, each limit (the present value to 600 years) a red point, the Gordon value dashed.

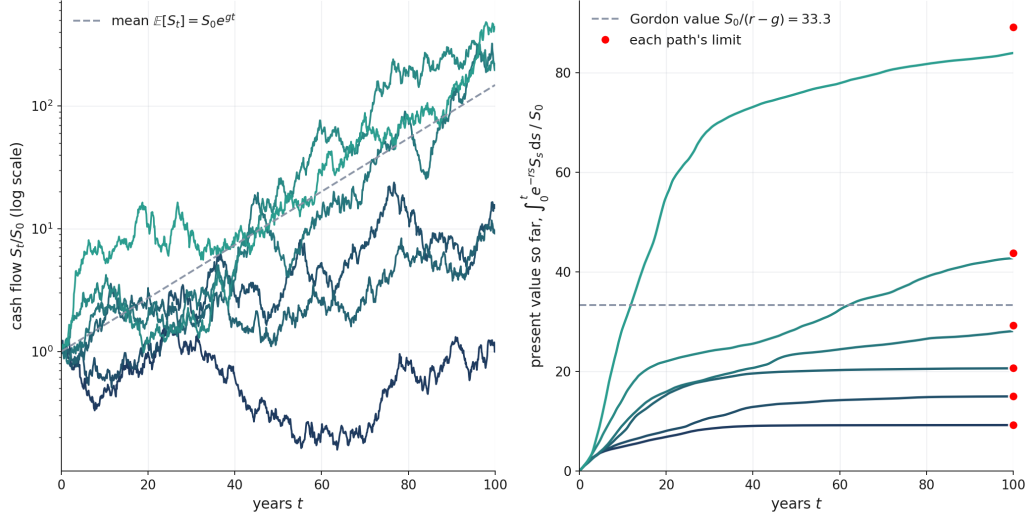


Figure 1: Six simulated cash flows growing at five per cent with twenty per cent volatility, and their running present values levelling off at different limits around the Gordon value.

The limits run from 9.26 to 89.10 times the initial rate, 0.28 to 2.67 of the Gordon value 33.3; four of the six end below it. By year 100 the paths have accrued 94.2 to 99.96 per cent of their limits, and the median path of the pool 99.4 per cent.

Script. `fig_density.py`: exact densities of $(\nu - 1)/\gamma_\nu$, and the histogram of the 10^6 shares simulated at $\sigma = 0.20$ by Algorithm 1.

Shows. The law of $P(r - g)/S_0$ at $r - g = 0.03$ and $\sigma = 0.10, 0.20, 0.30$, the mean dashed, each median a red point; the axis is clipped at a share of 3.

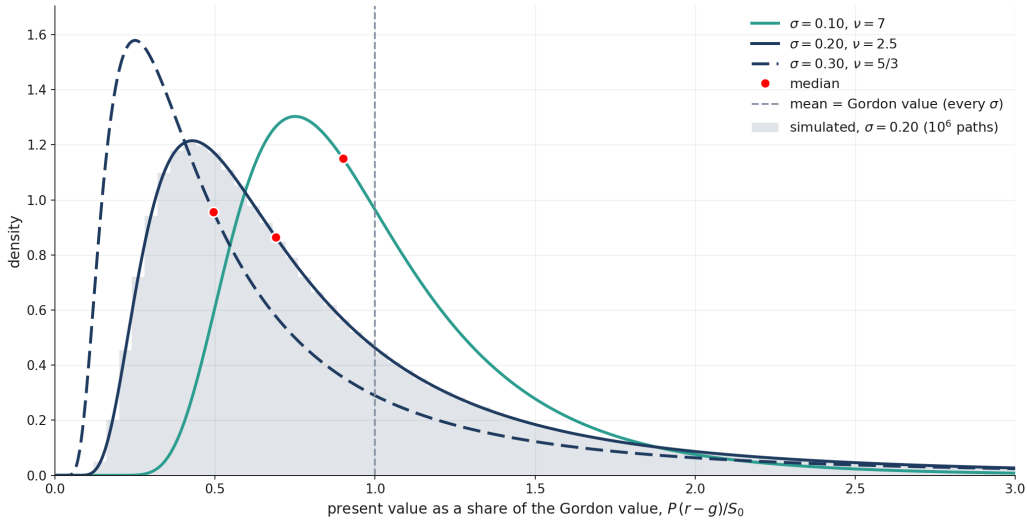


Figure 2: The density of the perpetuity as a share of the Gordon value at three volatilities, with each median marked, the common mean at one, and the axis clipped at three.

All three laws have mean one while their medians fall from 0.900 to 0.689 and 0.495. The clipped axis leaves out 0.45, 3.74 and 4.72 per cent of the mass, beyond three times the



Gordon value, and that thin, distant mass holds the mean at one. The $\sigma = 0.20$ histogram follows its curve.

Script. `fig_share.py`: exact median, 10 and 90 per cent quantiles and probability below the Gordon value on 351 volatilities from 0.05 to 0.40.

Shows. Left, the median and 10 to 90 per cent band at $r - g = 0.03$ (solid, shaded) and 0.05 (dashed), with $\sigma^* = \sqrt{2(r - g)}$ marked; right, the probability below the Gordon value.

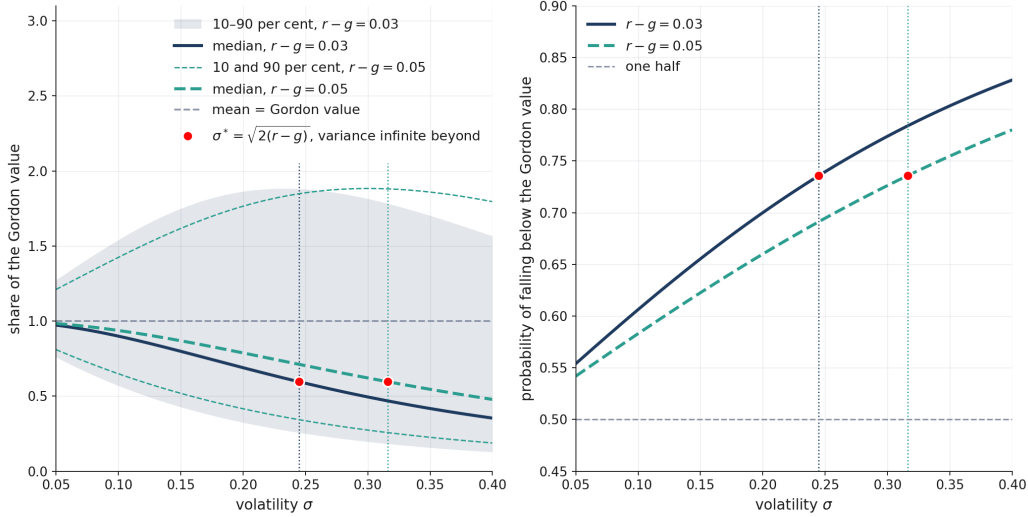


Figure 3: The median and the ten to ninety per cent band of the perpetuity as a share of the Gordon value against volatility, with the probability of falling below it.

At $r - g = 0.03$ the median falls from 0.973 to 0.354 across the range while the probability below rises from 0.554 to 0.828. The red points share their heights, a median of 0.596 and a probability of 0.736, at $\sigma^* = 0.245$ and 0.316: there $\nu = 2$ whatever the spread. The 90 per cent quantile peaks at 1.883 and falls back to 1.566 at $\sigma = 0.40$.

5. Discussion

Remark 1 (A terminal value or a pension liability). A terminal value or an indexed liability quoted at $S_0/(r - g)$ quotes the mean of a law skewed enough that most scenarios end below it. At a spread of 3 per cent and growth volatility of 20 to 30 per cent, the typical outcome is 69 to 49 per cent of the quoted number, 70 to 77 per cent of outcomes fall short of it, and at 30 per cent the variance is infinite. A ten-year model's terminal value has a median of 58 per cent of its own mean. Volatility does not move the mean and moves everything else.

The reading assumes a constant discount rate, geometric growth with constant volatility and no mean reversion; the flows of mature firms revert, which thins the upper tail. What survives is the structure: the Gordon value is an expectation, and its distance from the typical outcome is set by ν , through σ^2 against $2(r - g)$.



Remark 2 (What the note leaves out). Growth that reverts to a mean replaces the Brownian exponent by an Ornstein–Uhlenbeck one, and the inverse-gamma law is lost. Jumps lead to exponential functionals of Lévy processes, whose laws are known in closed form only in special cases; Bertoin and Yor survey them. A stochastic discount rate enters the exponent as a second process. The finite-horizon law, the integral to T without its tail, is Yor’s formula, which Matsumoto and Yor [3] survey and Geman and Yor [4] use for Asian options.

Proposition 1 needs only the infinite horizon, because the tail beyond any horizon is again a perpetuity, and Algorithm 1 uses the same fact to avoid truncation. Each extension keeps the question, how far the mean sits from the typical outcome, and changes the answer through the tail: thinner under mean reversion, heavier with jumps.

6. Conclusion

A growing cash flow discounted for ever is a perpetuity, an exponential functional of a drifted Brownian motion. Dufresne’s identity gives its law, $2S_0/(\sigma^2\gamma_\nu)$ with $\nu = 1 + 2(r - g)/\sigma^2$, and the Gordon value $S_0/(r - g)$ is its mean at every volatility. The rest depends on ν : as a share of the Gordon value the median is $(\nu - 1)/m(\nu)$, 0.689 at $r - g = 3$ per cent and $\sigma = 20$ per cent and 0.495 at $\sigma = 30$ per cent; the variance exists only while $\sigma^2 < 2(r - g)$; and the probability of falling below the Gordon value exceeds one half at every ν , 0.700 and 0.774 at those settings. A ten-year model’s terminal value has a median of 0.584 of its mean, not the 0.564 that multiplying medians suggests. A simulation free of the identity reproduces the inverse-gamma quantiles within 0.43 per cent, and a simulator that closes the tail exactly confirms the applied numbers.



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About the Desk

Stochastic Analysis Desk

Zamrik Quant Research

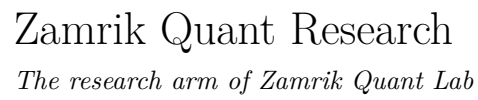
How random processes behave — Itô calculus, diffusions, local time, Fokker–Planck — in theory and in application.

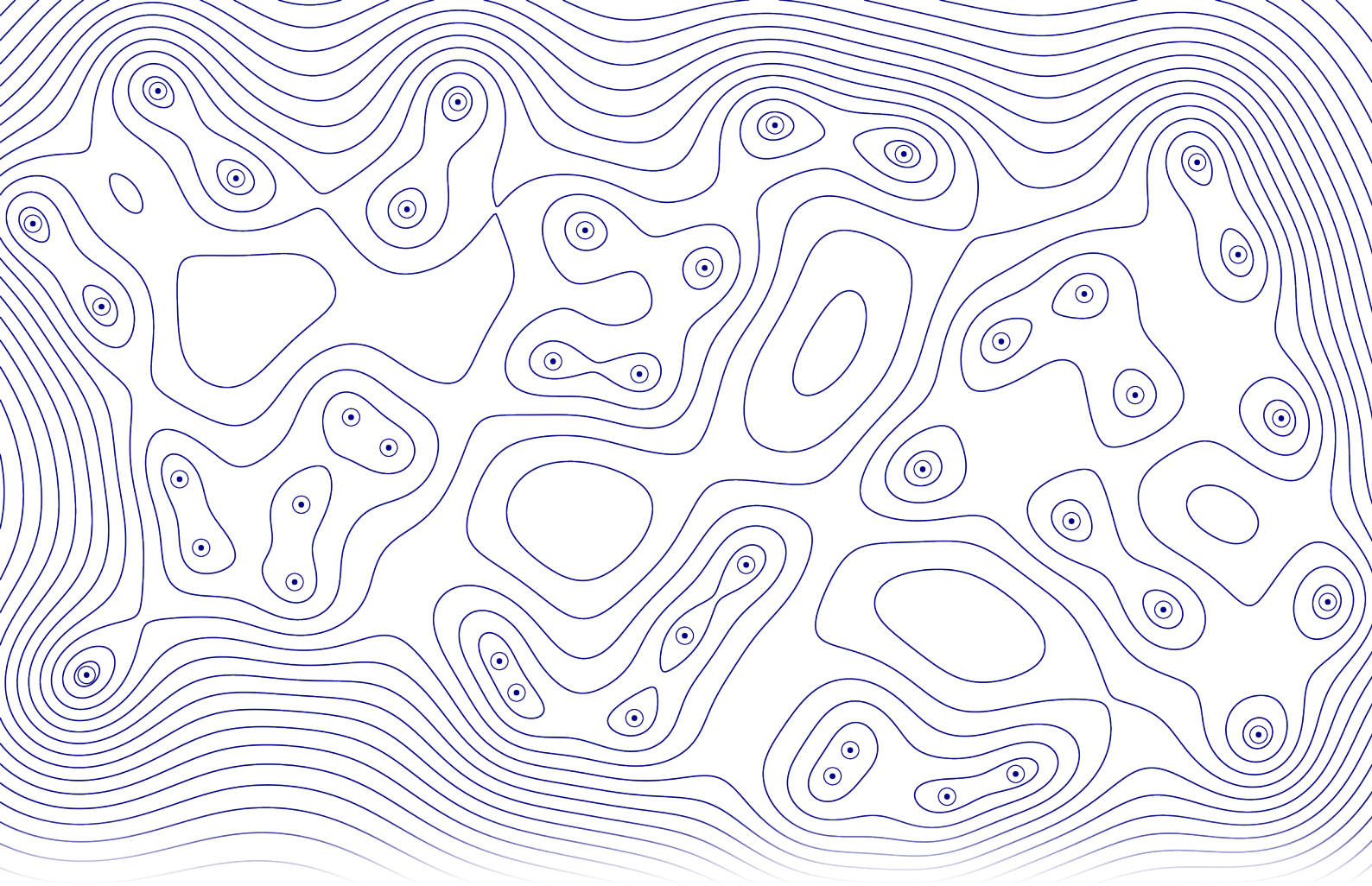
Stochastic analysis is the grammar of every other desk: a price, an order queue, a population or a value function is a functional of a random path, and what can be said about it depends on what can be proved about the path. The Stochastic Analysis desk works on those proofs and on the exact formulas they produce.

Its core questions are the classical ones, asked with care: when a diffusion first reaches a boundary, fixed or moving; how long it spends at each level before it does; and what happens at the boundary itself — reflection, killing, a sticky or skewed point, a reset to the start. The answers come through Itô calculus and martingale methods, local time and Tanaka’s formula, Ray–Knight theorems, time changes, and the Feynman–Kac and Fokker–Planck equations that turn questions about paths into equations for expectations and densities. Wherever the problem allows, the desk derives the law in closed form and then checks it against an independently built simulation.

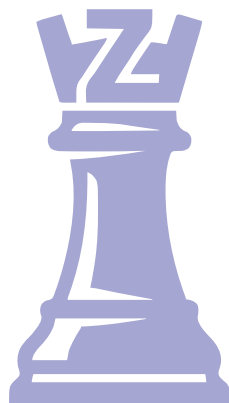
The results travel. The same calculus corrects a barrier that is checked once a day, explains why most stocks lose to the average stock, separates real correlation from the noise of random matrices, sets the acceptance rate of a Metropolis sampler, and models a neuron’s membrane and a day’s rainfall. The desk also studies the numerics that decide whether a simulated path tells the truth: the strong and weak convergence of Euler–Maruyama and Milstein schemes, Brownian-bridge corrections for barriers monitored in discrete time, and the error left when a continuous process is sampled on a grid.

The desk is supervised by **Dr. T. Zamrik**, Head of Zamrik Quant Research, PhD, Imperial College London (2011).





Power function of the Matérn interpolant: $P(\mathbf{x})^2 = \varphi(0) - \varphi_X(\mathbf{x})^\top A^{-1} \varphi_X(\mathbf{x})$, zero at every data point $\mathbf{x}_i$



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