

# The Russian Option: Optimal Stopping on the Running Maximum and Peskir's Maximality Principle

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## 1. Abstract

The Russian option, introduced by Shepp and Shiryaev (1993), is a perpetual American option whose payoff is the running maximum  $M_\tau = \sup_{s \leq \tau} X_s$  of the underlying stock price, discounted to the stopping time  $\tau$ . We analyse the perpetual problem for a geometric Brownian motion with drift  $\mu < r$ , and exploit the homogeneity of the payoff to reduce the two-dimensional state space  $(X_t, M_t)$  to the one-dimensional ratio process  $Y_t = X_t/M_t \in (0, 1]$ . In the continuation region  $(y^*, 1)$  the reduced value function  $v(y)$  solves an Euler–Cauchy ODE whose general solution is  $v(y) = Ay^{\beta_+} + By^{\beta_-}$ , and the four constants  $(A, B, y^*)$  are determined by a Neumann reflection condition at  $y = 1$  and smooth-pasting conditions at  $y^*$ . We present Peskir's maximality principle, which characterises  $y^*$  as the largest candidate threshold for which the ODE solution dominates the obstacle, and we identify the two Doob–Meyer compensators of the discounted value process as the local times of  $Y$  at  $y^*$  and at the reflecting barrier  $y = 1$ .

## 2. Introduction

The American put stops when the stock price falls too low; the holder trades a known intrinsic value against the option of waiting. The Russian option, introduced by Shepp and Shiryaev [1], poses a richer problem: the holder receives the running maximum  $M_\tau$  of the stock price at a stopping time  $\tau$  of their choice. Because the payoff grows with the historical peak, waiting is always tempting, yet discounting makes delay costly. The result is a non-trivial perpetual optimal stopping problem on a two-dimensional state space.

The key structural insight is that the value function  $V(x, m)$  is homogeneous of degree one in  $(x, m)$ , so the problem reduces to a one-dimensional ODE on the ratio  $y = x/m \in (0, 1]$ . The reflecting boundary at  $y = 1$  (when the stock sets a new maximum) introduces a Neumann condition that, together with smooth pasting at the free boundary  $y^*$ , pins down the solution completely.

Peskir [2] reformulated this structure as the maximality principle: the value function is identified as the pointwise largest solution to the ODE that remains above the obstacle. This geometric characterisation yields an explicit formula for  $y^*$  and avoids solving a free boundary equation

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directly.

### 3. Setup and the State Space

**Definition 3.1** (Stock price and running maximum). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered probability space carrying a standard Brownian motion  $W_t$ . The stock price satisfies

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x > 0,$$

with drift  $\mu \in \mathbb{R}$  and volatility  $\sigma > 0$ . The running maximum process is

$$M_t = \sup_{0 \leq s \leq t} X_s, \quad M_0 = m \geq x.$$

**Definition 3.2** (Russian option value). Fix discount rate  $r > \mu$ . The value of the perpetual Russian option is

$$V(x, m) = \sup_{\tau} \mathbb{E}_{x, m}[e^{-r\tau} M_{\tau}],$$

where the supremum is over all  $(\mathcal{F}_t)$ -stopping times  $\tau$ , and  $\mathbb{E}_{x, m}$  denotes expectation given  $X_0 = x, M_0 = m$ .

*Remark 3.3.* The condition  $r > \mu$  is necessary for the value to be finite. If  $\mu \geq r$ , the stock drifts upward fast enough that waiting always increases the expected discounted maximum, so the supremum is infinite.

### 4. Dimension Reduction via the Ratio Process

**Proposition 4.1** (Homogeneity). *The value function satisfies  $V(\lambda x, \lambda m) = \lambda V(x, m)$  for all  $\lambda > 0$ . Consequently there exists a function  $v : (0, 1] \rightarrow [1, \infty)$  such that*

$$V(x, m) = m v(y), \quad y = \frac{x}{m} \in (0, 1].$$

*Proof.* For any stopping time  $\tau$ ,  $\mathbb{E}_{\lambda x, \lambda m}[e^{-r\tau} M_{\tau}] = \lambda \mathbb{E}_{x, m}[e^{-r\tau} M_{\tau}]$  by the scaling property of GBM. Taking the supremum over  $\tau$  preserves the factor  $\lambda$ .  $\square$

**Definition 4.2** (Ratio process). Set  $Y_t = X_t/M_t$ . Then  $Y_t \in (0, 1]$  for all  $t \geq 0$ . By Itô's formula, in the interior  $\{Y_t < 1\}$ :

$$dY_t = \mu Y_t dt + \sigma Y_t dW_t,$$

and at  $\{Y_t = 1\}$  the process is instantaneously reflected downward by the local time  $L_t^1$  of  $Y$  at level 1, arising from the flat stretches of  $M_t$ .

*Remark 4.3.* The generator of  $Y$  acting on  $C^2$  functions in the interior is

$$\mathcal{L}_Y = \mu y \partial_y + \frac{1}{2} \sigma^2 y^2 \partial_{yy}.$$

The Skorokhod reflection at  $y = 1$  contributes a boundary term to the infinitesimal generator, leading to the Neumann condition derived in Section 4.

## 5. The ODE and Boundary Conditions

**Theorem 5.1** (Variational problem for  $v$ ). *The reduced value function  $v$  satisfies the obstacle problem*

$$\min\{\mathcal{L}_Y v - rv, v - 1\} = 0 \quad \text{on } (0, 1),$$

with boundary condition  $v'(1) = v(1)$  at the reflecting barrier. The state space splits into: - **Stopping region**  $\mathcal{S} = (0, y^*]$ :  $v(y) = 1$  (holder stops, receives  $M_\tau = m$ ). - **Continuation region**  $\mathcal{C} = (y^*, 1)$ :  $\mathcal{L}_Y v = rv$ .

*Proof.* In  $\mathcal{C}$ , the process  $e^{-rt}V(X_t, M_t) = e^{-rt}M_t v(Y_t)$  must be a martingale, yielding  $\mathcal{L}_Y v = rv$ . The Neumann condition at  $y = 1$  follows from  $\partial_m V(m, m) = 0$ : differentiating  $V(x, m) = mv(x/m)$  with respect to  $m$  and evaluating at  $x = m$  gives  $v(1) - v'(1) = 0$ .  $\square$

**Lemma 5.2** (Characteristic roots). *The Euler–Cauchy ODE  $\mathcal{L}_Y v = rv$ , i.e.*

$$\frac{1}{2}\sigma^2 y^2 v'' + \mu y v' - rv = 0,$$

has two real roots  $\beta_+ > 1 > 0 > \beta_-$  given by

$$\beta_\pm = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) \pm \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}}{\sigma^2}.$$

The general solution is  $v(y) = Ay^{\beta_+} + By^{\beta_-}$  with  $A, B > 0$ .

*Proof.* Substituting  $v(y) = y^\beta$  gives the characteristic equation  $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ . The discriminant  $(\mu - \frac{\sigma^2}{2})^2 + 2r\sigma^2 > 0$  ensures two real roots. That the product  $\beta_+\beta_- = -2r/\sigma^2 < 0$  confirms opposite signs. That  $\beta_+ > 1$  follows from evaluating the characteristic polynomial at  $\beta = 1$ :  $\mu - r < 0$  (since  $r > \mu$ ), so the polynomial is negative at  $\beta = 1$ , and since it is positive for large  $\beta$ , the positive root exceeds 1.  $\square$

## 6. The Smooth-Pasting System

**Theorem 6.1** (Four-equation system). *The constants  $(A, B, y^*)$  satisfy the system:*

$$Ay^{*\beta_+} + By^{*\beta_-} = 1, \quad (\text{value matching at } y^*)$$

$$A\beta_+ y^{*\beta_+ - 1} + B\beta_- y^{*\beta_- - 1} = 0, \quad (\text{smooth pasting at } y^*)$$

$$A(\beta_+ - 1) + B(\beta_- - 1) = 0. \quad (\text{Neumann condition at } y = 1)$$

**Proposition 6.2** (Explicit threshold). *From the smooth-pasting and Neumann conditions:*

$$\frac{A}{B} = \frac{1 - \beta_-}{\beta_+ - 1} = -\frac{\beta_-}{\beta_+} y^{*(\beta_- - \beta_+)}.$$

*Eliminating  $A/B$  gives the explicit formula*

$$y^* = \left( \frac{|\beta_-|(\beta_+ - 1)}{\beta_+ (|\beta_-| + 1)} \right)^{1/(\beta_+ - \beta_-)}.$$

*Proof.* The Neumann condition  $A(\beta_+ - 1) = B(1 - \beta_-)$  gives the first ratio. The smooth-pasting condition  $A\beta_+ y^{*\beta_+ - 1} = -B\beta_- y^{*\beta_- - 1}$  gives the second. Setting them equal and solving for  $y^{*(\beta_+ - \beta_-)}$  yields the stated formula.  $\square$

**Corollary 6.3.** *Once  $y^*$  is known, the constants are recovered as*

$$B = \frac{1}{\frac{1 - \beta_-}{\beta_+ - 1} y^{*\beta_+} + y^{*\beta_-}}, \quad A = \frac{1 - \beta_-}{\beta_+ - 1} B.$$

## 7. Peskir's Maximality Principle

**Definition 7.1** (Candidate family). For each  $\lambda \in (0, 1)$ , define  $v_\lambda$  as the solution to  $\mathcal{L}_Y v = rv$  in  $(\lambda, 1)$  satisfying the Neumann condition at  $y = 1$  and smooth pasting  $v'_\lambda(\lambda) = 0$  at  $\lambda$ . Explicitly,

$$v_\lambda(y) = A_\lambda y^{\beta_+} + B_\lambda y^{\beta_-},$$

where  $A_\lambda, B_\lambda$  are determined by the two boundary conditions at  $\lambda$  and  $y = 1$ .

**Theorem 7.2** (Maximality principle, Peskir [2]). *The optimal threshold  $y^*$  is the largest  $\lambda \in (0, 1)$  such that*

$$v_\lambda(y) \geq 1 \quad \text{for all } y \in [\lambda, 1].$$

*At  $\lambda = y^*$ , the candidate  $v_{y^*}$  just touches the obstacle at  $y^*$ :  $v_{y^*}(y^*) = 1$ . For  $\lambda > y^*$ ,  $v_\lambda$  dips strictly below 1 somewhere in  $[\lambda, 1]$ .*

*Remark 7.3.* The maximality principle replaces the free boundary equation. Instead of solving a nonlinear system for  $y^*$ , one identifies it geometrically: increase  $\lambda$  until the ODE solution can no longer dominate the obstacle. The threshold is the last point where dominance is possible — hence "maximal."

**Proof sketch.** For  $\lambda$  small,  $v_\lambda$  is large throughout  $[\lambda, 1]$  and dominates the obstacle easily. As  $\lambda$  increases, the continuation region shrinks and  $v_\lambda(\lambda)$  decreases monotonically. At  $\lambda = y^*$ , the value matching condition  $v_{y^*}(y^*) = 1$  is the unique crossing point. For  $\lambda > y^*$ , the smooth-pasting construction forces  $v_\lambda$  below 1 at  $\lambda$ , violating the obstacle constraint. Full details are in [2].

### 8. Doob–Meyer Decomposition of the Value Process

**Theorem 8.1** (Two-compensator decomposition). *Let  $Z_t = e^{-rt}V(X_t, M_t) = e^{-rt}M_t v(Y_t)$ . Then*

$$Z_t = Z_0 + M_t^Z + A_t^{\text{stop}} - A_t^{\text{refl}},$$

where  $M_t^Z$  is a local martingale,  $A_t^{\text{stop}}$  is the Doob–Meyer compensator accumulating in the stopping region  $\mathcal{S}$ , and  $A_t^{\text{refl}}$  is the Skorokhod local time of  $Y$  at the reflecting barrier  $y = 1$ .

*Proof.* Apply Itô’s formula to  $e^{-rt}M_tv(Y_t)$ :

$$d(e^{-rt}M_tv(Y_t)) = e^{-rt}[(\mathcal{L}_Y v - rv)M_t dt + v(Y_t) dM_t^c + \sigma M_t Y_t v'(Y_t) dW_t - v'(1)M_t dL_t^1]$$

where  $L_t^1$  is the local time of  $Y$  at 1 and  $dM_t^c$  denotes the continuous martingale part of  $dM_t$  (which vanishes since  $M$  is increasing). In  $\mathcal{C}$ :  $\mathcal{L}_Y v = rv$ , the  $dt$  term vanishes. In  $\mathcal{S}$ :  $\mathcal{L}_Y(1) - r < 0$ , contributing to  $A_t^{\text{stop}}$ . The local time term at  $y = 1$  contributes  $-v'(1)M_t dL_t^1$  to  $-A_t^{\text{refl}}$ ; by the Neumann condition  $v'(1) = v(1) > 0$  this term is positive, confirming  $A_t^{\text{refl}}$  is increasing. □

*Remark 8.2.* The two compensators have distinct probabilistic roles.  $A_t^{\text{stop}}$  accumulates time spent in the stopping region  $(0, y^*]$  and encodes the cost of suboptimal continuation.  $A_t^{\text{refl}}$  accumulates the local time of  $Y$  at 1 — the Skorokhod push each time the stock sets a new maximum. Both are zero on the optimal trajectory started from  $y_0 \in \mathcal{C}$  until the first exit time, confirming that  $Z_{t \wedge \tau^*}$  is a true martingale.

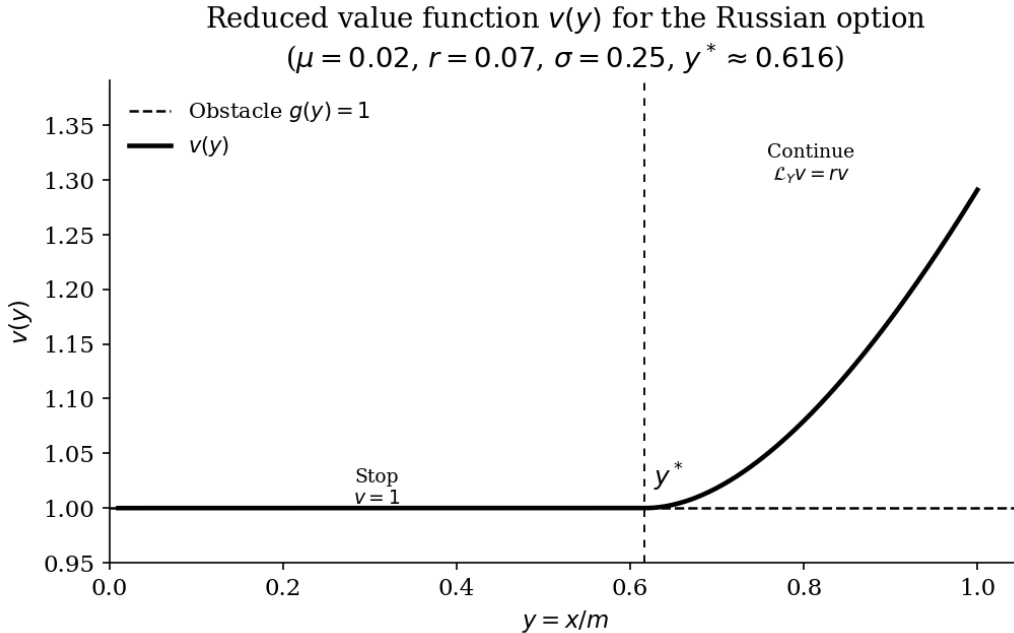


Figure 1: Reduced value function  $v(y)$  (solid) and obstacle  $g \equiv 1$  (dashed) for  $\mu = 0.02, r = 0.07, \sigma = 0.25$ . The optimal threshold  $y^* \approx 0.616$  is marked by a vertical dashed line; the holder stops whenever  $Y_t \leq y^*$ .

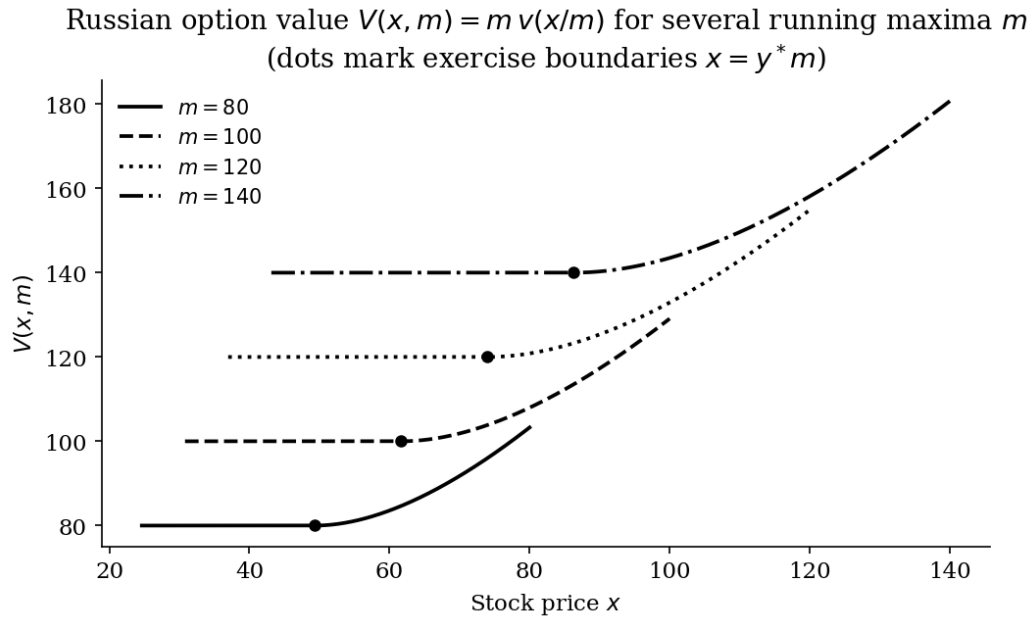


Figure 2: Russian option value  $V(x, m) = m v(x/m)$  as a function of stock price  $x$  for four values of the running maximum  $m$ . Filled circles mark the exercise boundaries  $x = y^* m$ . The linear scaling in  $m$  is a direct consequence of the homogeneity proved in Proposition 3.1.

## 9. Numerical Algorithm

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1 Input: mu, r, sigma
2
3 Step 1 - Characteristic roots
4 Solve  $(\sigma^2/2)*\beta^2 + (\mu - \sigma^2/2)*\beta - r = 0$ 
5 disc =  $(\mu - \sigma^2/2)^2 + 2*r*\sigma^2$ 
6 beta+ =  $(-(\mu - \sigma^2/2) + \sqrt{\text{disc}}) / \sigma^2$ 
7 beta- =  $(-(\mu - \sigma^2/2) - \sqrt{\text{disc}}) / \sigma^2$ 
8
9 Step 2 - Optimal threshold (explicit formula)
10 abm =  $|\beta^-|$ 
11  $y^* = (abm*(\beta^+ - 1) / (\beta^+*(abm + 1)))^{1/(\beta^+ - \beta^-)}$ 
12
13 Step 3 - Coefficients A, B
14 ratio =  $(1 - \beta^-) / (\beta^+ - 1)$  [A/B from Neumann condition]
15 B =  $1 / (\text{ratio} * y^{*\beta^+} + y^{*\beta^-})$ 
16 A = ratio * B
17
18 Step 4 - Value function
19 For y in  $(y^*, 1]$ :  $v(y) = A*y^{\beta^+} + B*y^{\beta^-}$ 
20 For y in  $(0, y^*]$ :  $v(y) = 1$ 
21  $V(x, m) = m * v(x/m)$ 
22
23 Output: beta+, beta-, y*, A, B, v, V

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## 10. References

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