The Russian Option: Optimal Stopping on the Running Maximum and Peskir's Maximality Principle

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1. Abstract

The Russian option, introduced by Shepp and Shiryaev (1993), is a perpetual American option whose payoff is the running maximum $M_\tau = \sup_{s \le \tau} X_s$ of the underlying stock price, discounted to the stopping time τ. We analyse the perpetual problem for a geometric Brownian motion with drift $\mu < r$, and exploit the homogeneity of the payoff to reduce the two-dimensional state space (X_t, M_t) to the one-dimensional ratio process $Y_t = X_t/M_t \in (0, 1]$. In the continuation region $(y^*, 1)$ the reduced value function $v(y)$ solves an Euler–Cauchy ODE whose general solution is $v(y) = Ay^{\beta_+} + By^{\beta_-}$, and the four constants (A, B, y^*) are determined by a Neumann reflection condition at $y = 1$ and smooth-pasting conditions at y^*. We present Peskir's maximality principle, which characterises y^* as the largest candidate threshold for which the ODE solution dominates the obstacle, and we identify the two Doob–Meyer compensators of the discounted value process as the local times of Y at y^* and at the reflecting barrier $y = 1$.

2. Introduction

The American put stops when the stock price falls too low; the holder trades a known intrinsic value against the option of waiting. The Russian option, introduced by Shepp and Shiryaev [1], poses a richer problem: the holder receives the running maximum M_τ of the stock price at a stopping time τ of their choice. Because the payoff grows with the historical peak, waiting is always tempting, yet discounting makes delay costly. The result is a non-trivial perpetual optimal stopping problem on a two-dimensional state space.

The key structural insight is that the value function $V(x, m)$ is homogeneous of degree one in (x, m), so the problem reduces to a one-dimensional ODE on the ratio $y = x/m \in (0, 1]$. The reflecting boundary at $y = 1$ (when the stock sets a new maximum) introduces a Neumann condition that, together with smooth pasting at the free boundary y^*, pins down the solution completely.

Peskir [2] reformulated this structure as the maximality principle: the value function is identified as the pointwise largest solution to the ODE that remains above the obstacle.

This geometric characterisation yields an explicit formula for y^* and avoids solving a free boundary equation directly.

3. Setup and the State Space

Definition 3.1 (Stock price and running maximum)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space carrying a standard Brownian motion W_t. The stock price satisfies

$$dX_t = \mu X_t\, dt + \sigma X_t\, dW_t, \qquad X_0 = x > 0,$$

with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$. The running maximum process is

$$M_t = \sup_{0\leq s\leq t} X_s, \qquad M_0 = m \geq x.$$

Definition 3.2 (Russian option value)**.** Fix discount rate $r > \mu$. The value of the perpetual Russian option is

$$V(x,m) = \sup_{\tau} \mathbb{E}_{x,m}[e^{-r\tau} M_\tau]\,,$$

where the supremum is over all $(\mathcal{F}_t)$-stopping times τ, and $\mathbb{E}_{x,m}$ denotes expectation given $X_0 = x$, $M_0 = m$.

Remark 3.3. The condition $r > \mu$ is necessary for the value to be finite. If $\mu \geq r$, the stock drifts upward fast enough that waiting always increases the expected discounted maximum, so the supremum is infinite.

4. Dimension Reduction via the Ratio Process

Proposition 4.1 (Homogeneity)**.** *The value function satisfies* $V(\lambda x, \lambda m) = \lambda V(x,m)$ *for all* $\lambda > 0$*. Consequently there exists a function* $v : (0,1] \to [1,\infty)$ *such that*

$$V(x,m) = m\, v(y), \qquad y = \frac{x}{m} \in (0,1].$$

Proof. For any stopping time τ, $\mathbb{E}_{\lambda x,\lambda m}[e^{-r\tau} M_\tau] = \lambda\, \mathbb{E}_{x,m}[e^{-r\tau} M_\tau]$ by the scaling property of GBM. Taking the supremum over τ preserves the factor λ. □

Definition 4.2 (Ratio process)**.** Set $Y_t = X_t/M_t$. Then $Y_t \in (0,1]$ for all $t \geq 0$. By Itô's formula, in the interior $\{Y_t < 1\}$:

$$dY_t = \mu Y_t\, dt + \sigma Y_t\, dW_t,$$

and at $\{Y_t = 1\}$ the process is instantaneously reflected downward by the local time L_t^1 of Y at level 1, arising from the flat stretches of M_t.

Remark 4.3. The generator of Y acting on C^2 functions in the interior is

$$\mathcal{L}_Y = \mu y\, \partial_y + \tfrac{1}{2}\sigma^2 y^2\, \partial_{yy}.$$

The Skorokhod reflection at $y = 1$ contributes a boundary term to the infinitesimal generator, leading to the Neumann condition derived in Section 4.

5. The ODE and Boundary Conditions

Theorem 5.1 (Variational problem for v). *The reduced value function v satisfies the obstacle problem*

$$\min\{\mathcal{L}_Y v - rv,\ v - 1\} = 0 \quad \text{on } (0,1),$$

*with boundary condition $v'(1) = v(1)$ at the reflecting barrier. The state space splits into: - **Stopping region** $\mathcal{S} = (0, y^*]$: $v(y) = 1$ (holder stops, receives $M_\tau = m$). - **Continuation region** $\mathcal{C} = (y^*, 1)$: $\mathcal{L}_Y v = rv$.*

Proof. In $\mathcal{C}$, the process $e^{-rt}V(X_t, M_t) = e^{-rt}M_t v(Y_t)$ must be a martingale, yielding $\mathcal{L}_Y v = rv$. The Neumann condition at $y = 1$ follows from $\partial_m V(m,m) = 0$: differentiating $V(x,m) = mv(x/m)$ with respect to m and evaluating at $x = m$ gives $v(1) - v'(1) = 0$. □

Lemma 5.2 (Characteristic roots). *The Euler–Cauchy ODE $\mathcal{L}_Y v = rv$, i.e.*

$$\tfrac{1}{2}\sigma^2 y^2 v'' + \mu y v' - rv = 0,$$

has two real roots $\beta_+ > 1 > 0 > \beta_-$ given by

$$\beta_\pm = \frac{-\left(\mu - \frac{\sigma^2}{2}\right) \pm \sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}}{\sigma^2}.$$

The general solution is $v(y) = Ay^{\beta_+} + By^{\beta_-}$ with $A, B > 0$.

Proof. Substituting $v(y) = y^\beta$ gives the characteristic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$. The discriminant $(\mu - \frac{\sigma^2}{2})^2 + 2r\sigma^2 > 0$ ensures two real roots. That the product $\beta_+\beta_- = -2r/\sigma^2 < 0$ confirms opposite signs. That $\beta_+ > 1$ follows from evaluating the characteristic polynomial at $\beta = 1$: $\mu - r < 0$ (since $r > \mu$), so the polynomial is negative at $\beta = 1$, and since it is positive for large β, the positive root exceeds 1. □

6. The Smooth-Pasting System

Theorem 6.1 (Four-equation system). *The constants (A, B, y^*) satisfy the system:*

$$Ay^{*\beta_+} + By^{*\beta_-} = 1, \qquad \textit{(value matching at } y^*\textit{)}$$

$$A\beta_+ y^{*\beta_+ - 1} + B\beta_- y^{*\beta_- - 1} = 0, \qquad \textit{(smooth pasting at } y^*\textit{)}$$

$$A(\beta_+ - 1) + B(\beta_- - 1) = 0. \qquad \textit{(Neumann condition at } y = 1\textit{)}$$

Proposition 6.2 (Explicit threshold)**.** *From the smooth-pasting and Neumann conditions:*

$$\frac{A}{B} = \frac{1-\beta_-}{\beta_+ - 1} = -\frac{\beta_-}{\beta_+}\, y^{*(\beta_- - \beta_+)}.$$

Eliminating A/B gives the explicit formula

$$y^* = \left(\frac{|\beta_-|\,(\beta_+ - 1)}{\beta_+\,(|\beta_-|+1)}\right)^{1/(\beta_+ - \beta_-)}.$$

Proof. The Neumann condition $A(\beta_+ - 1) = B(1-\beta_-)$ gives the first ratio. The smooth-pasting condition $A\beta_+ y^{*\beta_+ - 1} = -B\beta_- y^{*\beta_- - 1}$ gives the second. Setting them equal and solving for $y^{*(\beta_+ - \beta_-)}$ yields the stated formula. □

Corollary 6.3. *Once y^* is known, the constants are recovered as*

$$B = \frac{1}{\dfrac{1-\beta_-}{\beta_+ - 1}\, y^{*\beta_+} + y^{*\beta_-}}, \qquad A = \frac{1-\beta_-}{\beta_+ - 1}\, B.$$

7. Peskir's Maximality Principle

Definition 7.1 (Candidate family)**.** For each $\lambda \in (0,1)$, define v_λ as the solution to $\mathcal{L}_Y v = rv$ in $(\lambda, 1)$ satisfying the Neumann condition at $y = 1$ and smooth pasting $v'_\lambda(\lambda) = 0$ at λ. Explicitly,

$$v_\lambda(y) = A_\lambda y^{\beta_+} + B_\lambda y^{\beta_-},$$

where A_λ, B_λ are determined by the two boundary conditions at λ and $y = 1$.

Theorem 7.2 (Maximality principle, Peskir [2])**.** *] The optimal threshold y^* is the largest $\lambda \in (0,1)$ such that*

$$v_\lambda(y) \geq 1 \quad \textit{for all } y \in [\lambda, 1].$$

At $\lambda = y^*$, the candidate v_{y^*} just touches the obstacle at y^*: $v_{y^*}(y^*) = 1$. For $\lambda > y^*$, v_λ dips strictly below 1 somewhere in $[\lambda, 1]$.

Remark 7.3. The maximality principle replaces the free boundary equation. Instead of solving a nonlinear system for y^*, one identifies it geometrically: increase λ until the ODE solution can no longer dominate the obstacle. The threshold is the last point where dominance is possible — hence "maximal."

> ***Proof sketch.*** *For λ small, v_λ is large throughout $[\lambda, 1]$ and dominates the obstacle easily. As λ increases, the continuation region shrinks and $v_\lambda(\lambda)$ decreases monotonically. At $\lambda = y^*$, the value matching condition $v_{y^*}(y^*) = 1$ is the unique crossing point. For $\lambda > y^*$, the smooth-pasting construction forces v_λ below 1 at λ, violating the obstacle constraint. Full details are in [2].*

8. Doob–Meyer Decomposition of the Value Process

Theorem 8.1 (Two-compensator decomposition)**.** *Let $Z_t = e^{-rt}V(X_t, M_t) = e^{-rt}M_t v(Y_t)$. Then*

$$Z_t = Z_0 + M_t^Z + A_t^{\text{stop}} - A_t^{\text{refl}},$$

where M_t^Z is a local martingale, A_t^{stop} is the Doob–Meyer compensator accumulating in the stopping region $\mathcal{S}$, and A_t^{refl} is the Skorokhod local time of Y at the reflecting barrier $y = 1$.

Proof. Apply Itô's formula to $e^{-rt}M_t v(Y_t)$:

$$d(e^{-rt}M_t v(Y_t)) = e^{-rt}\big[(\mathcal{L}_Y v - rv)M_t\,dt + v(Y_t)\,dM_t^c + \sigma M_t Y_t v'(Y_t)\,dW_t - v'(1)M_t\,dL_t^1\big]$$

where L_t^1 is the local time of Y at 1 and dM_t^c denotes the continuous martingale part of dM_t (which vanishes since M is increasing). In $\mathcal{C}$: $\mathcal{L}_Y v = rv$, the dt term vanishes. In $\mathcal{S}$: $\mathcal{L}_Y(1) - r < 0$, contributing to A_t^{stop}. The local time term at $y = 1$ contributes $-v'(1)M_t\,dL_t^1$ to $-A_t^{\text{refl}}$; by the Neumann condition $v'(1) = v(1) > 0$ this term is positive, confirming A^{refl} is increasing. □

Remark 8.2. The two compensators have distinct probabilistic roles. A_t^{stop} accumulates time spent in the stopping region $(0, y^*]$ and encodes the cost of suboptimal continuation. A_t^{refl} accumulates the local time of Y at 1 — the Skorokhod push each time the stock sets a new maximum. Both are zero on the optimal trajectory started from $y_0 \in \mathcal{C}$ until the first exit time, confirming that $Z_{t\wedge\tau^*}$ is a true martingale.

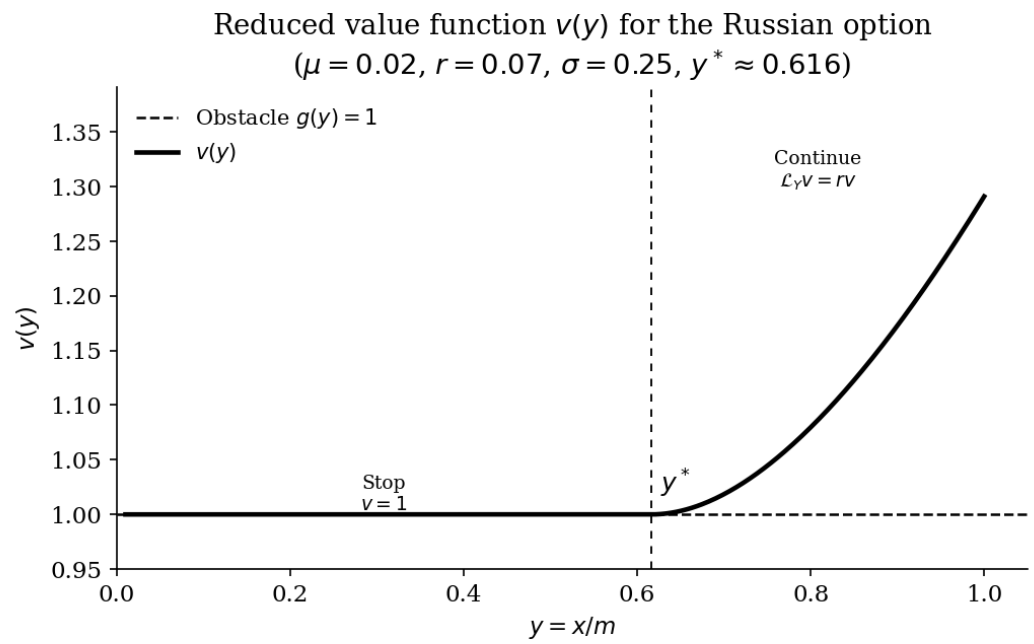

Figure 1: Reduced value function $v(y)$ (solid) and obstacle $g \equiv 1$ (dashed) for $\mu = 0.02$, $r = 0.07$, $\sigma = 0.25$. The optimal threshold $y^* \approx 0.616$ is marked by a vertical dashed line; the holder stops whenever $Y_t \leq y^*$.

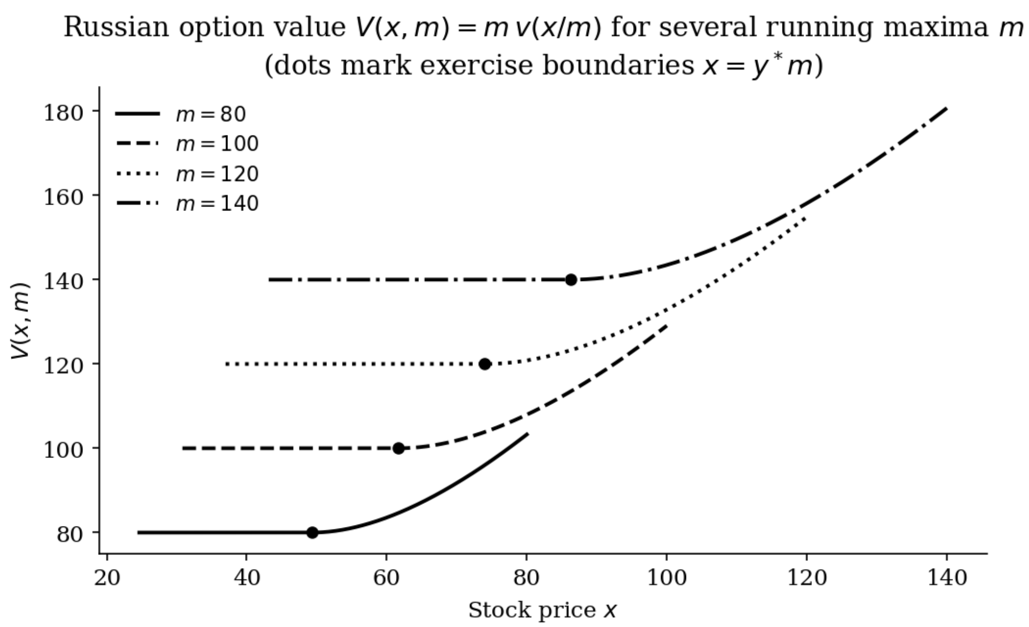

Figure 2: Russian option value $V(x,m) = m\,v(x/m)$ as a function of stock price x for four values of the running maximum m. Filled circles mark the exercise boundaries $x = y^* m$. The linear scaling in m is a direct consequence of the homogeneity proved in Proposition 3.1.

9. Numerical Algorithm

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Input: mu, r, sigma

Step 1 - Characteristic roots
  Solve (sigma^2/2)*beta^2 + (mu - sigma^2/2)*beta - r = 0
  disc  = (mu - sigma^2/2)^2 + 2*r*sigma^2
  beta+ = (-(mu - sigma^2/2) + sqrt(disc)) / sigma^2
  beta- = (-(mu - sigma^2/2) - sqrt(disc)) / sigma^2

Step 2 - Optimal threshold (explicit formula)
  abm   = |beta-|
  y*    = ( abm*(beta+ - 1) / (beta+*(abm + 1)) )^{1/(beta+ - beta-)}

Step 3 - Coefficients A, B
  ratio = (1 - beta-) / (beta+ - 1)         [A/B from Neumann condition]
  B     = 1 / (ratio * y*^{beta+} + y*^{beta-})
  A     = ratio * B

Step 4 - Value function
  For y in (y*, 1]:  v(y) = A*y^{beta+} + B*y^{beta-}
  For y in (0, y*]:  v(y) = 1
  V(x, m) = m * v(x/m)

Output: beta+, beta-, y*, A, B, v, V
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10. References

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