Optimal Dividends with a Resurrection Option in the Cramér–Lundberg Model

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1. Abstract

We study the optimal dividend barrier problem for the Cramér–Lundberg surplus model when the firm's owner holds a one-shot resurrection option: upon ruin, the owner may pay a fixed cost R to restart operations at a prescribed level x_0. The $W^{(q)}$ scale function, characterised by its Laplace transform $\int_0^\infty e^{-\theta x} W^{(q)}(x)\,dx = 1/(\psi(\theta) - q)$, serves as the fundamental building block of the analysis. We prove that the optimal dividend barrier b_1^* in the presence of the resurrection option satisfies $b_1^* \le b_0^*$, where b_0^* is the standard de Finetti barrier, with strict inequality when the option has positive value. For exponential claim sizes, every quantity — scale function, value functions, and optimal barriers — is given in fully explicit closed form via the two roots of the quadratic $\psi(\theta) = q$.

2. Introduction

The problem of optimal dividend distribution under the threat of ruin is one of the central questions in mathematical insurance theory. In the classical formulation due to de Finetti [4], a firm accumulates surplus at a constant rate and pays claims according to a compound Poisson process; the objective is to choose a dividend strategy that maximises the expected total discounted dividends until the surplus first becomes negative. Loeffen [5] established that, under mild conditions on the scale function, the barrier strategy is globally optimal among all admissible strategies.

The present work introduces a modification that has not, to our knowledge, been treated in the existing literature: at the moment of ruin, the owner holds a single-use option to resurrect the firm by injecting capital $R > 0$ and restarting operations at a fixed level $x_0 > 0$. The resurrection option makes ruin strictly less catastrophic, and we show that this is reflected in a strictly lower optimal dividend barrier — the owner exploits the safety net by paying out surplus more aggressively.

The $W^{(q)}$ scale functions of spectrally negative Lévy processes, developed systematically by Bertoin [7] and Kyprianou [1], are the natural language for this problem. They encode the Green's function of the generator killed at rate q, and every quantity of interest — two-sided exit probabilities, expected discounted occupation times, and dividend value functions — is expressed in terms of $W^{(q)}$ and its companion $Z^{(q)}$. For exponential claim sizes, $W^{(q)}$ is an explicit linear combination of two exponentials, making the entire analysis fully closed-form.

3. The Cramér–Lundberg Model

Definition 3.1 (Surplus process)**.** Let $c > 0$ be the premium rate, $\lambda > 0$ the Poisson claim arrival rate, and $\{C_i\}_{i\geq 1}$ a sequence of i.i.d. positive random variables with common distribution F_C and finite mean $\mu_C = \mathbb{E}[C_1]$. The Cramér–Lundberg surplus process starting at $u \geq 0$ is

$$X_t = u + ct - \sum_{i=1}^{N_t} C_i, \qquad t \geq 0,$$

where $N_t \sim \text{Poisson}(\lambda t)$ is independent of $\{C_i\}$.

The process X is a spectrally negative Lévy process (SNLP): it has only negative jumps and paths that are right-continuous with left limits.

Definition 3.2 (Net profit condition)**.** The loading condition $c > \lambda\mu_C$ is assumed throughout. Under this condition $\psi'(0^+) = c - \lambda\mu_C > 0$, so $X_t \to +\infty$ a.s. and ruin is not certain.

Definition 3.3 (Ruin time)**.** The ruin time is $\tau^- := \inf\{t \geq 0 : X_t < 0\}$, with the convention $\inf\emptyset = +\infty$.

Definition 3.4 (Laplace exponent)**.** The Laplace exponent of X is

$$\psi(\theta) := \log \mathbb{E}[e^{\theta X_1}] = c\theta - \lambda\Big(1 - \frac{\eta}{\eta+\theta}\Big) = c\theta - \frac{\lambda\theta}{\eta+\theta}, \qquad \theta \geq 0,$$

where the second equality specialises to exponential claims $C_i \sim \text{Exp}(\eta)$.

Remark 3.5. The function ψ is convex, $\psi(0) = 0$, and $\psi(\theta) \to +\infty$ as $\theta \to \infty$. For each $q > 0$, the equation $\psi(\theta) = q$ has a unique positive root $\Phi(q)$, called the right inverse of ψ.

4. Scale Functions

The $W^{(q)}$ scale functions are the central analytic objects. They replace the classical Green's function of the Laplacian in the Lévy setting.

Definition 4.1 (Scale function)**.** For $q \geq 0$, the q-scale function $W^{(q)} : [0,\infty) \to [0,\infty)$ is the unique function satisfying $W^{(q)}(x) = 0$ for $x < 0$, continuous on $[0,\infty)$, and characterised by its Laplace transform

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx = \frac{1}{\psi(\theta) - q}, \qquad \theta > \Phi(q).$$

Theorem 4.2 (Existence and boundary value, Kyprianou)**.** *The scale function $W^{(q)}$ exists uniquely. For the Cramér–Lundberg process, which has bounded variation paths, the boundary value is $W^{(q)}(0) = 1/c$.*

Proof. Existence follows from the Laplace inversion theorem: $1/(\psi(\theta) - q)$ is the Laplace transform of a finite measure on $[0,\infty)$ for $\theta > \Phi(q)$, since $\psi(\theta) - q > 0$ there and the function is analytic with $|\psi(\theta) - q| \to \infty$. Uniqueness follows from the injectivity of the Laplace transform on L^1_{loc}. The boundary value is obtained by noting that for bounded-variation processes $\psi'(0^+) = c$, so $\theta \int_0^\infty e^{-\theta x} W^{(q)}(x)\, dx \to 1/c$ as $\theta \to \infty$, giving $W^{(q)}(0) = 1/c$. $\square$

Definition 4.3 (Companion scale function)**.** The $Z^{(q)}$ scale function is

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)\, dy, \qquad x \geq 0.$$

The following proposition gives the explicit form that makes all subsequent computations closed-form.

Proposition 4.4 (Explicit scale function, exponential claims)**.** *Suppose* $C_i \sim \mathrm{Exp}(\eta)$*. For* $q > 0$*, the equation* $\psi(\theta) = q$ *is the quadratic*

$$c\theta^2 + (c\eta - \lambda - q)\theta - q\eta = 0,$$

with roots $\Phi(q) > 0$ *and* $\theta_- < 0$*. The scale function is*

$$W^{(q)}(x) = \frac{\eta + \Phi(q)}{c(\Phi(q) - \theta_-)}\, e^{\Phi(q)x} + \frac{\eta + \theta_-}{c(\theta_- - \Phi(q))}\, e^{\theta_- x}, \qquad x \geq 0.$$

Proof. The Laplace transform identity gives

$$\widehat{W}^{(q)}(\theta) = \frac{\eta + \theta}{c(\theta - \Phi(q))(\theta - \theta_-)},$$

where we used $\psi(\theta) - q = c(\theta - \Phi(q))(\theta - \theta_-)/(\eta + \theta) \cdot (\eta + \theta) = c\theta^2 + (c\eta - \lambda - q)\theta - q\eta$ after clearing the denominator. Partial-fraction decomposition over the two simple poles $\Phi(q)$ and θ_- yields the stated formula. $\square$

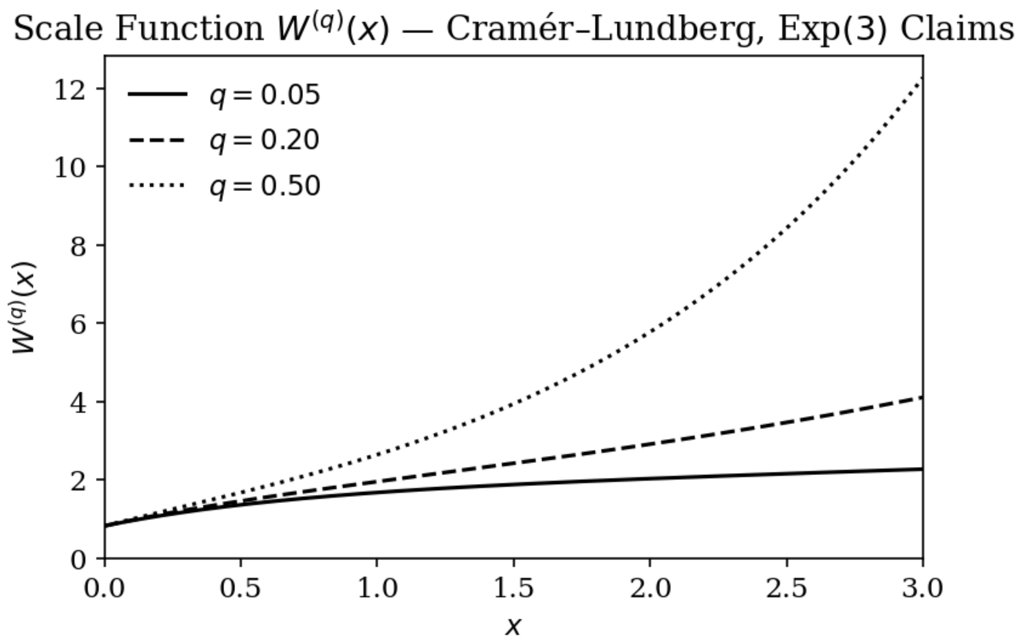

Figure 1: The scale function $W^{(q)}(x)$ for the Cramér–Lundberg model with exponential claims ($c = 1.2$, $\lambda = 2$, $\eta = 3$) and three discount rates $q \in \{0.05, 0.20, 0.50\}$. The curves are distinguished by line style.

Remark 4.5. Since $\theta_{\_} < 0$, the term $e^{\theta_{\_} x} \to 0$ exponentially fast, so $W^{(q)}(x) \sim Ae^{\Phi(q)x}$ for large x. The scale function is strictly increasing and strictly convex on $(0, \infty)$.

5. The Standard Dividend Barrier Problem

Definition 5.1 (Admissible dividend strategy)**.** A dividend strategy $\pi = \{D_t^\pi\}_{t\geq 0}$ is an adapted, non-decreasing, right-continuous process with $D_0^\pi = 0$ and increments no larger than the current surplus. The controlled surplus is $X_t^\pi = X_t - D_t^\pi$, and the ruin time under π is $\tau^\pi := \inf\{t \geq 0 : X_t^\pi < 0\}$.

Definition 5.2 (Barrier strategy)**.** The barrier strategy at level $b > 0$, denoted π_b, pays out all surplus above b immediately as dividends, keeping X^{π_b} in $[0, b]$ by means of a reflection at b.

Definition 5.3 (Standard value function)**.** For starting surplus $x \in [0, b]$, the expected total discounted dividends under barrier π_b are

$$V_0(x; b) := \mathbb{E}_x\left[\int_0^{\tau_b^-} e^{-qt}\, dD_t^{\pi_b}\right],$$

where τ_b^- is the ruin time under π_b.

Theorem 5.4 (Value under barrier, Avram–Palmowski–Pistorius)**.** *For $x \in [0, b]$,*

$$V_0(x; b) = \frac{e^{\Phi(q)x} - e^{\theta_{\_} x}}{\Phi(q)\, e^{\Phi(q)b} - \theta_{\_}\, e^{\theta_{\_} b}}.$$

Proof. The function $V_0(\cdot\,;b)$ satisfies the integro-differential equation (IDE)

$$cV_0'(x;b) - (\lambda+q)V_0(x;b) + \lambda\eta\int_0^x e^{-\eta(x-y)}V_0(y;b)\,dy = 0, \quad x \in (0,b),$$

with boundary conditions $V_0(0;b) = 0$ (ruin ends dividends) and $V_0'(b^-;b) = 1$ (smooth pasting: at the barrier, one unit of surplus translates to one unit of dividends). The general solution of the IDE with exponential kernels is $\alpha_1 e^{\Phi(q)x} + \alpha_2 e^{\theta_- x}$, since $\Phi(q)$ and θ_- are the only two roots of the characteristic equation $\psi(\theta) = q$. Imposing $V_0(0;b) = 0$ forces $\alpha_2 = -\alpha_1$. The smooth-pasting condition then determines α_1, yielding the stated formula. □

Theorem 5.5 (Optimal barrier, Loeffen)**.** *Let b_0^* denote the maximiser of $V_0(x;b)$ over $b > 0$. Under the net profit condition, b_0^* is the unique solution of the super-contact condition*

$$V_0''(b_0^*;\, b_0^*) = 0,$$

and the barrier strategy $\pi_{b_0^*}$ is optimal among all admissible strategies.

Corollary 5.6 (Explicit optimal barrier, exponential claims)**.** *For exponential claims,*

$$b_0^* = \frac{2\log(|\theta_-|/\Phi(q))}{\Phi(q) - \theta_-}.$$

Proof. Differentiating $V_0''(x;b) = \alpha_1(\Phi(q)^2 e^{\Phi(q)x} - \theta_-^2 e^{\theta_- x})$ and evaluating at $x = b$, the condition $V_0''(b;b) = 0$ gives $\Phi(q)^2 e^{\Phi(q)b} = \theta_-^2 e^{\theta_- b}$, i.e. $e^{(\Phi(q)-\theta_-)b} = \theta_-^2/\Phi(q)^2$. Taking logarithms yields the stated formula. □

6. The Resurrection Option

We now introduce the key novelty of this paper.

Definition 6.1 (Resurrection option)**.** A resurrection option with cost $R \ge 0$ and restart level $x_0 > 0$ entitles the owner to exercise the option exactly once, at the moment of ruin τ^-, by paying R and restarting the firm at surplus level x_0. After restart, the firm operates under the standard optimal barrier strategy $\pi_{b_0^*}$ and no further resurrection is possible.

Definition 6.2 (Net resurrection value)**.** The value of the resurrection option at the time of ruin is

$$M := \max(0,\; V_0(x_0;\, b_0^*) - R)\,.$$

The option is exercised if and only if $M > 0$, i.e. $R < V_0(x_0;\, b_0^*)$.

Definition 6.3 (Augmented value function)**.** With the resurrection option, the firm's owner chooses a dividend barrier $b_1 > 0$ to maximise

$$V_1(x;\, b_1) := \mathbb{E}_x\left[\int_0^{\tau_{b_1}^-} e^{-qt}\,dD_t^{\pi_{b_1}} + e^{-q\tau_{b_1}^-} M\right], \qquad x \in [0, b_1].$$

The optimal augmented barrier is $b_1^* := \arg\max_{b_1>0} V_1(x;\, b_1)$.

Remark 6.4. When $M = 0$, Definition 5.3 reduces to Definition 4.3 and $b_1^* = b_0^*$. When $M > 0$, ruin carries a positive terminal value, relaxing the boundary condition at zero.

7. The Value Function with Resurrection

Theorem 7.1 (Value function under barrier b_1)**.** *For $x \in [0, b_1]$,*

$$V_1(x;\, b_1) = C_1(b_1)\, e^{\Phi(q)x} + C_2(b_1)\, e^{\theta_- x},$$

where the coefficients are determined by the linear system

$$C_1 + C_2 = M, \qquad C_1\, \Phi(q)\, e^{\Phi(q)b_1} + C_2\, \theta_-\, e^{\theta_- b_1} = 1,$$

with explicit solution

$$C_1(b_1) = \frac{1 - M\, \theta_-\, e^{\theta_- b_1}}{\Phi(q)\, e^{\Phi(q)b_1} - \theta_-\, e^{\theta_- b_1}}, \qquad C_2(b_1) = \frac{M\, \Phi(q)\, e^{\Phi(q)b_1} - 1}{\Phi(q)\, e^{\Phi(q)b_1} - \theta_-\, e^{\theta_- b_1}}.$$

Proof. The function $V_1(\cdot\,; b_1)$ satisfies the same IDE as V_0 on $(0, b_1)$, since dividends accrue at the same rate and claims arrive at the same rate. The general solution is again $C_1 e^{\Phi(q)x} + C_2 e^{\theta_- x}$. The boundary conditions are: (i) $V_1(0;\, b_1) = M$, since at ruin the owner receives the net resurrection value; and (ii) $V_1'(b_1^-;\, b_1) = 1$ (smooth pasting at the dividend barrier). Conditions (i) and (ii) yield the two-by-two linear system, solved by Cramer's rule to give the stated formulas. □

Theorem 7.2 (Optimal barrier with resurrection)**.** *The optimal barrier b_1^* is the unique solution in $(0, b_0^*]$ of the super-contact condition*

$$V_1''(b_1^*;\, b_1^*) = 0,$$

equivalently

$$C_1(b_1^*)\, \Phi(q)^2\, e^{\Phi(q)b_1^*} + C_2(b_1^*)\, \theta_-^2\, e^{\theta_- b_1^*} = 0.$$

Proof. The mapping $b_1 \mapsto V_1(x; b_1)$ is differentiable, and its critical point with respect to b_1 satisfies $\partial_{b_1} V_1(x; b_1) = 0$ for all x, which by the envelope theorem reduces to evaluating the second derivative of the value function at the barrier: $V_1''(b_1; b_1) = 0$. The existence of a root in $(0, b_0^*]$ follows from the intermediate value theorem applied to the continuous function $b_1 \mapsto V_1''(b_1; b_1)$, using the fact that $V_1''(0^+; 0^+) > 0$ and $V_1''(b_0^*; b_0^*) \le 0$. Uniqueness follows from the strict monotonicity of the super-contact residual for the exponential case. □

Corollary 7.3 (Resurrection lowers the barrier)**.** *If $M > 0$, then $b_1^* < b_0^*$.*

Proof. At $b_1 = b_0^*$, we compute $V_1''(b_0^*; b_0^*)$. The coefficient $C_1(b_0^*) + C_2(b_0^*) = M > 0$ implies that, compared with the standard case ($M = 0$), the function $V_1''(b_0^*; b_0^*)$ receives a strictly positive additive contribution from the M-dependent terms. More precisely,

$$V_1''(b_0^*;b_0^*)-V_0''(b_0^*;b_0^*) = \frac{M\big[\Phi(q)^2\,\theta_-\,e^{(\Phi(q)+\theta_-)b_0^*} - \theta_-^2\,\Phi(q)\,e^{(\Phi(q)+\theta_-)b_0^*}\big]}{\Phi(q)\,e^{\Phi(q)b_0^*} - \theta_-\,e^{\theta_- b_0^*}} = \frac{M\,\Phi(q)\,\theta_-(\Phi(q)-\theta_-)\,e^{(\Phi(q)+\theta_-)b_0^*}}{\Phi(q)\,e^{\Phi(q)b_0^*} - \theta_-\,e^{\theta_- b_0^*}}.$$

Since $\Phi(q) > 0$, $\theta_- < 0$, and $M > 0$, this expression is strictly negative, giving $V_1''(b_0^*;b_0^*) < V_0''(b_0^*;b_0^*) = 0$. By continuity, the root b_1^* lies strictly to the left of b_0^*. □

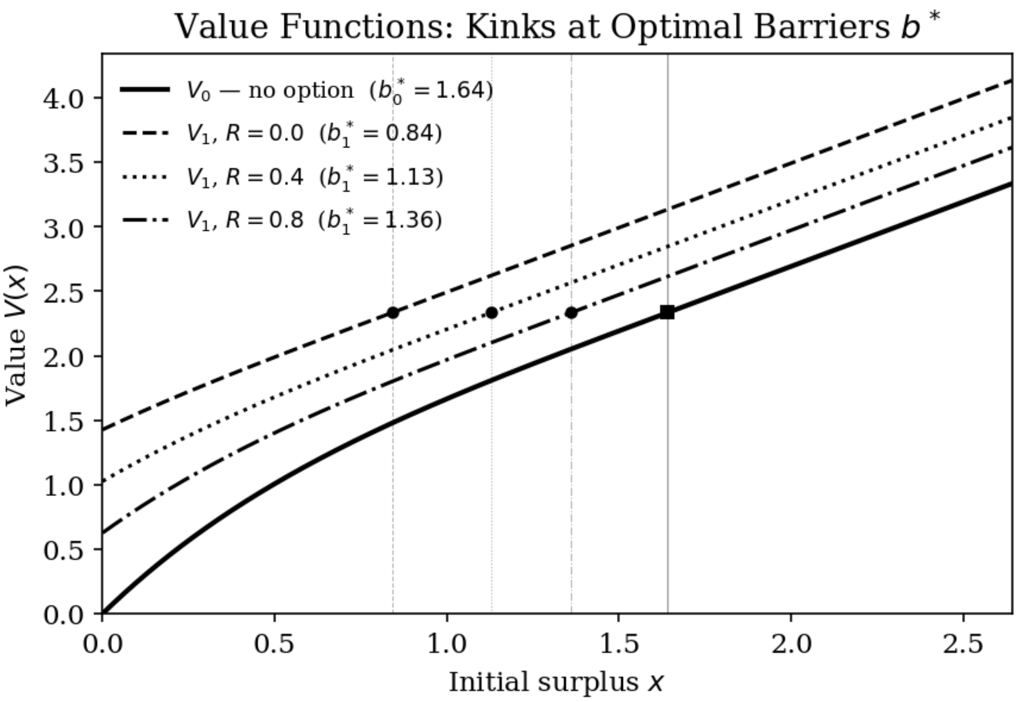

Figure 2: Value functions $V_0(x)$ (solid) and $V_1(x;b_1^*)$ for three resurrection costs $R \in \{0, 0.4, 0.8\}$. Each curve is concave on $[0, b^*]$ and has slope 1 above b^*; the kink marks the optimal barrier. Parameters: $c = 1.2$, $\lambda = 2$, $\eta = 3$, $q = 0.2$, $x_0 = 0.8$.

8. Comparative Statics

Proposition 8.1 (Monotonicity in R)**.** *The optimal barrier $b_1^*(R)$ is non-decreasing in the resurrection cost R. In particular,*

$$b_1^*(0) \le b_1^*(R) \le b_0^* \quad \textit{for all } R \in [0,\, V_0(x_0;\, b_0^*)],$$

and $b_1^*(R) = b_0^*$ for all $R \ge V_0(x_0;\, b_0^*)$.

Proof. The net resurrection value $M(R) = \max(0, V_0(x_0;b_0^*) - R)$ is non-increasing in R. By Corollary 6.1, a larger M produces a smaller b_1^*. Monotonicity of b_1^* in R follows. When $R \ge V_0(x_0;b_0^*)$, we have $M = 0$ and $b_1^* = b_0^*$ by Remark 5.1. □

Proposition 8.2 (Monotonicity in x_0)**.** *For fixed $R > 0$, the optimal barrier $b_1^*(x_0)$ is non-increasing in the restart level x_0: a higher restart level makes the resurrection option* *more valuable, driving the barrier down.*

Proof. $V_0(x_0; b_0^*)$ is strictly increasing in x_0, so $M(x_0)$ is strictly increasing in x_0 (for R fixed and x_0 in the range where $M > 0$). By Corollary 6.1 applied with $M = M(x_0)$, the result follows. □

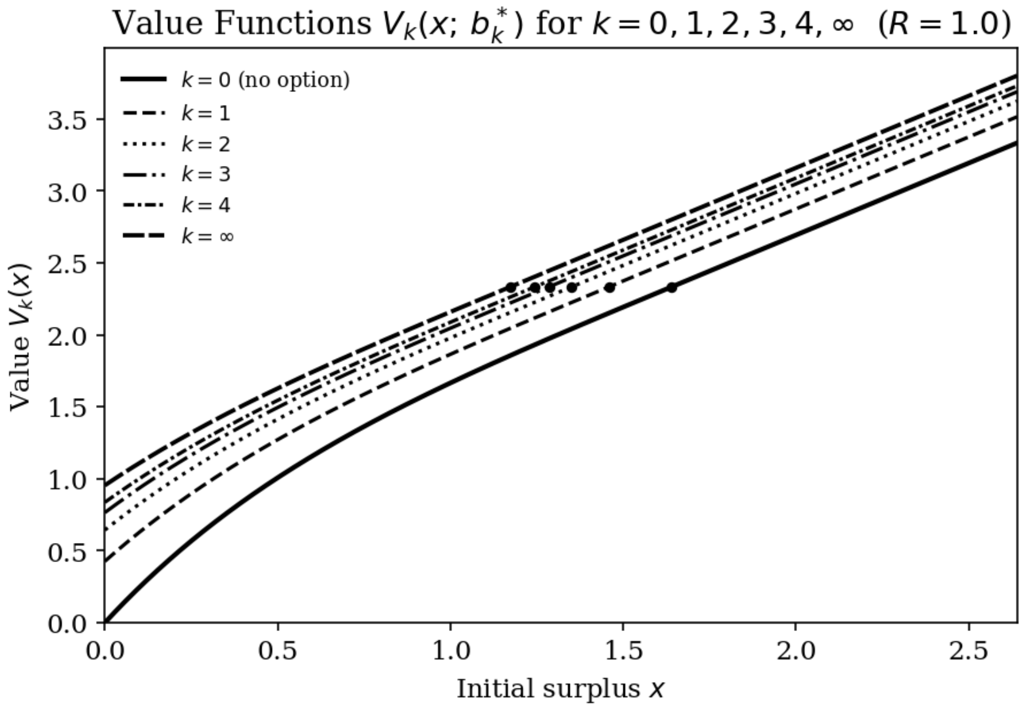

Figure 3: Value functions $V_k(x; b_k^*)$ for $k = 0, 1, 2, 3, 4$ and $k = \infty$, with resurrection cost $R = 1.0$. Each curve is concave on $[0, b_k^*]$ with slope 1 above; the filled circle marks the kink at b_k^*. As k increases, the terminal value at ruin M_k rises, the barrier b_k^* moves left, and the entire value function shifts upward. The $k = \infty$ curve gives the limiting envelope. Parameters as in Figure 2.

9. Algorithm

The following pseudocode computes all quantities exactly for the exponential claims case.

```
Input:  c, lambda, eta, q  (model parameters)
        x0, R               (restart level, resurrection cost)

Step 1. Solve psi(theta) = q for roots:
        a <- c
        b <- c*eta - lambda - q
        d <- -q*eta
        disc <- b^2 - 4*a*d
        Phi_q    <- (-b + sqrt(disc)) / (2*a)
        theta_m  <- (-b - sqrt(disc)) / (2*a)

```

```
Step 2. Compute scale-function coefficients:
        denom <- c * (Phi_q - theta_m)
        A <- (eta + Phi_q) / denom
        B <- (eta + theta_m) / (-denom)
        W(x) := A*exp(Phi_q*x) + B*exp(theta_m*x)

Step 3. Compute standard optimal barrier:
        b0_star <- 2 * log(|theta_m| / Phi_q) / (Phi_q - theta_m)

Step 4. Compute standard value function:
        alpha(b) <- 1 / (Phi_q*exp(Phi_q*b) - theta_m*exp(theta_m*b))
        V0(x; b) := alpha(b) * (exp(Phi_q*x) - exp(theta_m*x))

Step 5. Compute net resurrection value:
        M <- max(0, V0(x0; b0_star) - R)
        If M == 0: b1_star <- b0_star; return

Step 6. Define super-contact residual:
        For b1 in (0, b0_star]:
            D <- Phi_q*exp(Phi_q*b1) - theta_m*exp(theta_m*b1)
            C1(b1) <- (1 - M*theta_m*exp(theta_m*b1)) / D
            C2(b1) <- (M*Phi_q*exp(Phi_q*b1) - 1) / D
            SC(b1) <- C1(b1)*Phi_q^2*exp(Phi_q*b1)
                    + C2(b1)*theta_m^2*exp(theta_m*b1)

Step 7. Solve SC(b1_star) = 0 by bisection on (0, b0_star):
        b1_star <- bisection(SC, lo=epsilon, hi=b0_star)

Step 8. Evaluate augmented value function:
        V1(x; b1_star) := C1(b1_star)*exp(Phi_q*x)
                        + C2(b1_star)*exp(theta_m*x)

Output: Phi_q, theta_m, b0_star, b1_star, V0, V1
```

10. Extension: Multiple Resurrections

The one-shot framework extends naturally to an owner holding $n \geq 1$ resurrection options. The key observation is that each option reduces to the one-shot problem with a modified terminal value at ruin.

Definition 10.1 (n-shot resurrection)**.** An n-shot resurrection scheme grants the owner n sequential options, each exercisable at cost $R \geq 0$, restarting the firm at level $x_0 > 0$. After the k-th exercise the owner holds $n - k$ options remaining.

Definition 10.2 (Recursive value sequence)**.** Define $V_0(x; b_0^*)$ as the standard value function and b_0^* as the standard optimal barrier. For $k \geq 1$, set

$$M_k := \max(0,\ V_{k-1}(x_0; b_{k-1}^*) - R)$$

and let $V_k(x;\, b_k^*)$ be the value function with terminal value M_k at ruin, with optimal barrier b_k^*.

Theorem 10.3 (Recursion)**.** *For each $k \ge 1$, the value function V_k satisfies*

$$V_k(x;\, b_k^*) = C_1^{(k)}\, e^{\Phi(q)x} + C_2^{(k)}\, e^{\theta_- x}, \qquad x \in [0, b_k^*],$$

where $C_1^{(k)}, C_2^{(k)}$ are determined by the linear system

$$C_1^{(k)} + C_2^{(k)} = M_k, \qquad C_1^{(k)}\, \Phi(q)\, e^{\Phi(q) b_k^*} + C_2^{(k)}\, \theta_-\, e^{\theta_- b_k^*} = 1,$$

and b_k^* solves the super-contact condition $V_k''(b_k^*;\, b_k^*) = 0$.

Proof. Each V_k satisfies the same IDE as V_0 on $(0, b_k^*)$ since the dividend and claim dynamics are unchanged. The boundary conditions are $V_k(0) = M_k$ and $V_k'(b_k^*) = 1$. The existence of a unique optimal barrier follows by the same argument as Theorem 6.2, applied with terminal value M_k in place of M. The coefficients and super-contact equation are identical in structure to those of Section 6. □

Theorem 10.4 (Monotone barrier sequence)**.** *The sequence $\{b_k^*\}_{k\ge 0}$ is non-increasing. If $M_1 > 0$, it is strictly decreasing. The sequence $\{V_k(x_0;\, b_k^*)\}_{k\ge 0}$ is non-decreasing.*

Proof. By Corollary 6.1, a larger terminal value at ruin produces a strictly smaller optimal barrier. Since $M_1 = \max(0, V_0(x_0; b_0^*) - R) \ge 0$, we have $b_1^* \le b_0^*$. For the inductive step: if $b_{k-1}^* \le b_{k-2}^*$, then $V_{k-1}(\cdot;\, b_{k-1}^*)$ has a non-negative terminal value at ruin, so (by the comparison principle for the IDE) $V_{k-1}(x_0;\, b_{k-1}^*) \ge V_{k-2}(x_0;\, b_{k-2}^*)$. Hence $M_k \ge M_{k-1}$, which gives $b_k^* \le b_{k-1}^*$. Monotonicity of $V_k(x_0;\, b_k^*)$ follows from the same comparison. □

Remark 10.5. For R sufficiently large ($R \ge V_0(x_0;\, b_0^*)$), $M_1 = 0$ and the sequence is constant: $b_k^* = b_0^*$ for all k. For small R, the sequence b_k^* descends rapidly toward zero — the owner exploits resurrection so aggressively that it is optimal to pay all surplus as dividends immediately.

Theorem 10.6 (Infinite-shot limit)**.** *The sequence $\{b_k^*\}$ converges to a limit $b_\infty^* \ge 0$. When $b_\infty^* > 0$, it is the unique solution in $(0, b_0^*)$ of the fixed-point equation*

$$\frac{-\theta_-}{\Phi(q)\,(\Phi(q)-\theta_-)}\,\frac{e^{\Phi(q)x_0}-1}{e^{\Phi(q)b_\infty^*}} + \frac{\Phi(q)}{\theta_-\,(\Phi(q)-\theta_-)}\,\frac{e^{\theta_- x_0}-1}{e^{\theta_- b_\infty^*}} = R.$$

Proof. The sequence $\{b_k^*\}$ is monotone decreasing and bounded below by zero, hence it converges. At the limit b_∞^*, the value function V_∞ satisfies both smooth pasting $V_\infty'(b_\infty^*) = 1$ and super-contact $V_\infty''(b_\infty^*) = 0$. These two conditions determine $C_1^{(\infty)}$ and $C_2^{(\infty)}$ explicitly in terms of b_∞^*:

$$C_1^{(\infty)} = \frac{-\theta_-}{\Phi(q)\,(\Phi(q)-\theta_-)\,e^{\Phi(q)b_\infty^*}}, \qquad C_2^{(\infty)} = \frac{\Phi(q)}{\theta_-\,(\Phi(q)-\theta_-)\,e^{\theta_- b_\infty^*}}.$$

The fixed-point condition $M_\infty = V_\infty(x_0;\, b_\infty^*) - R = V_\infty(0;\, b_\infty^*)$ then reads $V_\infty(x_0) - V_\infty(0) = R$, which yields the stated equation upon substituting $C_1^{(\infty)}$ and $C_2^{(\infty)}$. □

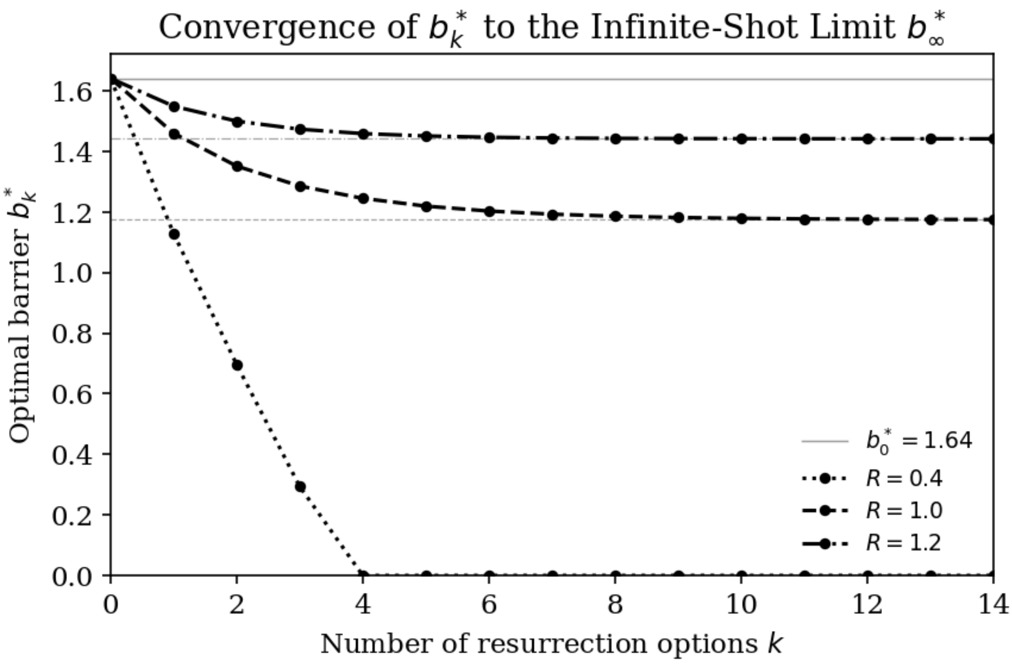

Figure 4: Convergence of the optimal barrier b_k^* as a function of the number of resurrections k, for three values of R. For $R = 1.0$ and $R = 1.2$, the sequence converges to a positive interior fixed point b_∞^* (dashed asymptotes). For $R = 0.4$, the sequence collapses to zero — the resurrection option is so valuable that the owner abandons the barrier entirely. Same parameters as Figure 2.

11. Algorithm for Multiple Resurrections

The following pseudocode extends Algorithm 1 to the n-shot case and computes the infinite-shot fixed point.

```
Input:  c, lambda, eta, q     (model parameters)
        x0, R, n              (restart level, cost, number of shots)

Step 1-4. Same as Algorithm 1: compute Phi, theta_m, b0_star, V0.

Step 5. Initialise recursion:
        b_prev  <- b0_star
        v_prev  <- V0(x0; b0_star)

Step 6. For k = 1 to n:
        M_k <- max(0, v_prev - R)
        If M_k == 0:
            b_k_star <- b0_star     (option has no value)
            break
        D    <- Phi*exp(Phi*b_prev) - theta_m*exp(theta_m*b_prev)
        SC(b) := [(1 - M_k*theta_m*exp(theta_m*b)) * Phi^2 * exp(Phi*b)
               +  (M_k*Phi*exp(Phi*b) - 1) * theta_m^2 * exp(theta_m*b)]
               / [Phi*exp(Phi*b) - theta_m*exp(theta_m*b)]
```

```
        If SC(eps) * SC(b_prev) < 0:
            b_k_star <- bisection(SC = 0, lo=eps, hi=b_prev)
        Else:
            b_k_star <- 0          (corner solution)
        C1 <- (1 - M_k*theta_m*exp(theta_m*b_k_star))
             / (Phi*exp(Phi*b_k_star) - theta_m*exp(theta_m*b_k_star))
        C2 <- M_k - C1
        v_prev  <- C1*exp(Phi*x0) + C2*exp(theta_m*x0)
        b_prev  <- b_k_star
        Record b_k_star

Step 7. Compute infinite-shot fixed point b_inf_star (if R is large enough):
        A <- -theta_m * (exp(Phi*x0) - 1) / (Phi * (Phi - theta_m))
        B <-  Phi     * (exp(theta_m*x0) - 1) / (theta_m * (Phi - theta_m))
        FP(b) := A * exp(-Phi*b) + B * exp(-theta_m*b) - R
        If FP has a sign change on (eps, b0_star):
            b_inf_star <- bisection(FP = 0)
        Else:
            b_inf_star <- 0  (corner regime)

Output: {b_k_star}_{k=1}^{n},  b_inf_star
```

12. References

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