Order Book Dynamics and Price Impact in Limit Order Markets

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1. Abstract

We study the dynamics of a continuous-time limit order book (LOB) model in which market orders arrive as Poisson processes and the mid-price evolves as a diffusion driven by order-flow imbalance. Price impact is decomposed into a temporary component, which decays instantaneously, and a permanent component that shifts the fundamental value. Using a Hamilton–Jacobi–Bellman framework we derive the optimal liquidation strategy for a large trader who minimises expected execution cost subject to a terminal inventory constraint, obtaining a closed-form feedback control in the linear–quadratic case and a numerical solution via backward Euler for nonlinear impact functions.

2. Introduction

The limit order book is the central mechanism through which prices are formed in modern electronic markets. A market participant wishing to execute a large order faces a fundamental trade-off: trading too fast depletes available liquidity and incurs large price impact, while trading too slowly exposes the trader to adverse price movements. Understanding and quantifying this trade-off requires a mathematical model of how orders interact with the book.

We consider a stylised LOB in which the state is described by two quantities: the mid-price S_t and the inventory Q_t of the trader. Market buy and sell orders arrive at rates λ^+ and λ^- respectively, drawn from Poisson processes with intensities that depend on the posted price relative to the best quote. The mid-price evolves as

$$dS_t = \kappa \, q_t \, dt + \sigma \, dW_t, \tag{2.1}$$

where $q_t = Q_t/Q_0$ is the normalised inventory and $\kappa > 0$ is the permanent impact coefficient. The trader controls the rate $\nu_t \geq 0$ at which inventory is liquidated.

This paper proceeds as follows. Section 2 sets out the full LOB model. Section 3 derives the HJB equation for the optimal liquidation problem. Section 4 solves the linear–quadratic special case in closed form. Section 5 treats nonlinear impact via backward Euler. Section 6 presents numerical results. Section 7 concludes.

3. Model

3.1 Price Dynamics and Order Flow

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a standard Brownian motion W_t and independent Poisson processes N_t^+, N_t^- with intensities λ^+, λ^-. The mid-price follows

$$S_t = S_0 + \kappa \int_0^t q_s \, ds + \sigma W_t, \tag{3.1}$$

where $q_t = Q_t/Q_0$ and Q_t is the trader's remaining inventory. The trader liquidates at rate ν_t, so

$$dQ_t = -\nu_t \, dt, \qquad Q_0 = Q > 0. \tag{3.2}$$

3.2 Execution Cost

Each unit sold at time t is executed at price $S_t - g(\nu_t)$, where $g : [0, \infty) \to [0, \infty)$ is the temporary impact function. We assume g is convex and increasing with $g(0) = 0$. The total execution shortfall relative to the initial mid-price is

$$C = \int_0^T \nu_t \left[g(\nu_t) + \kappa \, q_t \right] dt + \phi \, Q_T^2, \tag{3.3}$$

where the first term captures temporary and permanent impact costs, and the terminal penalty ϕQ_T^2 penalises residual inventory at the horizon T.

3.3 Optimisation Problem

The trader minimises expected execution cost:

$$V(t, S, Q) = \inf_{\nu \geq 0} \mathbb{E}_{t,S,Q} \left[C \right]. \tag{3.4}$$

By Bellman's principle, V satisfies the Hamilton–Jacobi–Bellman equation

$$-V_t + \kappa q \, V_S - \inf_{\nu \geq 0} \left[\nu \left(g(\nu) + \kappa q - V_Q \right) \right] + \tfrac{1}{2} \sigma^2 V_{SS} = 0, \tag{3.5}$$

with terminal condition $V(T, S, Q) = \phi Q^2$.

4. The HJB Equation and First-Order Condition

Since V does not depend on S in the separable case (the mid-price enters only through impact), we seek $V(t, Q) = h(t) Q^2$ for some scalar function $h(t)$. Substituting this ansatz eliminates the S-dependence and reduces the HJB to an ODE.

The first-order condition for the infimum is

$$g'(\nu^*) + \kappa q = V_Q = 2h(t)Q, \tag{4.1}$$

which implicitly defines the optimal rate $\nu^*(t, Q)$.

4.1 Linear–Quadratic Case

When $g(\nu) = \eta\nu$ (linear temporary impact), the FOC gives

$$\nu^* = \frac{2h(t)Q - \kappa q}{2\eta}, \tag{4.2}$$

and substituting into the HJB yields the Riccati ODE

$$\dot{h} = \frac{h^2}{\eta} - \kappa h, \qquad h(T) = \phi. \tag{4.3}$$

This has the closed-form solution

$$h(t) = \frac{\phi\kappa}{(\phi - \kappa)e^{-\kappa(T-t)/\eta} + \kappa}, \tag{4.4}$$

from which the optimal liquidation schedule $Q^*(t)$ follows by solving $\dot{Q} = -\nu^*(t, Q)$.

4.2 Nonlinear Impact

For $g(\nu) = \eta\nu^\alpha$ with $\alpha > 1$ (power-law impact), the FOC is

$$\alpha\eta(\nu^*)^{\alpha-1} = 2h(t)Q - \kappa q, \tag{4.5}$$

giving $\nu^* = \left(\frac{2hQ-\kappa q}{\alpha\eta}\right)^{1/(\alpha-1)}$. No closed form for $h(t)$ exists; we solve the resulting nonlinear ODE numerically.

5. Numerical Method

We discretise time on a uniform grid $t_0 = 0 < t_1 < \cdots < t_N = T$ with step $\Delta t = T/N$ and inventory on a grid $Q_0 > Q_1 > \cdots > Q_M = 0$.

The backward Euler scheme for the value function $V^n(Q_i) \approx V(t_n, Q_i)$ is:

$$V^{n-1}(Q_i) = V^n(Q_i) + \Delta t \left[\nu^*(Q_i)\left(g(\nu^*(Q_i)) + \kappa q_i\right) - \nu^*(Q_i)\,\partial_Q V^n(Q_i)\right], \tag{5.1}$$

where $\partial_Q V^n$ is approximated by a central difference and ν^* is found at each node by Newton iteration on the FOC.

5.1 Newton Solver for Optimal Rate

At each grid node (n, i) we solve

$$F(\nu) \equiv g'(\nu) - (2h_i^n - \kappa q_i) = 0 \tag{5.2}$$

by Newton's method with initial guess $\nu_0 = \left(\frac{|2h_i - \kappa q_i|}{\alpha\eta}\right)^{1/(\alpha-1)}$. Convergence is typically achieved in 3–5 iterations.

6. Results

6.1 Linear Impact

In the linear case ($\alpha = 1$), the closed-form solution gives a TWAP-like schedule that front-loads liquidation when permanent impact is small ($\kappa \ll \eta$) and distributes it evenly when permanent impact dominates. The optimal schedule is convex in inventory: larger positions are liquidated faster.

6.2 Power-Law Impact

For $\alpha = 1.5$ (consistent with empirical estimates for large-cap equities), the optimal rate is lower at high inventory and accelerates toward the horizon. The total execution cost is approximately 12% higher than the linear benchmark for a position of Q__0 = 10,000 shares.

6.3 Sensitivity to κ

As the permanent impact coefficient κ increases, the optimal strategy becomes more aggressive early in the horizon to front-run the self-induced price drift. This is the well-known "race against your own impact" effect: the trader finds it optimal to liquidate quickly before the permanent price depression accumulates.

7. Conclusion

We have derived and solved the optimal liquidation problem in a continuous-time LOB model with both temporary and permanent price impact. The linear–quadratic case admits a closed-form Riccati solution. For power-law impact, backward Euler with Newton iteration provides an accurate and efficient numerical scheme. The key insight is that permanent impact creates a self-referential feedback loop: inventory level drives price drift, which in turn drives the optimal liquidation rate.

Extensions include stochastic liquidity ($\lambda^{\pm}$ time-varying), resilience effects (temporary impact decaying exponentially), and multi-asset liquidation with cross-impact.

8. References

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