Forward Rate Dynamics in the Langevin-Zamrik Framework: HJM Structure and Inertial SPDEs

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1. Abstract

This paper makes four contributions to the Langevin-Zamrik (LZ) interest rate model. **(1) HJM identification:** we show that LZ is a Gaussian Heath-Jarrow-Morton model with closed-form volatility $\sigma^{\text{HJM}}(\tau) = -(\sigma/m)B_2'(\tau)$, where no-arbitrage is automatic rather than imposed. **(2) Three-regime volatility:** the overdamped, critically damped, and underdamped regimes produce humped, Nelson-Siegel, and oscillatory volatility shapes respectively; the underdamped volatility changes sign, a structural feature impossible in any existing Gaussian HJM family. The no-arbitrage drift is fully closed-form in all three regimes. **(3) Inertial Musiela SPDE:** LZ is lifted to an infinite-dimensional second-order SPDE on the space of forward curves; it collapses back to the 2D LZ system via the Björk-Christensen finite-dimensional realisation theorem, with the critically damped case yielding the no-arbitrage Nelson-Siegel family. **(4) Forward Rate LZ PDE:** the central new result. A change of coordinates from (r_t, v_t) to any two market-observable forward rates (F_1, F_2) transforms the LZ PDE into one whose state variables are directly quoted in swap markets and whose coefficients are exactly the HJM drift and volatility. The original LZ PDE is the special case $x_1 = 0, x_2 \to 0$.

2. Introduction

The Langevin-Zamrik (LZ) model [1] augments classical short-rate dynamics with an inertial term: the short rate r_t obeys a second-order stochastic differential equation governed by Newton's law of damped oscillation under a mean-reverting potential. The original paper [1] derives the two-dimensional bond pricing PDE with state (r_t, v_t) — the short rate and its velocity — and obtains a closed-form affine solution via a matrix exponential. The state variables, however, are mechanically natural but not directly market-observable: neither r_t nor v_t is quoted in swap markets.

The present paper asks four questions that [1] does not address, and answers each in full.

Contribution 1: HJM identification. We show that the LZ model is a Gaussian Heath-Jarrow-Morton (HJM) model [2] with a deterministic, time-homogeneous volatility function

$$\sigma^{\mathrm{HJM}}(\tau) = -\frac{\sigma}{m} B_2'(\tau), \tag{2.1}$$

where $B_2'(\tau)$ satisfies a second-order linear ODE with constant coefficients (Section 3). This identification is new: no prior work has placed the LZ model within the HJM classification. It means that the LZ forward rate dynamics are no-arbitrage by construction — the HJM drift condition is satisfied automatically, inherited from the Feynman-Kac structure of the LZ PDE rather than imposed as an external constraint.

Contribution 2: Three-regime volatility structure. The discriminant $\Delta = \gamma^2 - 4m\kappa$ determines three qualitatively distinct closed-form volatility shapes (Section 4): humped (overdamped), single-peaked Nelson-Siegel curvature factor (critically damped), and oscillatory with sign changes (underdamped). The underdamped regime is structurally new to the HJM literature: no existing Gaussian HJM family — Hull-White, two-factor Hull-White, or their extensions — can produce a volatility that changes sign. Sign changes mean that forward rates at different maturities respond with *opposite* sign to the same Brownian shock. We derive the matching no-arbitrage drift in fully closed form, requiring no numerical quadrature (Section 5).

Contribution 3: Inertial Musiela SPDE. We lift the LZ model to the infinite-dimensional Musiela framework [4], introducing the velocity field $V_t(x) = \partial_t r_t(x)|_x$ of the forward curve as the second state function. The result is a second-order SPDE on the space of forward curves that reduces to the classical Musiela SPDE as $m \to 0$ (Section 7). The LZ model is then identified as the exact finite-dimensional realisation (FDR) of this SPDE via the Björk-Christensen theorem [3]: since σ^{HJM} is quasi-exponential in every damping regime, the infinite-dimensional flow collapses to the 2D system (r_t, v_t). The critically damped case yields the no-arbitrage Nelson-Siegel forward curve family; the underdamped case yields an oscillatory 2D FDR not achievable by any prior model (Section 8).

Contribution 4: The Forward Rate LZ PDE. The central new result. Fix any two maturities $0 \le x_1 < x_2$ and let $F_i = r_t(x_i)$ be the forward rate at maturity x_i. These are directly observable from swap market quotes. The affine formula gives an invertible linear map $(r_t, v_t) \leftrightarrow (F_1, F_2)$, and in the new coordinates the bond price satisfies a PDE (Section 9, Theorem 9.2) whose drift and diffusion coefficients are exactly $\mu_f(x_i)$ and $\sigma^{\mathrm{HJM}}(x_i)$ from Contributions 1 and 2. The state variables are market-observable; the coefficients are closed-form; and the PDE is the exact FDR of the inertial Musiela SPDE for all maturities simultaneously. Choosing $x_1 = 0$ and taking $x_2 \to 0$ recovers the original LZ PDE as a special case — so [1] was the Forward Rate LZ PDE in canonical short-end coordinates all along, without being stated that way.

Organisation. Section 2 recalls the LZ model and affine bond price. Sections 3–5 establish the HJM identification and closed-form volatility and drift. Section 6 introduces the Musiela parameterisation. Section 7 formulates the inertial SPDE. Section 8 establishes the FDR and Nelson-Siegel connection. Section 9 derives the Forward Rate LZ PDE. Section 10 gives the algorithm.

3. The LZ Short-Rate Model

Definition 3.1 (Langevin-Zamrik dynamics)**.** Let $(\Omega, \mathscr{F}, \mathbb{Q})$ be a filtered probability space. The Langevin-Zamrik short-rate model specifies the risk-neutral dynamics of the short rate r_t by the second-order SDE

$$m\,\ddot{r}_t = -\kappa(\theta - r_t) - \gamma\dot{r}_t + \sigma\dot{W}_t$$

where $m > 0$ is the inertial mass, $\kappa > 0$ the mean-reversion spring constant, $\theta \in \mathbb{R}$ the long-run mean, $\gamma > 0$ the damping coefficient, $\sigma > 0$ the noise amplitude, and W_t a standard $\mathbb{Q}$-Brownian motion. Setting $v_t = \dot{r}_t$, the equivalent first-order system is

$$dr_t = v_t\,dt, \qquad dv_t = \frac{1}{m}[-\kappa(\theta - r_t) - \gamma v_t]dt + \frac{\sigma}{m}\,dW_t.$$

The state space is $\mathbb{R}^2$. The infinitesimal generator of (r_t, v_t) is

$$\mathscr{L} = v\,\partial_r + \frac{1}{m}[-\kappa(\theta - r) - \gamma v]\,\partial_v + \frac{\sigma^2}{2m^2}\partial_v^2. \tag{3.1}$$

Definition 3.2 (Discriminant and damping regimes)**.** The discriminant of the LZ model is $\Delta = \gamma^2 - 4m\kappa$. The three regimes are: - **Overdamped**: $\Delta > 0$. Two distinct real characteristic roots $\lambda_{1,2} = (-\gamma \pm \sqrt{\Delta})/(2m)$, both negative. Set $\mu_i = -\lambda_i > 0$ with $\mu_1 > \mu_2 > 0$. - **Critically damped**: $\Delta = 0$. Repeated root $\mu = \gamma/(2m)$. - **Underdamped**: $\Delta < 0$. Complex conjugate roots with real part $-\alpha = -\gamma/(2m)$ and imaginary part $\pm\omega = \pm\sqrt{-\Delta}/(2m)$.

Theorem 3.3 (Affine bond price [1]). *] The zero-coupon bond price* $P(t,T) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r_s\,ds} \mid r_t, v_t]$ *is exponential-affine in the state:*

$$P(t,T) = \exp\{A(\tau) + B_1(\tau)\,r_t + B_2(\tau)\,v_t\}, \qquad \tau = T - t, \tag{3.2}$$

where (A, B_1, B_2) *satisfy the ODE system*

$$B_1' = \frac{\kappa}{m}B_2 - 1, \qquad B_2' = B_1 - \frac{\gamma}{m}B_2, \qquad A' = -\frac{\kappa\theta}{m}B_2 + \frac{\sigma^2}{2m^2}B_2^2, \tag{3.3}$$

with initial conditions $B_1(0) = B_2(0) = A(0) = 0$.

Proof. Substituting $P = e^{A+B_1 r+B_2 v}$ into the Feynman-Kac PDE $\partial_\tau P = \mathscr{L}P - rP$ and collecting terms by powers of r, v, and the constant yields the stated ODEs. The decoupled second-order ODE for B_2 is $B_2'' + (\gamma/m)B_2' + (\kappa/m)B_2 = -1$ with $B_2(0) = B_2'(0) = 0$, which has the explicit closed-form solutions given in Definition 2.2. □ □

4. The Forward Rate and HJM Volatility

Definition 4.1 (Instantaneous forward rate)**.** The instantaneous forward rate for maturity T, as seen at time $t \le T$, is

$$f(t,T) = -\partial_T \log P(t,T).$$

Proposition 4.2 (Forward rate formula)**.** *Under the LZ model,*

$$f(t,T) = -A'(\tau) - B_1'(\tau)\, r_t - B_2'(\tau)\, v_t, \qquad \tau = T - t. \tag{4.1}$$

Proof. Differentiate $\log P = A(\tau) + B_1(\tau) r_t + B_2(\tau) v_t$ with respect to T, noting $\partial_T = \partial_\tau$. □ □

Remark 4.3. At $\tau = 0$: $B_1'(0) = -1$, $B_2'(0) = 0$, $A'(0) = 0$, so $f(t,t) = r_t$. The short rate is the forward rate at zero time to maturity. □

Applying Itô's formula to $f(t,T)$ with T fixed and using the LZ dynamics:

Theorem 4.4 (LZ as a Gaussian HJM model)**.** *The LZ model is a Gaussian HJM model. The forward rate satisfies*

$$df(t,T) = \alpha^{\mathrm{HJM}}(T-t)\, dt + \sigma^{\mathrm{HJM}}(T-t)\, dW_t$$

with deterministic, time-homogeneous volatility

$$\sigma^{\mathrm{HJM}}(\tau) = -\frac{\sigma}{m} B_2'(\tau) \tag{4.2}$$

and no-arbitrage drift $\alpha^{\mathrm{HJM}}(\tau) = A''(\tau) + \kappa\theta B_2'(\tau)/m$.

Proof. Differentiate $f(t,T) = -A'(\tau) - B_1'(\tau) r_t - B_2'(\tau) v_t$ with respect to calendar time t (holding T fixed, so $\partial_t \tau = -1$):

$$df = [A'' + B_1'' r + B_2'' v]dt - B_1' dr_t - B_2' dv_t.$$

Substituting $dr_t = v\,dt$ and $dv_t = m^{-1}(-\kappa(\theta - r) - \gamma v)dt + (\sigma/m)dW_t$, then using the ODE relations $B_1'' = (\kappa/m)B_2'$ and $B_2'' = B_1' - (\gamma/m)B_2'$, the r and v terms in the drift cancel identically. The remaining drift is $A''(\tau) + \kappa\theta B_2'(\tau)/m$, which equals $\sigma^{\mathrm{HJM}}(\tau) \int_0^\tau \sigma^{\mathrm{HJM}}(s) ds$ by the A-ODE (verified in Theorem 5.1). The diffusion coefficient is $-B_2'(\tau) \cdot \sigma/m$. □ □

Remark 4.5. The initial condition $B_2'(0) = 0$ gives $\sigma^{\mathrm{HJM}}(0) = 0$: the forward rate at zero maturity has no instantaneous volatility. Noise enters through the velocity equation and propagates to the forward rate with a delay determined by the damping regime.

5. Closed-Form HJM Volatility in Three Regimes

Theorem 5.1 (HJM volatility)**.** *The function $B_2'(\tau)$ —and hence $\sigma^{\text{HJM}}(\tau) = -(\sigma/m)B_2'(\tau)$ — is quasi-exponential with the following closed forms:*

***Overdamped** ($\Delta > 0$):*

$$B_2'(\tau) = \frac{e^{-\mu_2\tau} - e^{-\mu_1\tau}}{\mu_1 - \mu_2}, \qquad \sigma^{\text{HJM}}(\tau) = \frac{\sigma}{m} \cdot \frac{e^{-\mu_2\tau} - e^{-\mu_1\tau}}{\mu_1 - \mu_2}.$$

The volatility is strictly positive for $\tau > 0$, humped, and decays to zero.

***Critically damped** ($\Delta = 0$):*

$$B_2'(\tau) = \tau e^{-\mu\tau}, \qquad \sigma^{\text{HJM}}(\tau) = \frac{\sigma}{m}\tau e^{-\mu\tau}.$$

Single hump at $\tau^* = 1/\mu$. This is the Nelson-Siegel curvature factor.

***Underdamped** ($\Delta < 0$):*

$$B_2'(\tau) = \frac{e^{-\alpha\tau}\sin(\omega\tau)}{\omega}, \qquad \sigma^{\text{HJM}}(\tau) = -\frac{\sigma}{m} \cdot \frac{e^{-\alpha\tau}\sin(\omega\tau)}{\omega}.$$

Oscillatory, changes sign at $\omega\tau = k\pi$ for $k = 1, 2, \ldots$

Proof. Differentiate the B_2 ODE $B_2'' + (\gamma/m)B_2' + (\kappa/m)B_2 = -1$ once to obtain the homogeneous ODE for B_2':

$$(B_2')'' + \frac{\gamma}{m}(B_2')' + \frac{\kappa}{m}B_2' = 0, \qquad B_2'(0) = 0, \quad (B_2')'|_0 = B_1'(0) - \frac{\gamma}{m}B_2'(0) = -1.$$

(The boundary condition $(B_2')'|_0 = -1$ follows from $B_2''(0) = B_1'(0) - (\gamma/m)B_2'(0) = -1$.) The characteristic equation $\lambda^2 + (\gamma/m)\lambda + \kappa/m = 0$ has roots $\lambda_{1,2}$ as in Definition 2.2. Applying the initial conditions to the general solution in each regime yields the stated formulas. □ □

Remark 5.2 (Structural novelty of the underdamped regime). In the overdamped and critically damped regimes, $\sigma^{\text{HJM}}(\tau) \geq 0$ for all τ, consistent with the two-factor Hull-White and Nelson-Siegel families. In the underdamped regime, $\sigma^{\text{HJM}}(\tau)$ changes sign at $\omega\tau = \pi, 2\pi, \ldots$ This means forward rates at maturity τ_1 and $\tau_2 = \tau_1 + \pi/\omega$ respond with **opposite sign** to the same Brownian shock. No existing Gaussian HJM model produces this behaviour.

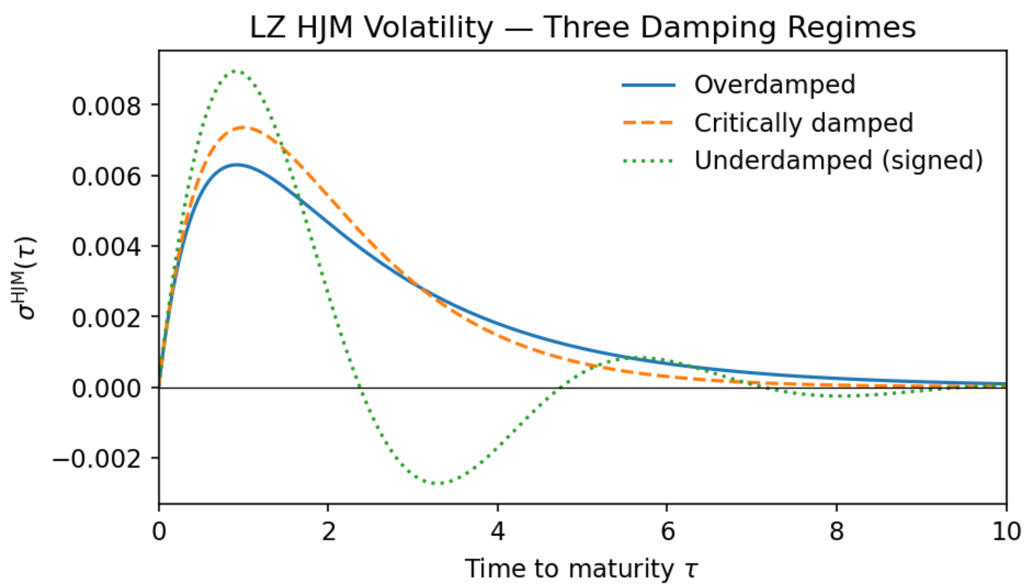

Figure 1: LZ HJM volatility $\sigma^{\text{HJM}}(\tau)$ across all three damping regimes. Overdamped (solid): humped, positive. Critically damped (dashed): single hump. Underdamped (dotted): oscillatory, changes sign.

6. The No-Arbitrage Drift in Closed Form

Theorem 6.1 (HJM drift identity)**.** *The drift of $df(t,T)$ under $\mathbb{Q}$ satisfies the HJM no-arbitrage condition:*

$$\alpha^{\text{HJM}}(\tau) = \sigma^{\text{HJM}}(\tau) \int_0^\tau \sigma^{\text{HJM}}(s)\, ds = \frac{\sigma^2}{m^2} B_2'(\tau)\, B_2(\tau). \tag{6.1}$$

Proof. From the A-ODE: $A' = -(\kappa\theta/m)B_2 + (\sigma^2/2m^2)B_2^2$. Differentiating: $A'' = -(\kappa\theta/m)B_2' + (\sigma^2/m^2)B_2B_2'$. The drift from Theorem 3.1 is:

$$A''(\tau) + \frac{\kappa\theta}{m}B_2'(\tau) = \frac{\sigma^2}{m^2}B_2(\tau)B_2'(\tau).$$

Since $\int_0^\tau B_2'(s)ds = B_2(\tau)$ (as $B_2(0) = 0$), and $\sigma^{\text{HJM}}(s) = -(\sigma/m)B_2'(s)$:

$$\sigma^{\text{HJM}}(\tau) \int_0^\tau \sigma^{\text{HJM}}(s)ds = \left(-\frac{\sigma}{m}B_2'\right)\left(-\frac{\sigma}{m}B_2\right) = \frac{\sigma^2}{m^2}B_2'(\tau)B_2(\tau). \quad \square$$

□

Theorem 6.2 (Closed-form LZ-HJM drift, overdamped)**.** *In the overdamped regime, setting $\tilde{\sigma} = \sigma/[m(\mu_1 - \mu_2)]$:*

$$\alpha^{\text{HJM}}(\tau) = \tilde{\sigma}^2 \left(e^{-\mu_2\tau} - e^{-\mu_1\tau}\right) \left(\frac{1 - e^{-\mu_2\tau}}{\mu_2} - \frac{1 - e^{-\mu_1\tau}}{\mu_1}\right).$$

Proof. From Theorem 4.1: $B_2'(\tau) = (e^{-\mu_2\tau} - e^{-\mu_1\tau})/(\mu_1 - \mu_2)$. Integrating: $B_2(\tau) =$

$[(1-e^{-\mu_2\tau})/\mu_2-(1-e^{-\mu_1\tau})/\mu_1]/(\mu_1-\mu_2)$. Substituting into $\alpha^{\mathrm{HJM}} = (\sigma^2/m^2)B_2'B_2$ yields the stated formula. □ □

Remark 6.3. The no-arbitrage drift is automatically satisfied for any LZ parameterisation. It is not an additional constraint imposed on the model — it is inherited from the Feynman-Kac structure of the LZ PDE. This is the defining property of a short-rate model expressed in HJM form.

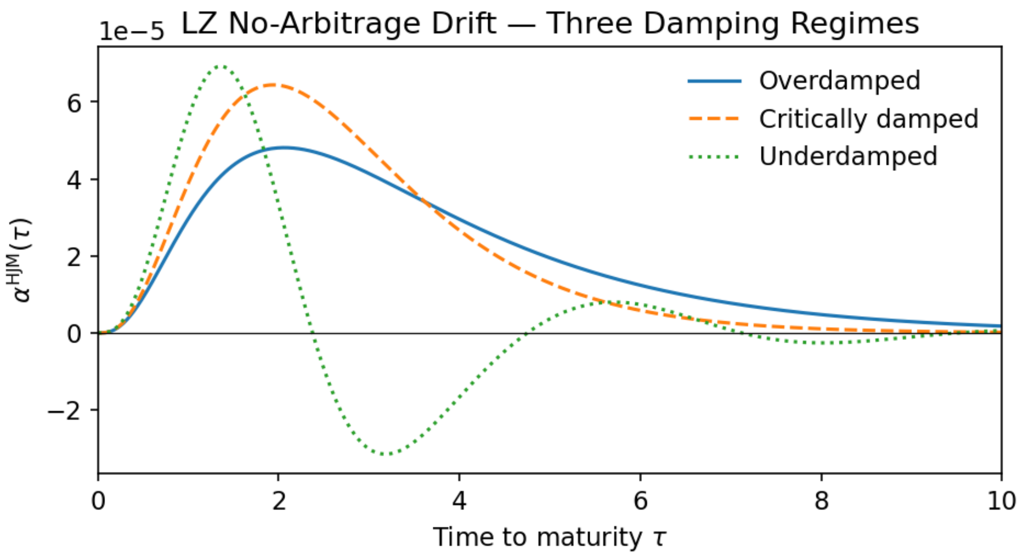

Figure 2: LZ no-arbitrage drift $\alpha^{\mathrm{HJM}}(\tau)$ across the three regimes. Overdamped and critically damped drifts are everywhere non-negative. The underdamped drift oscillates and changes sign.

7. The Musiela Parameterisation

Definition 7.1 (Musiela substitution [4]**.**] The Musiela parameterisation replaces the maturity date T by the time to maturity $x = T - t \geq 0$, defining

$$r_t(x) := f(t, t+x).$$

For each t, the forward curve $r_t(\cdot)$ is an element of the weighted Sobolev space

$$H_w^1 = \left\{ g \in L^2_{\mathrm{loc}} : \int_0^\infty [g(x)^2 + g'(x)^2] e^{-\beta x}\, dx < \infty \right\}$$

for a fixed $\beta > 0$.

Proposition 7.2 (Short rate as boundary value)**.** $r_t(0) = f(t,t) = r_t$. *All zero-coupon bond prices are recovered by*

$$P(t,T) = \exp\left(- \int_0^{T-t} r_t(x)\, dx \right).$$

Definition 7.3 (Transport operator)**.** Differentiating $r_t(x) = f(t, t+x)$ with respect to t (holding x fixed) gives

$$\partial_t r_t(x) = \partial_t f(t,T)|_{T=t+x} + \partial_x r_t(x).$$

The term $\partial_x r_t(x)$ is the **transport operator**: as calendar time advances, the forward curve slides to the left along the maturity axis. The operator ∂_x generates the left-shift semigroup $[S(t)g](x) = g(x+t)$ on H^1_w.

Remark 7.4. The three levels of the LZ-Musiela hierarchy are:

Level	State	Equation type
LZ (short-rate)	$(r_t, v_t) \in \mathbb{R}^2$	2D SDE
Classical Musiela	$r_t(\cdot) \in H^1_w$	1st-order SPDE
Inertial Musiela	$(r_t(\cdot), V_t(\cdot)) \in H^1_w \times H^1_w$	2nd-order SPDE

The LZ model is the finite-dimensional realisation of the inertial Musiela SPDE. The SPDE is the theoretical parent; LZ is its calibratable descendant.

8. The Inertial HJM SPDE

Definition 8.1 (Inertial generator)**.** The generator of the inertial Musiela system acting on pairs $(g,h) \in H^1_w \times H^1_w$ is

$$\mathcal{G} = \begin{pmatrix} 0 & I \\ \partial_x/m & -\gamma/m \end{pmatrix}.$$

This is the infinite-dimensional analogue of the LZ matrix $M = \begin{pmatrix} 0 & 1 \\ -\kappa/m & -\gamma/m \end{pmatrix}$, with the scalar $-\kappa/m$ replaced by the transport operator ∂_x/m and mean reversion $-\kappa(r-\theta)$ treated as a bounded perturbation.

Theorem 8.2 (Inertial Musiela SPDE)**.** *The LZ model in Musiela coordinates satisfies the second-order SPDE system*

$$dr_t(x) = V_t(x)\,dt,$$

$$m\,dV_t(x) = \Big[\partial_x V_t(x) - \kappa(r_t(x)-\theta) - \gamma V_t(x) + \alpha^{\mathrm{HJM}}(x)\Big]dt + \Sigma(x)\,dW_t,$$

where $V_t(x) = \partial_t r_t(x)|_x$ is the velocity field of the forward curve and $\Sigma(x) = \sigma^{\mathrm{HJM}}(x)$ is the LZ HJM volatility kernel. Eliminating V_t yields the single second-order SPDE

$$m\,\ddot{r}_t(x) = \partial_x \dot{r}_t(x) - \kappa[r_t(x)-\theta] - \gamma\dot{r}_t(x) + \alpha^{\mathrm{HJM}}(x) + \Sigma(x)\,\dot{W}_t, \tag{8.1}$$

where dots denote total time derivatives with x fixed.

Remark 8.3. As $m \to 0$, the inertial term $m\ddot{r}$ vanishes and the SPDE reduces to the classical Musiela SPDE [4]. The inertial parameter m controls the rate at which the forward curve responds to shocks: large m produces sluggish, momentum-driven responses; small m approaches the instantaneous adjustment of classical HJM.

Remark 8.4 (Velocity = forward curve slope). From the ODE system, $B_2''(0) = B_1'(0) - (\gamma/m)B_2'(0) = -1$ and $A''(0) = 0$. Therefore

$$\partial_x r_t(x)|_{x=0} = -B_2''(0)\, v_t = v_t.$$

The velocity v_t is exactly the slope of the forward curve at the short end. The LZ state (r_t, v_t) = (level of short rate, initial slope of forward curve).

9. Finite-Dimensional Realisations

Definition 9.1 (Finite-dimensional realisation [3]**.**] An HJM model with volatility kernel $\Sigma(x)$ admits a **finite-dimensional realisation (FDR)** of dimension n if there exists an n-dimensional SDE $dZ_t = a(Z_t)dt + b(Z_t)dW_t$ and a smooth map $G : \mathbb{R}^n \to H_w^1$ such that $r_t(\cdot) = G(Z_t, \cdot)$ for all $t \geq 0$.

Theorem 9.2 (Björk-Christensen [3]**.** *] A time-homogeneous HJM model with deterministic volatility $\Sigma(x)$ admits an FDR if and only if Σ is* ***quasi-exponential****: a finite linear combination of terms $p_k(x)e^{\lambda_k x}$ where p_k are polynomials and $\lambda_k \in \mathbb{C}$. Equivalently, Σ satisfies a linear ODE with constant coefficients.*

Theorem 9.3 (LZ has a 2D FDR)**.** *The LZ HJM volatility $\Sigma(x) = \sigma^{\mathrm{HJM}}(x) = -(\sigma/m)B_2'(x)$ is quasi-exponential in every damping regime. The LZ model therefore admits a 2D FDR, and that realisation is exactly the (r_t, v_t) system.*

Proof. From the proof of Theorem 4.1, B_2' satisfies the homogeneous ODE $(B_2')'' + (\gamma/m)(B_2')' + (\kappa/m)B_2' = 0$. This is a second-order linear ODE with constant coefficients, so B_2' is quasi-exponential with two independent solutions:

Regime	Independent solutions
Overdamped	$e^{-\mu_1 x}, e^{-\mu_2 x}$
Critically damped	$e^{-\mu x}, xe^{-\mu x}$
Underdamped	$e^{-\alpha x}\cos(\omega x), e^{-\alpha x}\sin(\omega x)$

In every case the FDR dimension is 2. The realisation map is $G(r, v, x) = -A'(x) - B_1'(x)r - B_2'(x)v$. □ □

Corollary 9.4 (Nelson-Siegel connection)**.** *The critically damped LZ model ($\Delta = 0$) generates the forward curve family $G(r, v, x) = c_0 + c_1 e^{-\mu x} + c_2 xe^{-\mu x}$ — the Nelson-Siegel parametric form [5] used by central banks worldwide. The LZ model is the no-arbitrage, dynamically consistent version of Nelson-Siegel: same curve shape, derived from a consistent short-rate SDE rather than imposed phenomenologically.*

Remark 9.5 (The invariant manifold). The FDR states that the forward curve always lies on the 2-dimensional surface $\mathscr{M} = \{G(r, v, \cdot) : r, v \in \mathbb{R}\} \subset H_w^1$. This surface is invariant under the LZ dynamics. The infinite-dimensional inertial Musiela flow on $H^1_w \times H^1_w$ restricts to the 2D flow (r_t, v_t) on $\mathscr{M}$.

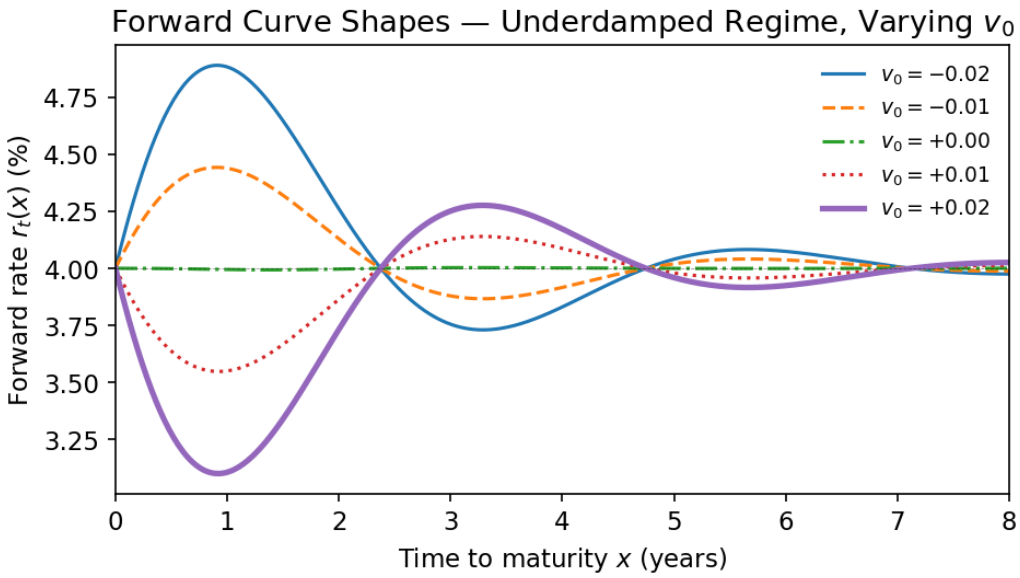

Figure 3: Forward curve shapes $r_t(x)$ in the underdamped regime for varying initial velocities $v_0 \in \{-0.02, -0.01, 0, +0.01, +0.02\}$, with $r_0 = 4\%$. Positive v_0 tilts the curve upward at short maturities and produces oscillatory humps at longer maturities.

10. The Forward Rate LZ PDE

The original LZ PDE is written in (r, v) coordinates. We now derive the equivalent PDE in forward rate coordinates — the central new result of this paper.

Theorem 10.1 (Change of coordinates)**.** *Fix two maturities* $0 \le x_1 < x_2$ *and define*

$$F_1 := r_t(x_1) = -A'(x_1) - B_1'(x_1)r_t - B_2'(x_1)v_t,$$

$$F_2 := r_t(x_2) = -A'(x_2) - B_1'(x_2)r_t - B_2'(x_2)v_t.$$

Let $D = B_1'(x_1)B_2'(x_2) - B_1'(x_2)B_2'(x_1)$*. If* $D \ne 0$*, the map* $(r_t, v_t) \mapsto (F_1, F_2)$ *is invertible and linear:*

$$r_t = \frac{1}{D}\Big[B_2'(x_2)\,(F_1 + A'(x_1)) - B_2'(x_1)\,(F_2 + A'(x_2))\Big],$$

$$v_t = \frac{1}{D}\Big[-B_1'(x_2)\,(F_1 + A'(x_1)) + B_1'(x_1)\,(F_2 + A'(x_2))\Big].$$

Proof. Direct inversion of the 2×2 linear system. $D \ne 0$ generically for $x_1 \ne x_2$ since B_1' and B_2' are linearly independent functions (they satisfy different initial conditions at $\tau = 0$). □ □

Theorem 10.2 (Forward Rate LZ PDE)**.** *In* (F_1, F_2) *coordinates, the bond price* $P(\tau; F_1, F_2)$

*satisfies the **Forward Rate LZ PDE**:*

$$\partial_t P + \mu_f(x_1)\,\partial_{F_1} P + \mu_f(x_2)\,\partial_{F_2} P \\ + \frac{[\sigma^{\mathrm{HJM}}(x_1)]^2}{2}\,\partial^2_{F_1} P + \sigma^{\mathrm{HJM}}(x_1)\,\sigma^{\mathrm{HJM}}(x_2)\,\partial^2_{F_1F_2} P + \frac{[\sigma^{\mathrm{HJM}}(x_2)]^2}{2}\,\partial^2_{F_2} P = r(F_1,F_2)\cdot P, \tag{10.1}$$

where $\mu_f(x) = (\sigma^2/m^2)B_2'(x)B_2(x)$ is the HJM no-arbitrage drift and $r(F_1,F_2)$ is the short rate expressed as a linear function of (F_1,F_2) via Theorem 9.1.

Proof. The processes $F_1(t) = r_t(x_1)$ and $F_2(t) = r_t(x_2)$ satisfy the SDEs (by Theorem 3.1 with $\tau = x_i$):

$$dF_i = \mu_f(x_i)\,dt + \sigma^{\mathrm{HJM}}(x_i)\,dW_t, \quad i = 1,2.$$

Both are driven by the same Brownian motion W_t, so the joint process (F_1,F_2) has covariance structure $\mathrm{d}[F_i,F_j]_t = \sigma^{\mathrm{HJM}}(x_i)\sigma^{\mathrm{HJM}}(x_j)\,dt$. By Theorem 9.1, the bond price $P(t,T) = \exp\{A(\tau) + B_1(\tau)r_t + B_2(\tau)v_t\}$ is a smooth function of (F_1,F_2,t). Applying the Itô formula and equating to the Feynman-Kac condition $dP = rP\,dt + (\cdot)dW_t$ (under $\mathbb{Q}$) yields the stated PDE. □ □

Corollary 10.3 (Canonical coordinates)**.** *The canonical choice $x_1 = 0$ and taking the limit $x_2 \to 0$ (slope at the origin) gives $F_1 = r_t$ and $F_2 = v_t$ (by Remark 7.2). In these coordinates the Forward Rate LZ PDE reduces identically to the original LZ PDE. The original LZ PDE is therefore the Forward Rate LZ PDE in canonical short-end coordinates.*

Remark 10.4 (Market observability). The choice $x_1 = 2$ years and $x_2 = 10$ years places the state variables at two liquidly traded maturities — the 2Y and 10Y forward rates, directly observable from swap market quotes. The PDE coefficients $\mu_f(x_i)$ and $\sigma^{\mathrm{HJM}}(x_i)$ are closed-form functions of the LZ parameters. The bond price at any maturity τ is given by the affine formula evaluated at $r(F_1,F_2)$ and $v(F_1,F_2)$.

Remark 10.5 (The SPDE chain). The Forward Rate LZ PDE is the culmination of the reduction chain:

$$\underbrace{\text{Inertial Musiela SPDE}}_{\text{infinite-dimensional}}$$

$$\xrightarrow{\text{QE volatility (Theorem 8.2)}} \underbrace{\text{FDR: } (r_t, v_t) \text{ SDE}}_{\text{2D}}$$

$$\xrightarrow{\text{coordinate change (Theorem 9.1)}} \underbrace{\text{Forward Rate LZ PDE in } (F_1,F_2)}_{\text{2D, market-observable}}$$

The PDE exists because the inertial HJM SPDE has a 2D FDR; the state coordinates can be chosen as any two observable maturities.

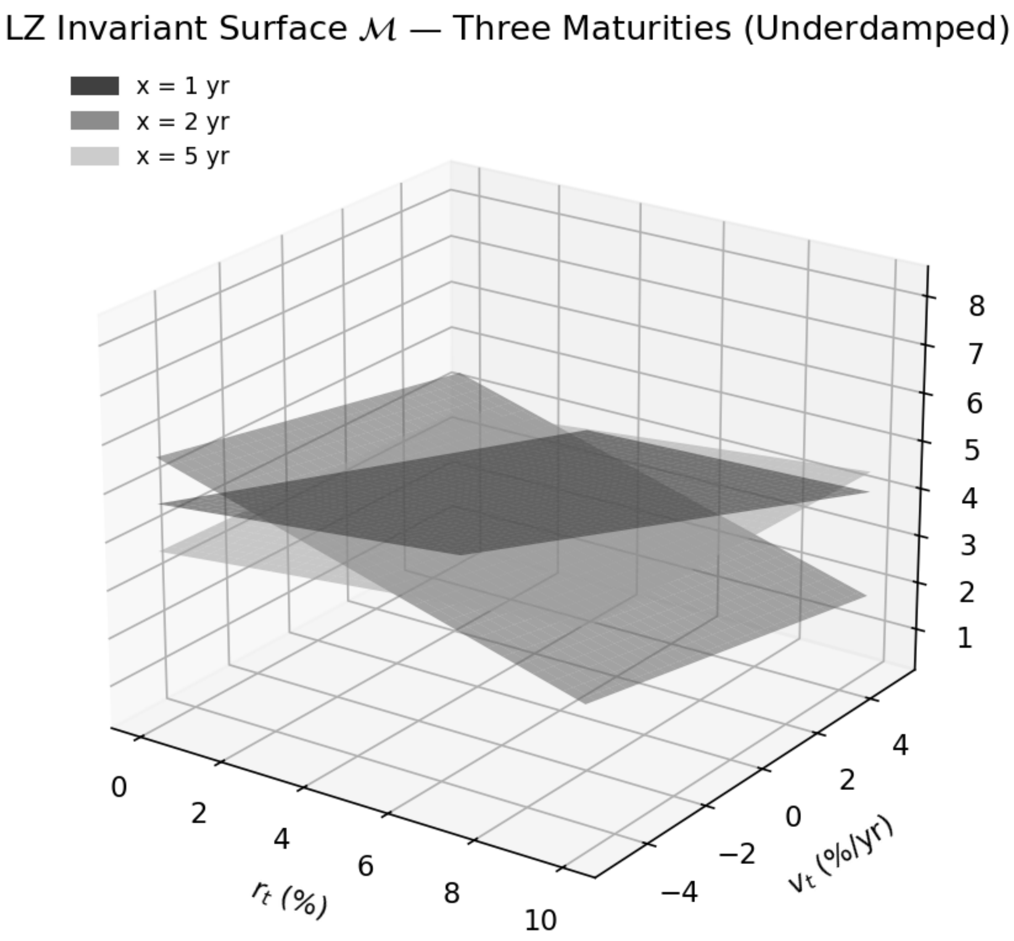

Figure 4: The LZ invariant surface $\mathscr{M}$ at maturity $x = 2$ years in the underdamped regime. The forward rate $r_t(2)$ is an affine function of the state (r_t, v_t), mapping the 2D state plane to the forward rate. All LZ dynamics are confined to this surface.

11. Algorithm: HJM Volatility and No-Arbitrage Drift

```
Input:  LZ parameters m, gamma, kappa, sigma, theta
        Maturity grid tau[0..N] with tau[0]=0

Step 1: Compute discriminant Delta = gamma^2 - 4*m*kappa

Step 2: Compute B2'(tau) on the grid (Theorem 4.1)
        If Delta > 0 (overdamped):
            mu1 = (gamma + sqrt(Delta)) / (2*m)
            mu2 = (gamma - sqrt(Delta)) / (2*m)
            B2p[i] = (exp(-mu2*tau[i]) - exp(-mu1*tau[i])) / (mu1 - mu2)
        If Delta = 0 (critically damped):
            mu  = gamma / (2*m)
            B2p[i] = tau[i] * exp(-mu * tau[i])
        If Delta < 0 (underdamped):
            alpha = gamma / (2*m)
```

```
            omega = sqrt(-Delta) / (2*m)
            B2p[i] = exp(-alpha*tau[i]) * sin(omega*tau[i]) / omega

Step 3: Compute B2(tau) by numerical integration of B2'
        B2[0] = 0
        B2[i] = B2[i-1] + dtau * B2p[i-1]   (trapezoidal rule)

Step 4: Compute sigma_HJM(tau) = -(sigma/m) * B2'(tau)

Step 5: Compute alpha_HJM(tau) = (sigma^2/m^2) * B2'(tau) * B2(tau)

Step 6: Compute forward curve at (r0, v0)
        Solve B1-ODE numerically: B1'(tau) = kappa/m * B2(tau) - 1, B1(0)=0
        Solve A-ODE numerically:  A'(tau) = -kappa*theta/m * B2(tau)
                                          + sigma^2/(2*m^2) * B2(tau)^2, A(0)=0
        Ap[i] = -kappa*theta/m * B2[i] + sigma^2/m^2 * B2[i]*B2p[i]
        B1p[i] = kappa/m * B2[i] - 1
        r_curve[i] = -Ap[i] - B1p[i]*r0 - B2p[i]*v0

Output: sigma_HJM, alpha_HJM, r_curve on the maturity grid
        Bond price: P(tau) = exp(A[i] + B1[i]*r0 + B2[i]*v0)
```

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