Forward Rate Derivatives Pricing under the Langevin-Zamrik Model

Pricing Caplets, Bond Options, and Swaptions in LZ-HJM Form

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1. Abstract

We price interest rate derivatives — caplets, floorlets, bond options, and swaptions — under the Langevin-Zamrik (LZ) PDE model, using its representation as a Gaussian Heath-Jarrow-Morton (HJM) model in forward-rate space. Since the LZ forward rate is normally distributed (not lognormal), caplets are priced exactly as put options on zero-coupon bonds — whose log-prices are Gaussian — giving a Black formula on the lognormal bond price with variance $\sigma_P^2(\tau_0, \delta)$. Swaptions use the Bachelier (normal) formula on the approximately-Gaussian swap rate. We derive closed-form expressions for the bond option variance $\sigma_P^2(\tau_0, \tau_S)$, the HJM variance profile $V(\tau) = \int_0^\tau [\sigma^{\text{HJM}}(s)]^2\,ds$, and the swaption cross-covariance $C_{ij}(\tau_0)$ in all three LZ damping regimes and in 1FHW. A key structural finding is that in the underdamped regime the bond option volatility is non-monotone in bond tenor τ_S — it can decrease as tenor increases near resonance periods $n\pi/\omega$ — a phenomenon impossible in any Gaussian HJM model with positive σ^{HJM}. A unified LZ-HJM pricer algorithm computes all instruments in a single pass.

2. Introduction

This paper is the third in the Langevin-Zamrik (LZ) series. Paper 1 [1] established the affine bond pricing framework: the short-rate system

$$dr_t = v_t\,dt, \qquad m\,dv_t = -\gamma v_t\,dt - \kappa(r_t - \theta)\,dt + \sigma\,dW_t \tag{2.1}$$

admits an explicit bond price $P(t,T) = \exp\{A(\tau) + B_1(\tau)r_t + B_2(\tau)v_t\}$ where $\tau = T - t$ and the coefficient functions satisfy a linear ODE cascade. Paper 2 [2] lifted the LZ model to forward-rate space: the forward rates satisfy a Musiela SPDE driven by a single Brownian motion, the HJM volatility function is $\sigma^{\text{HJM}}(\tau) = (\sigma/m)B_2'(\tau)$, and no-arbitrage holds automatically. Paper 2 also showed that LZ is the inertial generalization of the one-factor Hull-White (1FHW) model: as inertia vanishes ($m \to 0$), the LZ dynamics collapse to the 1FHW short-rate SDE.

The natural question left open by Paper 2 is: what does the LZ HJM volatility structure imply for the prices of standard interest rate derivatives? Paper 2 identified the volatility function; this paper computes its integral consequences.

Contribution 1. Market caplets and floorlets are priced exactly as bond options (Theorem 3.1). Since $\log P(T, T+\delta)$ is Gaussian in the LZ model, the bond option formula (Black formula on lognormal bond prices) is exact. The HJM variance profile $V(\tau) = \int_0^\tau [\sigma^{\text{HJM}}(s)]^2\, ds$ is derived in closed form in all three LZ regimes (Theorems 3.3–3.6).

Contribution 2. We derive a closed-form expression for the bond option variance $\sigma_P^2(\tau_0, \tau_S)$ in all three LZ regimes (Theorems 4.3–4.5). The central formula is $\sigma_P^2 = (\sigma^2/m^2) \int_0^{\tau_0} [B_2(r+\tau_S) - B_2(r)]^2\, dr$, which reduces to single known integrals in each regime. In the underdamped case the bond vol is non-monotone in tenor (Corollary 4.8) — a structural prediction with no analogue in 1FHW.

Contribution 3. Swaptions are priced using the Bachelier (normal) formula on the approximately-Gaussian swap rate (Theorem 5.2). The swap rate is normally distributed as a linear combination of Gaussian forward rates — not lognormal as in the BGM framework. Closed-form cross-covariance integrals $C_{ij}(\tau_0)$ are derived for all four models in Theorems 5.3–5.6.

Contribution 4. We establish the systematic comparison between LZ and 1FHW as Gaussian HJM models. All pricing differences are traced to a single structural gap: the inertial delay $\sigma_{\text{LZ}}^{\text{HJM}}(0) = 0$ versus $\sigma_{\text{HW}}^{\text{HJM}}(0) = \sigma > 0$.

Contribution 5. We identify the underdamped implied vol signature: oscillatory caplet vol and non-monotone swaption vol surfaces that no Gaussian HJM model with positive σ^{HJM} can replicate.

The paper is structured as follows. Section 2 sets up both models in HJM forward-rate form and states the structural comparison. Sections 3–5 derive the closed-form pricing formulas for caplets, bond options, and swaptions respectively. Section 6 presents the unified algorithm. Section 7 gives numerical examples for all four models.

3. Models in Forward-Rate Form

3.1 The Langevin-Zamrik Model

The LZ model [1] is the second-order short-rate system:

Definition 3.1 (LZ Dynamics)**.** The risk-neutral short-rate process satisfies

$$dr_t = v_t\, dt, \qquad m\, dv_t = -\gamma v_t\, dt - \kappa(r_t - \theta)\, dt + \sigma\, dW_t \tag{3.1}$$

with parameters $m > 0$ (inertia), $\gamma > 0$ (damping), $\kappa > 0$ (mean reversion), $\theta > 0$ (long-run mean), $\sigma > 0$ (volatility), and a standard $\mathbb{Q}$-Brownian motion W_t. The state vector is (r_t, v_t).

The characteristic polynomial of the drift matrix is $\lambda^2 + (\gamma/m)\lambda + \kappa/m = 0$, with discriminant $\Delta = \gamma^2 - 4m\kappa$. Three damping regimes arise: overdamped ($\Delta > 0$), critically damped ($\Delta = 0$), underdamped ($\Delta < 0$).

Paper 2 [2] established that LZ is a single-factor Gaussian HJM model. The HJM volatility function is:

Theorem 3.2 (LZ HJM Volatility, [2]). *] The instantaneous forward rate $f(t,T) = -\partial_T \log P(t,T)$ satisfies $df(t,T) = \alpha^{\mathrm{HJM}}(\tau)\,dt + \sigma^{\mathrm{HJM}}(\tau)\,dW_t$ where $\tau = T - t$ and*

$$\sigma^{\mathrm{HJM}}_{\mathrm{LZ}}(\tau) = \frac{\sigma}{m} B_2'(\tau) \tag{3.2}$$

The no-arbitrage drift is $\alpha^{\mathrm{HJM}}(\tau) = \sigma^{\mathrm{HJM}}(\tau) \int_0^\tau \sigma^{\mathrm{HJM}}(s)\,ds$. The function $B_2'(\tau)$ is the derivative of the bond-price coefficient $B_2(\tau)$, given in the three regimes by:

Regime	$\sigma^{\mathrm{HJM}}_{\mathrm{LZ}}(\tau)$
Overdamped	$\frac{\sigma}{m} \frac{e^{-\mu_2\tau} - e^{-\mu_1\tau}}{\mu_1 - \mu_2}$
Critically damped	$\frac{\sigma}{m} \tau e^{-\mu\tau}$
Underdamped	$\frac{\sigma}{m} \frac{e^{-\alpha\tau} \sin(\omega\tau)}{\omega}$

where $\mu_{1,2} = (\gamma \pm \sqrt{\Delta})/(2m)$ (overdamped), $\mu = \gamma/(2m)$ (critical), $\alpha = \gamma/(2m)$ and $\omega = \sqrt{-\Delta}/(2m)$ (underdamped).

The B_2(τ) functions themselves are:

Proposition 3.3 (B Closed Forms). *The bond-price coefficient $B_2(\tau)$ satisfies:*

$$B_2^{\mathrm{od}}(\tau) = \frac{1}{\mu_1 - \mu_2}\left[\frac{1 - e^{-\mu_2\tau}}{\mu_2} - \frac{1 - e^{-\mu_1\tau}}{\mu_1}\right], \qquad B_2^{\mathrm{cd}}(\tau) = \frac{1 - e^{-\mu\tau}(1 + \mu\tau)}{\mu^2}$$
$$B_2^{\mathrm{ud}}(\tau) = \frac{1 - e^{-\alpha\tau}(\cos\omega\tau + \frac{\alpha}{\omega}\sin\omega\tau)}{\alpha^2 + \omega^2} \tag{3.3}$$

with $B_2(0) = 0$ and $B_2(\tau) \to 1/(\alpha^2 + \omega^2)$ as $\tau \to \infty$ in the underdamped case.

3.2 One-Factor Hull-White as a Gaussian HJM Model

The correct benchmark is the 1FHW model in HJM form — not in short-rate form. This places both models on identical footing as single-driver Gaussian HJM models.

Definition 3.4 (1FHW HJM Form). The one-factor Hull-White model is fully specified by its HJM volatility:

$$\sigma^{\mathrm{HJM}}_{\mathrm{HW}}(\tau) = \sigma e^{-a\tau}, \qquad \tau \geq 0 \tag{3.4}$$

with mean reversion $a > 0$ and volatility $\sigma > 0$. The no-arbitrage drift is $\alpha^{\mathrm{HJM}}_{\mathrm{HW}}(\tau) = (\sigma^2/a)\, e^{-a\tau}(1 - e^{-a\tau})$. The 1FHW bond-price coefficient is $B_{\mathrm{HW}}(\tau) = -(1 - e^{-a\tau})/a$.

3.3 The Structural Comparison

The two models share the same mathematical structure (single-driver Gaussian HJM) but differ in one fundamental way:

Remark 3.5 (Inertial Delay). In the 1FHW model $\sigma_{\mathrm{HW}}^{\mathrm{HJM}}(0) = \sigma > 0$: a Brownian shock instantly displaces the spot rate and all forward rates. In the LZ model $\sigma_{\mathrm{LZ}}^{\mathrm{HJM}}(0) = B_2'(0)(\sigma/m) = 0$ in every damping regime, because $B_2'(0) = 0$ follows from $B_2(0) = 0$ and the initial conditions of the ODE cascade. The spot rate $r_t = f(t,t)$ has zero instantaneous volatility — it is momentarily rigid — and the shock propagates outward along the maturity axis. The LZ volatility rises from zero, peaks at $\tau^* = \arctan(\omega/\alpha)/\omega$ (underdamped) or $\tau^* = 1/\mu$ (critically damped), then decays. The 1FHW volatility is maximal at $\tau = 0$ and strictly decreasing. This is the inertial delay in the volatility domain. It is the source of all pricing differences between the two models.

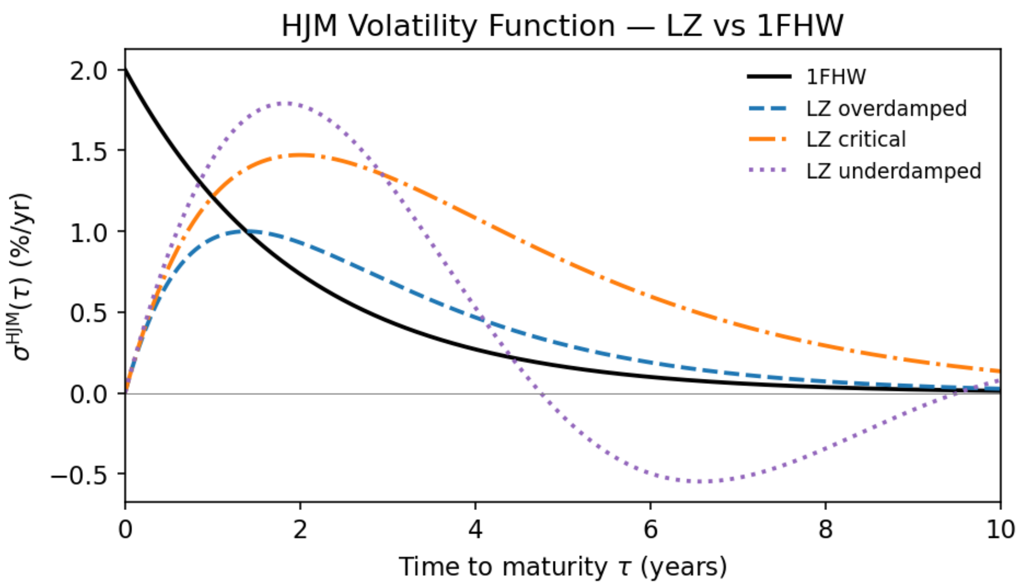

Figure 1: HJM volatility functions for 1FHW (black, solid), LZ overdamped (blue, dashed), LZ critically damped (orange, dash-dot), and LZ underdamped (purple, dotted). Parameters: $\sigma = 0.02$, $m = 1$; LZ overdamped $\gamma = 1.5$, $\kappa = 0.5$; LZ critical $\gamma = 1.0$, $\kappa = 0.25$; LZ underdamped $\gamma = 0.5$, $\kappa = 0.5$; 1FHW $a = 0.5$.

4. Caplet and Floorlet Pricing

4.1 Market Caplets as Bond Options

A market caplet resets at T and pays at $T+\delta$ the amount $\delta\max(L(T)-K,0)$, where $L(T) = (1/\delta)(1/P(T,T+\delta)-1)$ is the simply-compounded LIBOR rate. The forward rate $f(T,T) = r_T$ is Gaussian in the LZ model — normally distributed, not lognormal. The LIBOR rate $L(T)$ is a nonlinear function of the Gaussian state and is also not lognormal. However, the caplet admits an exact algebraic reduction to a bond put option.

Theorem 4.1 (Caplet / Floorlet as Bond Options). *In any affine Gaussian term structure model:*

$$\delta \max(L(T) - K, 0) = (1 + \delta K) \max(K^* - P(T, T + \delta), 0), \qquad K^* = \frac{1}{1 + \delta K} \quad (4.1)$$

Therefore the caplet price at time t is exactly:

$$\text{Caplet}(t, T, \delta, K) = (1 + \delta K) \cdot \text{Put}(t, T, T + \delta, K^*) \quad (4.2)$$

where $\text{Put}(t, T, T + \delta, K^*)$ *is the bond put price from Theorem 4.1 with* $\tau_0 = T - t$, $\tau_S = \delta$, *and bond variance* $\sigma_P^2(\tau_0, \delta)$ *from Theorems 4.4–4.7. The floorlet is priced analogously as a bond call.* ***Proof.*** *Since* $L(T) = (1/\delta)(1/P(T, T + \delta) - 1)$, *we have* $\delta L(T) = 1/P(T, T + \delta) - 1$, *so* $\delta(L(T) - K) = 1/P(T, T + \delta) - (1 + \delta K) = (1 + \delta K)(1/(1 + \delta K) - P(T, T + \delta))/P(T, T + \delta) \cdot P(T, T + \delta)$. *Multiplying through:* $\delta \max(L(T) - K, 0) = (1 + \delta K) \max(K^* - P(T, T + \delta), 0)$. *Since* $\log P(T, T + \delta) = A(\delta) + B_1(\delta) r_T + B_2(\delta) v_T$ *is a linear combination of the Gaussian state,* $P(T, T + \delta)$ *is lognormal. The bond put formula (Theorem 4.1) therefore applies exactly.* ***Remark.*** *The key step is that the bond price* $P(T, T + \delta)$ *is lognormal (log of an affine function of a Gaussian vector), even though the forward rate* $r_T = f(T, T)$ *is only normally distributed. The lognormal structure of P is what allows the Black formula to apply — but it applies to the bond, not to the forward rate directly.*

4.2 The HJM Variance Profile

The bond variance $\sigma_P^2(\tau_0, \delta)$ appearing in the caplet formula involves a single integral over $[0, \tau_0]$ of $[B_2(r + \delta) - B_2(r)]^2$. For small cap period δ, this simplifies via $B_2(r + \delta) - B_2(r) \approx \delta B_2'(r)$ to $\sigma_P^2 \approx \delta^2 V(\tau_0)$ where $V(\tau_0) = \int_0^{\tau_0} [\sigma^{\text{HJM}}(s)]^2 \, ds$. The quantity $V(\tau)$ therefore characterises the HJM variance accumulated up to maturity τ. It enters the swaption formula directly (Section 5) and defines the long-run vol profile of the model. We derive $V(\tau)$ in closed form for all four models.

4.3 Accumulated Variance — 1FHW

Theorem 4.2 (1FHW Accumulated Variance). *For the 1FHW model with parameters* (a, σ)*:*

$$V_{\text{HW}}(\tau) = \frac{\sigma^2}{2a} (1 - e^{-2a\tau}) \quad (4.3)$$

$V_{\text{HW}}(\tau)$ *is monotone increasing, bounded by* $\sigma^2/(2a)$, *with slope* σ^2 *at* $\tau = 0$. ***Proof.*** *Direct integration:* $\int_0^\tau \sigma^2 e^{-2as} \, ds = [\sigma^2 e^{-2as}/(-2a)]_0^\tau = \sigma^2(1 - e^{-2a\tau})/(2a)$.

4.4 Accumulated Variance — LZ Overdamped

Theorem 4.3 (LZ Overdamped Accumulated Variance). *Let* $\tilde{\sigma} = \sigma/[m(\mu_1 - \mu_2)]$. *Then:*

$$V_{\text{od}}(\tau) = \tilde{\sigma}^2 \left[\frac{1 - e^{-2\mu_2 \tau}}{2\mu_2} + \frac{1 - e^{-2\mu_1 \tau}}{2\mu_1} - \frac{2(1 - e^{-(\mu_1 + \mu_2)\tau})}{\mu_1 + \mu_2} \right] \quad (4.4)$$

Proof. *Expand* $[\sigma_{\mathrm{od}}^{\mathrm{HJM}}(s)]^2 = \tilde{\sigma}^2(e^{-\mu_2 s} - e^{-\mu_1 s})^2 = \tilde{\sigma}^2(e^{-2\mu_2 s} - 2e^{-(\mu_1+\mu_2)s} + e^{-2\mu_1 s})$*. Integrate term by term.*

4.5 Accumulated Variance — LZ Critically Damped

Theorem 4.4 (LZ Critically Damped Accumulated Variance)**.** *With* $\mu = \gamma/(2m)$*:*

$$V_{\mathrm{cd}}(\tau) = \frac{\sigma^2}{m^2} \cdot \frac{1 - e^{-2\mu\tau}(1 + 2\mu\tau + 2\mu^2\tau^2)}{4\mu^3} \tag{4.5}$$

Proof. *We need* $\int_0^\tau s^2 e^{-2\mu s}\,ds$*. Apply integration by parts twice:* $\int_0^\tau s^2 e^{-2\mu s}\,ds = [-s^2 e^{-2\mu s}/(2\mu)]_0^\tau + (1/\mu)\int_0^\tau s\,e^{-2\mu s}\,ds$*. The inner integral evaluates to* $(1-e^{-2\mu\tau}(1+2\mu\tau))/(4\mu^2)$*. Combining gives the result.*

4.6 Accumulated Variance — LZ Underdamped

Theorem 4.5 (LZ Underdamped Accumulated Variance)**.** *With* $\alpha = \gamma/(2m)$*,* $\omega = \sqrt{-\Delta}/(2m)$*:*

$$V_{\mathrm{ud}}(\tau) = \frac{\sigma^2}{2m^2\omega^2}\left[\frac{1 - e^{-2\alpha\tau}}{2\alpha} - \frac{e^{-2\alpha\tau}(2\alpha\cos 2\omega\tau + 2\omega\sin 2\omega\tau - 2\alpha)}{4(\alpha^2+\omega^2)}\right] \tag{4.6}$$

Proof. *Write* $[\sigma_{\mathrm{ud}}^{\mathrm{HJM}}(s)]^2 = (\sigma^2/m^2\omega^2)e^{-2\alpha s}\sin^2\omega s = (\sigma^2/2m^2\omega^2)e^{-2\alpha s}(1-\cos 2\omega s)$*. Integrate* $\int_0^\tau e^{-2\alpha s}\,ds = (1-e^{-2\alpha\tau})/(2\alpha)$*. For* $\int_0^\tau e^{-2\alpha s}\cos 2\omega s\,ds$*, use the antiderivative* $e^{-2\alpha s}(-2\alpha\cos 2\omega s + 2\omega\sin 2\omega s)/(4\alpha^2+4\omega^2)$ *and evaluate at* $s = 0, \tau$*.*

4.7 HJM Variance Profile Comparison

The quantity $\sqrt{V(\tau)/\tau}$ is the root-mean-square HJM volatility over $[0,\tau]$. For small cap period δ, the caplet bond vol satisfies $\sigma_P(\tau_0,\delta)/(\delta\sqrt{\tau_0}) \approx \sqrt{V(\tau_0)/\tau_0}$, so this profile characterises how the caplet vol level scales with expiry at any cap period.

Corollary 4.6 (HJM Variance Profile)**.** *The profile* $\sqrt{V(\tau)/\tau}$ *satisfies:*

Model	*Behaviour of* $\sqrt{V(\tau)/\tau}$
1FHW	*value* σ *at* $\tau = 0$*; strictly decreasing*
LZ overdamped	*value* 0 *at* $\tau = 0$*; rises to a hump, then decreases*
LZ critically damped	*value* 0 *at* $\tau = 0$*; rises to a hump, then decreases*
LZ underdamped	*value* 0 *at* $\tau = 0$*; non-monotone with oscillations*

The inertial delay $\sigma_{\mathrm{LZ}}^{\mathrm{HJM}}(0) = 0$ *forces* $V(\tau) = O(\tau^3)$ *as* $\tau \to 0^+$*, so the profile starts at zero in all LZ regimes. The 1FHW profile starts at* σ *because* $V_{\mathrm{HW}}(\tau) \to \sigma^2\tau$*.*
Proof. *For LZ:* $\sigma^{\mathrm{HJM}}(\tau) = O(\tau)$ *as* $\tau \to 0^+$*, so* $V(\tau) = \int_0^\tau O(s^2)\,ds = O(\tau^3)$*, giving* $\sqrt{V(\tau)/\tau} = O(\tau) \to 0$*. For 1FHW:* $V(\tau) \to \sigma^2\tau$*, so* $\sqrt{V(\tau)/\tau} \to \sigma$*.*

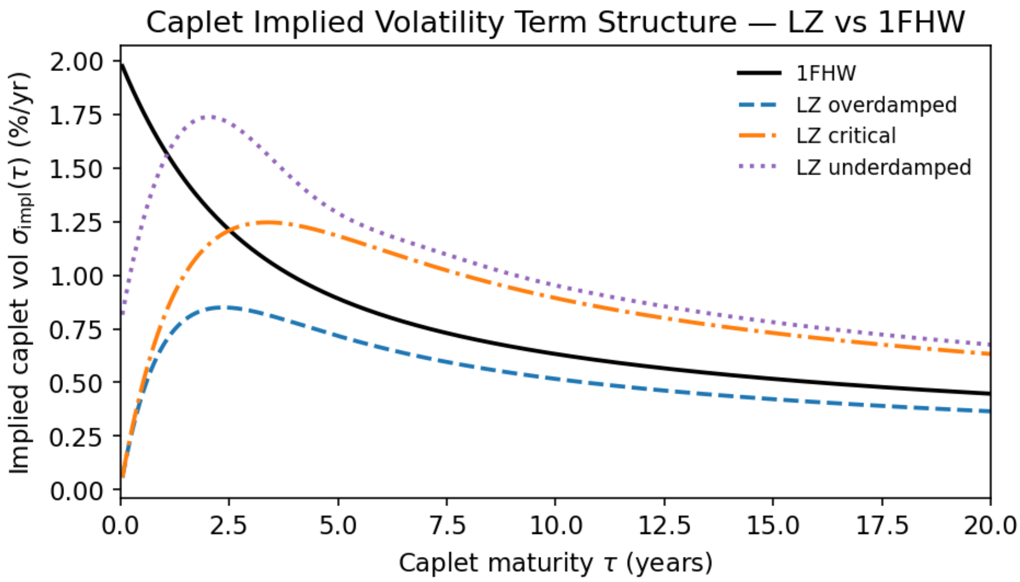

Figure 2: HJM variance rate profile $\sqrt{V(\tau)/\tau}$ for all four models. In the small-cap-period limit this equals $\sigma_P(\tau_0, \delta)/(\delta\sqrt{\tau_0})$ — the caplet bond vol scaled by δ. All LZ profiles start at zero (inertial delay); the 1FHW profile starts at σ. Same parameters as Figure 1.

5. Bond Option Pricing

A bond call option gives the right to purchase a zero-coupon bond $P(T,S)$ at time T for strike K. In a Gaussian HJM model, $\log P(T,S)$ is normal, and the option price is a Black formula.

5.1 General Gaussian Bond Option Formula

Theorem 5.1 (Bond Option Price)**.** *In any Gaussian HJM model, the bond call price at $t < T < S$ is:*

$$C(t,T,S,K) = P(t,S)\,\Phi(d_+) - K\,P(t,T)\,\Phi(d_-) \tag{5.1}$$

$$d_\pm = \frac{\log[P(t,S)/(K\,P(t,T))] \pm \frac{1}{2}\sigma_P^2}{\sigma_P}, \qquad \sigma_P^2 = \mathrm{Var}_t[\log P(T,S)] \tag{5.2}$$

The put price is $\mathrm{Put} = K\,P(t,T)\,\Phi(-d_-) - P(t,S)\,\Phi(-d_+)$. ***Proof.*** *In the LZ (and any affine Gaussian HJM) model, the log bond price is affine in the state:*

$$\log P(T,S) = A(T,S) + B_1(\tau_S)\,r_T + B_2(\tau_S)\,v_T$$

where $A(T,S)$ is deterministic and $\tau_S = S - T$. Since (r_T, v_T) is jointly Gaussian under the risk-neutral measure $\mathbb{Q}$ (the SDE (1)–(2) is linear with additive noise), any linear combination of (r_T, v_T) is Gaussian. Hence $\log P(T,S)$ is Gaussian, so $P(T,S)$ is ***lognormal*** *under $\mathbb{Q}$. To price under the forward measure $\mathbb{Q}^T$ with numeraire $P(t,T)$, we note that $P(T,S)/P(t,S)$ is a $\mathbb{Q}^T$-martingale. Under $\mathbb{Q}^T$, $\log P(T,S)$ remains*

Gaussian with variance $\sigma_P^2 = \mathrm{Var}_t[\log P(T,S)]$ *(the drift shifts but the variance is unchanged). The call price is:*

$$C(t,T,S,K) = P(t,T)\,\mathbb{E}^{\mathbb{Q}^T}[\max(P(T,S)-K,0)]$$

Integrating the lognormal density gives the Black formula (9).

Remark 5.2 (Gaussian rates, lognormal bonds: a consistency note). The LZ model has **Gaussian forward rates** (the HJM dynamics are linear with additive Brownian noise), so the short rate r_T and every forward rate $f(T,u)$ are normally distributed. One might expect Bachelier (normal) pricing throughout. However, bond prices are **lognormal**: $\log P(T,S)$ is Gaussian because it is an affine function of the Gaussian state (r_T, v_T), so $P(T,S) = e^{\text{Gaussian}}$ is lognormal. These two facts coexist: - **Instruments written on bond prices** (bond options, LIBOR caplets via the bond-put identity): the underlying $P(T,S)$ is lognormal ⇒ **Black formula is exact**. - **Instruments written on swap rates** (swaptions): the swap rate is a *ratio* of sums of lognormal bond prices — not exactly Gaussian, not exactly lognormal. Under the **linear swap rate approximation** (Section 5.2) it is approximately Gaussian ⇒ **Bachelier (normal) formula is an approximation**. The approximation is standard for all multi-factor Gaussian HJM models and is typically accurate to within 1–3 bp. The Black formula for bond options is exact in both the LZ and the 1FHW models — it is not specific to HW. What differs between the two models is the bond variance formula σ_P^2, not the pricing formula structure. For swaptions, the 1FHW model admits an additional exact formula via Jamshidian decomposition (exploiting the 1-factor monotonicity); the 2-factor LZ model does not. In this paper we use the Bachelier approximation for swaptions uniformly across all four models to ensure a consistent comparison.

5.2 Bond Price Variance via HJM

Theorem 5.3 (Bond Price Variance Formula)**.** *In a Gaussian HJM model with volatility* $\sigma^{\text{HJM}}(\tau)$*, the bond price variance accumulated over* $[t,T]$ *is:*

$$\sigma_P^2(\tau_0,\tau_S) = \int_0^{\tau_0}\left[\int_0^{\tau_S}\sigma^{\text{HJM}}(r+x)\,dx\right]^2 dr, \quad \tau_0 = T-t, \quad \tau_S = S-T \tag{5.3}$$

Proof. *The log bond price is* $\log P(T,S) = -\int_T^S f(T,u)\,du$*. Under the HJM dynamics* $df(t,u) = \alpha^{\text{HJM}}(u-t)\,dt + \sigma^{\text{HJM}}(u-t)\,dW_t$*, the stochastic part of* $\log P(T,S) - \log P(t,S)$ *is* $-\int_T^S\int_t^T \sigma^{\text{HJM}}(u-s)\,dW_s\,du$*. By the stochastic Fubini theorem this equals* $-\int_t^T\left[\int_T^S \sigma^{\text{HJM}}(u-s)\,du\right]dW_s$*. By the Itô isometry* $\sigma_P^2 = \int_t^T\left[\int_T^S \sigma^{\text{HJM}}(u-s)\,du\right]^2 ds$*. The change of variables* $r = s-t$*,* $x = u-T$ *gives equation (11).*

For the LZ model, since $\sigma^{\text{HJM}}(\tau) = (\sigma/m)B_2'(\tau)$ and $\int_0^{\tau_S} B_2'(r+x)\,dx = B_2(r+\tau_S) - B_2(r)$, equation (11) immediately reduces to:

Corollary 5.4 (LZ Central Bond Variance Formula). *Under the LZ model:*

$$\sigma_P^{2,\text{LZ}}(\tau_0,\tau_S) = \frac{\sigma^2}{m^2}\int_0^{\tau_0}\left[B_2(r+\tau_S) - B_2(r)\right]^2 dr \tag{5.4}$$

This is the master formula for LZ bond options. Its explicit evaluation in each regime is given in Theorems 4.4–4.6.

5.3 Bond Variance — 1FHW

Theorem 5.5 (1FHW Bond Variance). *For the 1FHW model with $B_{\text{HW}}(\tau) = -(1 - e^{-a\tau})/a$:*

$$\sigma_P^{2,\text{HW}}(\tau_0,\tau_S) = \frac{\sigma^2\,(1-e^{-a\tau_S})^2}{2a^3}\,(1-e^{-2a\tau_0}) \tag{5.5}$$

Proof. *We have $B_{\text{HW}}(r+\tau_S) - B_{\text{HW}}(r) = -(e^{-ar}/a)\,(1-e^{-a\tau_S})$. Substituting into (12): $\sigma_P^{2,\text{HW}} = (\sigma^2/a^2)(1-e^{-a\tau_S})^2\int_0^{\tau_0} e^{-2ar}\,dr = (\sigma^2(1-e^{-a\tau_S})^2/(2a^3))\,(1-e^{-2a\tau_0})$.*

The key observation: $\sigma_P^{2,\text{HW}}$ factorises as $f(\tau_S)\times g(\tau_0)$. In particular, $\partial_{\tau_S}\sigma_P^{2,\text{HW}} = 2\sigma^2(1-e^{-a\tau_S})e^{-a\tau_S}g(\tau_0)/a^2 > 0$: the 1FHW bond variance is **strictly increasing** in both arguments.

5.4 Bond Variance — LZ Overdamped

Theorem 5.6 (LZ Overdamped Bond Variance). *Let $\tilde{\sigma} = \sigma/[m(\mu_1-\mu_2)]$, $c_k = (1 - e^{-\mu_k\tau_S})/\mu_k$ for $k = 1,2$, and $I_{kl} = (1-e^{-(\mu_k+\mu_l)\tau_0})/(\mu_k+\mu_l)$. Then:*

$$\sigma_P^{2,\text{LZ,od}}(\tau_0,\tau_S) = \tilde{\sigma}^2\left[\frac{c_2^2}{2\mu_2}\left(1-e^{-2\mu_2\tau_0}\right) + \frac{c_1^2}{2\mu_1}\left(1-e^{-2\mu_1\tau_0}\right) - 2c_1c_2I_{12}\right] \tag{5.6}$$

Proof. *The overdamped B_2 difference is: $B_2^{\text{od}}(r+\tau_S) - B_2^{\text{od}}(r) = \frac{1}{\mu_1-\mu_2}[c_2e^{-\mu_2 r} - c_1e^{-\mu_1 r}]$. Squaring and integrating: $\int_0^{\tau_0}[c_2e^{-\mu_2 r} - c_1e^{-\mu_1 r}]^2\,dr = c_2^2(1-e^{-2\mu_2\tau_0})/(2\mu_2) + c_1^2(1-e^{-2\mu_1\tau_0})/(2\mu_1) - 2c_1c_2I_{12}$.*

5.5 Bond Variance — LZ Critically Damped

Theorem 5.7 (LZ Critically Damped Bond Variance). *Define $g_S = 1-(1+\mu\tau_S)e^{-\mu\tau_S}$ and $h_S = 1-e^{-\mu\tau_S}$. Define the integrals:*

$$\mathcal{I}_0 = \frac{1-e^{-2\mu\tau_0}}{2\mu},\quad \mathcal{I}_1 = \frac{1-e^{-2\mu\tau_0}}{4\mu^2} - \frac{\tau_0 e^{-2\mu\tau_0}}{2\mu},\quad \mathcal{I}_2 = \frac{1-e^{-2\mu\tau_0}}{4\mu^3} - e^{-2\mu\tau_0}\left(\frac{\tau_0^2}{2\mu} + \frac{\tau_0}{2\mu^2}\right)$$

Then:

$$\sigma_P^{2,\text{LZ,cd}}(\tau_0,\tau_S) = \frac{\sigma^2}{m^2\mu^4}\left[g_S^2\,\mathcal{I}_0 + 2g_S\mu h_S\,\mathcal{I}_1 + \mu^2h_S^2\,\mathcal{I}_2\right] \tag{5.7}$$

Proof. *The critically damped B_2 difference factors as $B_2^{\text{cd}}(r+\tau_S) - B_2^{\text{cd}}(r) = (e^{-\mu r}/\mu^2)(g_S + \mu h_S r)$. Squaring: $(e^{-2\mu r}/\mu^4)(g_S^2 + 2g_S\mu h_S r + \mu^2h_S^2r^2)$. The three standard integrals $\int_0^{\tau_0} e^{-2\mu r}\,dr$, $\int_0^{\tau_0} r\,e^{-2\mu r}\,dr$, $\int_0^{\tau_0} r^2\,e^{-2\mu r}\,dr$ give $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ by integration by parts.*

5.6 Bond Variance — LZ Underdamped

Theorem 5.8 (LZ Underdamped Bond Variance). *Let* $D = \alpha^2+\omega^2$, $P_S = 1-e^{-\alpha\tau_S}\cos\omega\tau_S$, $Q_S = e^{-\alpha\tau_S}\sin\omega\tau_S$, $A_S = P_S + Q_S\alpha/\omega$, $B_S = P_S\alpha/\omega - Q_S$. *Define:*

$$\mathcal{I}_0 = \frac{1-e^{-2\alpha\tau_0}}{2\alpha}$$
$$\mathcal{I}_c = \frac{\alpha - e^{-2\alpha\tau_0}(\alpha\cos 2\omega\tau_0 - \omega\sin 2\omega\tau_0)}{2D}, \qquad \mathcal{I}_s = \frac{\omega - e^{-2\alpha\tau_0}(\alpha\sin 2\omega\tau_0 + \omega\cos 2\omega\tau_0)}{2D}$$

Then:

$$\sigma_P^{2,\mathrm{LZ,ud}}(\tau_0,\tau_S) = \frac{\sigma^2}{m^2D^2}\left[\frac{A_S^2+B_S^2}{2}\mathcal{I}_0 + \frac{A_S^2-B_S^2}{2}\mathcal{I}_c + A_SB_S\mathcal{I}_s\right] \tag{5.8}$$

Proof. *The underdamped* B_2 *difference is* $B_2^{\mathrm{ud}}(r+\tau_S)-B_2^{\mathrm{ud}}(r) = (e^{-\alpha r}/D)[A_S\cos\omega r + B_S\sin\omega r]$ *(derived by expanding* $B_2(r+\tau_S)$ *using the addition formulas and collecting cosine and sine terms). Squaring and using* $\cos^2\omega r = (1+\cos 2\omega r)/2$, $\sin^2\omega r = (1-\cos 2\omega r)/2$, $\sin\omega r\cos\omega r = \sin 2\omega r/2$ *gives integrands* $e^{-2\alpha r}(1, \cos 2\omega r, \sin 2\omega r)$. *The three integrals* $\mathcal{I}_0, \mathcal{I}_c, \mathcal{I}_s$ *are computed using standard antiderivatives of* $e^{-2\alpha r}\cos 2\omega r$ *and* $e^{-2\alpha r}\sin 2\omega r$.

5.7 Non-Monotonicity in the Underdamped Regime

Corollary 5.9 (Non-Monotone Bond Vol). *In the underdamped LZ regime,* $\sigma_P^{2,\mathrm{LZ,ud}}(\tau_0,\tau_S)$ *is non-monotone in* τ_S. *Specifically, when* $\omega\tau_S = 2n\pi$ *for integer* $n \geq 1$, *we have* $B_2(r+\tau_S) \approx B_2(r)$ *for all* r, *so* $\sigma_P \approx 0$ *(near-zero bond option variance at resonance tenors).* ***Proof.*** *At* $\tau_S = 2n\pi/\omega$: $P_S = 1 - e^{-2n\pi\alpha/\omega}\cos(2n\pi) = 1 - e^{-2n\pi\alpha/\omega} > 0$ *but small for large* n; $Q_S = e^{-2n\pi\alpha/\omega}\sin(2n\pi) = 0$. *Thus* $A_S = P_S$ *and* $B_S = P_S\alpha/\omega$, *with* $A_S^2 + B_S^2 = P_S^2(1+\alpha^2/\omega^2) = P_S^2D/\omega^2$. *For large* n, $P_S \approx 0$ *and* $\sigma_P \approx 0$. *Between resonance tenors the variance is positive.* ***Remark 4.9.*** *This phenomenon is structurally impossible in any Gaussian HJM model where* $\sigma^{\mathrm{HJM}}(\tau) > 0$ *for all* $\tau > 0$. *In such models* $B(r+\tau_S) - B(r)$ *has a definite sign and is bounded away from zero for all* $\tau_S > 0$. *The LZ underdamped regime is exceptional precisely because* $\sigma_{\mathrm{LZ}}^{\mathrm{HJM}}$ *changes sign — the bond's net exposure to the Brownian driver cancels at resonance maturities.*

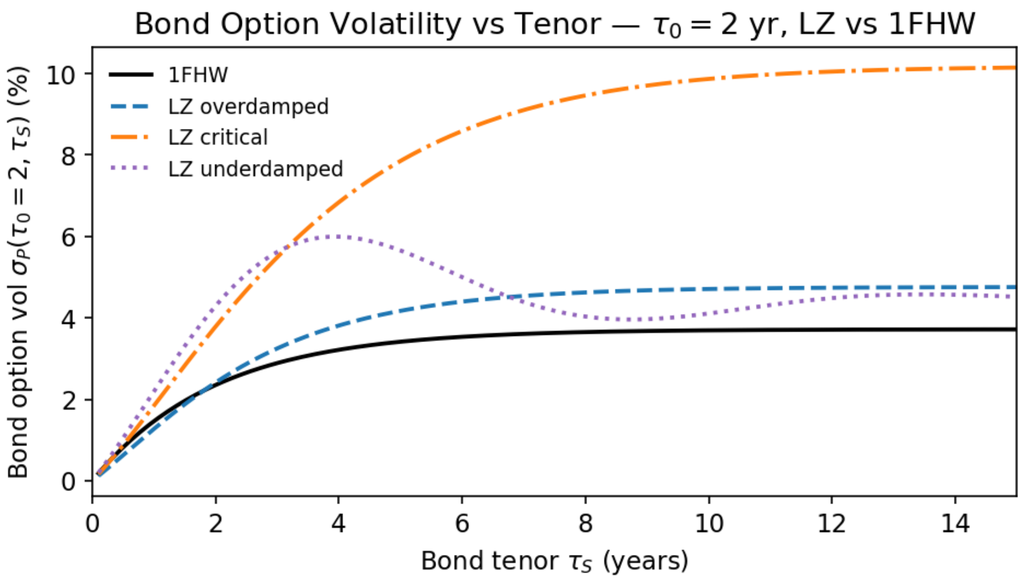

Figure 3: Bond option volatility $\sigma_P(\tau_0 = 2, \tau_S) = \sqrt{\sigma_P^2}$ as a function of bond tenor τ_S, for all four models. The 1FHW, LZ overdamped, and LZ critical curves are monotone increasing. The LZ underdamped curve (purple dotted) is non-monotone, with local minima near resonance periods $2\pi/\omega \approx 4.75$ yr.

6. Swaption Pricing

6.1 Setup

A payer swaption expiring at T_0 on a swap with payment dates $T_1 < \cdots < T_n$ and year fractions $\delta_i = T_i - T_{i-1}$ pays at T_0:

Definition 6.1 (Payer Swaption)**.** The payer swaption payoff is:

$$\text{Payer}(T_0) = \text{Annuity}(T_0) \cdot \max(S(T_0) - K, 0) \tag{6.1}$$

where $S(T_0) = [P(T_0, T_0) - P(T_0, T_n)] / \sum_{i=1}^{n} \delta_i P(T_0, T_i)$ is the par swap rate and $\text{Annuity}(T_0) = \sum_{i=1}^{n} \delta_i P(T_0, T_i)$.

6.2 Gaussian Swap Rate Approximation

The swap rate $S(T_0) = [P(T_0, T_0) - P(T_0, T_n)] / \sum_i \delta_i P(T_0, T_i)$ is a ratio of sums of lognormal bond prices. It is neither Gaussian nor lognormal — there is no known closed-form density for $S(T_0)$ in a multi-factor Gaussian HJM model. An exact swaption price requires integrating the payoff against the joint distribution of (r_{T_0}, v_{T_0}), which is 2D Gaussian with closed-form mean and covariance. This 2D quadrature is efficient but not a simple formula.

The standard approach — used here and in all Gaussian HJM literature including the G2++ model of Brigo-Mercurio [8] — replaces the exact ratio with a first-order Taylor expansion around today's forward rates.

Assumption 6.2 (Linear Swap Rate Approximation)**.** The swap rate is approximated by:

$$S(T_0) \approx S_0 + \sum_{i=1}^{n} w_i [f(T_0, T_i) - f(t, T_i)]$$

where $S_0 = [P(t, T_0) - P(t, T_n)]/\text{Annuity}(t)$ is the current forward swap rate and $w_i = -\delta_i P(t, T_i)/\text{Annuity}(t)$ are the annuity-weighted sensitivities, treated as deterministic (frozen at t).

Under Assumption 5.1 the increments $f(T_0, T_i) - f(t, T_i) = \int_t^{T_0} \sigma^{\text{HJM}}(T_i - s)\, dW_s$ are Gaussian in any Gaussian HJM model, so $S(T_0)$ becomes a linear combination of Gaussians — exactly Gaussian under the approximation.

Theorem 6.3 (Swaption Price — Bachelier Approximation)**.** *Under Assumption 5.1, the linearised swap rate $S(T_0)$ is normally distributed. The payer swaption price is the Bachelier (normal) formula:*

$$\text{Payer}(t) = \text{Annuity}(t) \cdot [(S_0 - K)\, \Phi(d) + \Sigma_{\text{swap}} \sqrt{\tau_0}\, \varphi(d)], \qquad d = \frac{S_0 - K}{\Sigma_{\text{swap}} \sqrt{\tau_0}} \quad (6.2)$$

$$\Sigma^2_{\text{swap}} = \frac{1}{\tau_0} \sum_{i,j=1}^{n} w_i w_j C_{ij}(\tau_0), \qquad C_{ij}(\tau_0) = \int_0^{\tau_0} \sigma^{\text{HJM}}(\tau_i - u)\, \sigma^{\text{HJM}}(\tau_j - u)\, du \quad (6.3)$$

where $\tau_i = T_i - t$, $w_i = -\delta_i P(t, T_i)/\text{Annuity}(t)$, S_0 is the current forward swap rate, and φ is the standard normal density. ***Proof.*** *Linearise $S(T_0) \approx S_0 + \sum_i w_i [f(T_0, T_i) - f(t, T_i)]$. Each increment $f(T_0, T_i) - f(t, T_i) = \int_t^{T_0} \sigma^{\text{HJM}}(T_i - s)\, dW_s$ is Gaussian; their weighted sum $\sum_i w_i \delta f_i$ is Gaussian with variance $\sum_{i,j} w_i w_j C_{ij}(\tau_0)$. Under the forward swap measure $S(T_0)$ is therefore $\mathcal{N}(S_0, \Sigma^2_{\text{swap}} \tau_0)$, and integrating the normal density against the call payoff* $\max(S - K, 0)$ *yields the Bachelier formula.*

Remark 6.4 (Approximation quality and why Fourier machinery is not needed)*. (i) Accuracy.* The linear swap rate approximation introduces an error because the frozen weights w_i are stochastic in the exact model. Empirically the error is 1–3 bp for at-the-money swaptions and grows for deep in- or out-of-the-money strikes. This level of accuracy is standard and accepted throughout the Gaussian HJM literature [8].

(ii) Black-76 vs Bachelier. Black-76 prices swaptions by treating the swap rate as log-normal — this is the BGM/LMM assumption. In a Gaussian HJM model the forward rates are normally distributed, so Black-76 applied directly to the swap rate is theoretically inconsistent. The Bachelier formula is the correct formula for a normally distributed underlying. It is used here for all four models (LZ and 1FHW) to ensure a consistent comparison.

(iii) Jamshidian exact formula for 1FHW. For the 1-factor Hull-White model, Jamshidian (1989) provides an exact swaption formula by decomposing the payer swaption into a portfolio of bond puts, exploiting the fact that the bond price is a monotone decreasing function of the single state variable r_{T_0}. This decomposition does not extend to

the 2-factor LZ model: the payoff boundary $\{(r,v) : S(T_0;r,v) = K\}$ is a curve in $\mathbb{R}^2$, not a threshold, so no monotone decomposition exists. We therefore use the Bachelier approximation for 1FHW as well, to maintain a uniform comparison framework across all four models.

(iv) No Fourier inversion needed. Fourier-based methods (characteristic function inversion, Lewis formula, Gil-Pelaez) are necessary when the payoff's underlying has no closed-form density — for example in jump-diffusion or stochastic-volatility models. In the LZ model the joint density of (r_{T_0}, v_{T_0}) is explicitly multivariate Gaussian, with mean and covariance computable in closed form from the linear SDE. An exact swaption price is therefore a 2D Gaussian integral, efficiently evaluated by 20-point Gauss-Hermite quadrature — no characteristic function inversion needed. The Bachelier approximation goes one step further, making $S(T_0)$ explicitly Gaussian and avoiding even this quadrature.

The key quantity is the matrix of cross-covariance integrals $C_{ij}(\tau_0)$. We compute these in closed form for all four models.

6.3 Cross-Covariance — 1FHW

Theorem 6.5 (1FHW Cross-Covariance)**.** *For the 1FHW model:*

$$C_{ij}^{\mathrm{HW}}(\tau_0) = \frac{\sigma^2}{2a} e^{-a(\tau_i+\tau_j)} (e^{2a\tau_0} - 1) \tag{6.4}$$

Proof. $C_{ij}^{\mathrm{HW}} = \sigma^2 e^{-a(\tau_i+\tau_j)} \int_0^{\tau_0} e^{2au}\,du = \sigma^2 e^{-a(\tau_i+\tau_j)}(e^{2a\tau_0} - 1)/(2a)$. ***Remark.*** *The 1FHW cross-covariance factorises:* $C_{ij}^{\mathrm{HW}} = h(\tau_i)h(\tau_j) \cdot g(\tau_0)$ *with* $h(\tau) = e^{-a\tau}$. *This rank-1 structure means the swaption variance simplifies to* $\Sigma^2_{\mathrm{swap}} = [\sum_i w_i h(\tau_i)]^2 g(\tau_0)/\tau_0$, *a geometric series. The LZ analogues do not factorise.*

6.4 Cross-Covariance — LZ Overdamped

Theorem 6.6 (LZ Overdamped Cross-Covariance)**.** *With* $\tilde{\sigma} = \sigma/[m(\mu_1 - \mu_2)]$*:*

$$C_{ij}^{\mathrm{od}}(\tau_0) = \tilde{\sigma}^2 \Bigg[\frac{e^{-\mu_2(\tau_i+\tau_j)}(e^{2\mu_2\tau_0} - 1)}{2\mu_2} + \frac{e^{-\mu_1(\tau_i+\tau_j)}(e^{2\mu_1\tau_0} - 1)}{2\mu_1} - \frac{(e^{(\mu_1+\mu_2)\tau_0} - 1)(e^{-\mu_2\tau_i-\mu_1\tau_j} + e^{-\mu_1\tau_i-\mu_2\tau_j})}{\mu_1+\mu_2} \Bigg] \tag{6.5}$$

Proof. *Expand* $\sigma_{\mathrm{od}}^{\mathrm{HJM}}(\tau_i - u)\sigma_{\mathrm{od}}^{\mathrm{HJM}}(\tau_j - u) = \tilde{\sigma}^2(e^{-\mu_2(\tau_i-u)} - e^{-\mu_1(\tau_i-u)})(e^{-\mu_2(\tau_j-u)} - e^{-\mu_1(\tau_j-u)})$. *This yields four exponential terms; integrating each from* 0 *to* τ_0 *gives the four terms in (22).*

6.5 Cross-Covariance — LZ Critically Damped

Theorem 6.7 (LZ Critically Damped Cross-Covariance)**.** *Define* $\mathscr{J}_0 = (e^{2\mu\tau_0}-1)/(2\mu)$*,* $\mathscr{J}_1 = \tau_0 e^{2\mu\tau_0}/(2\mu) - (e^{2\mu\tau_0}-1)/(4\mu^2)$*, and* $\mathscr{J}_2 = \tau_0^2 e^{2\mu\tau_0}/(2\mu) - \mathscr{J}_1/\mu$*. Then:*

$$C_{ij}^{\text{cd}}(\tau_0) = \frac{\sigma^2}{m^2} e^{-\mu(\tau_i+\tau_j)} \left[\tau_i\tau_j \mathscr{J}_0 - (\tau_i+\tau_j)\mathscr{J}_1 + \mathscr{J}_2\right] \tag{6.6}$$

Proof. $\sigma_{\text{cd}}^{\text{HJM}}(\tau_i-u)\sigma_{\text{cd}}^{\text{HJM}}(\tau_j-u) = (\sigma^2/m^2)(\tau_i-u)(\tau_j-u)e^{-\mu(\tau_i+\tau_j-2u)}$*. Factor out* $e^{-\mu(\tau_i+\tau_j)}$ *and expand the polynomial* $(\tau_i-u)(\tau_j-u) = \tau_i\tau_j - (\tau_i+\tau_j)u + u^2$*. Integrate* $\int_0^{\tau_0} e^{2\mu u}\,du = \mathscr{J}_0$*,* $\int_0^{\tau_0} u\,e^{2\mu u}\,du = \mathscr{J}_1$*,* $\int_0^{\tau_0} u^2 e^{2\mu u}\,du = \mathscr{J}_2$ *by standard integration by parts.*

6.6 Cross-Covariance — LZ Underdamped

Theorem 6.8 (LZ Underdamped Cross-Covariance)**.** *Define:*

$$\mathscr{K}_0 = \frac{e^{2\alpha\tau_0}-1}{2\alpha}$$

$$\mathscr{K}_c = \frac{e^{2\alpha\tau_0}(\alpha\cos 2\omega\tau_0 + \omega\sin 2\omega\tau_0) - \alpha}{2D}, \qquad \mathscr{K}_s = \frac{e^{2\alpha\tau_0}(\alpha\sin 2\omega\tau_0 - \omega\cos 2\omega\tau_0) + \omega}{2D}$$

Then:

$$C_{ij}^{\text{ud}}(\tau_0) = \frac{\sigma^2}{2m^2\omega^2} e^{-\alpha(\tau_i+\tau_j)} \left[\cos\omega(\tau_i-\tau_j)\,\mathscr{K}_0 - \cos\omega(\tau_i+\tau_j)\,\mathscr{K}_c - \sin\omega(\tau_i+\tau_j)\,\mathscr{K}_s\right] \tag{6.7}$$

Proof. *Use* $\sigma_{\text{ud}}^{\text{HJM}}(\tau) = (\sigma/m\omega)e^{-\alpha\tau}\sin\omega\tau$ *and the product formula* $\sin A\sin B = [\cos(A-B)-\cos(A+B)]/2$*:* $\sigma_{\text{ud}}^{\text{HJM}}(\tau_i-u)\sigma_{\text{ud}}^{\text{HJM}}(\tau_j-u) = (\sigma^2/2m^2\omega^2)e^{-\alpha(\tau_i+\tau_j-2u)}[\cos\omega(\tau_i-\tau_j) - \cos\omega(\tau_i+\tau_j-2u)]$*. For the first term:* $\cos\omega(\tau_i-\tau_j)$ *is constant in* u*, and* $\int_0^{\tau_0} e^{2\alpha u}\,du = \mathscr{K}_0$*. For the second term: expand* $\cos\omega(\tau_i+\tau_j-2u) = \cos\omega(\tau_i+\tau_j)\cos 2\omega u + \sin\omega(\tau_i+\tau_j)\sin 2\omega u$ *and use* $\int_0^{\tau_0} e^{2\alpha u}\cos 2\omega u\,du = \mathscr{K}_c$*,* $\int_0^{\tau_0} e^{2\alpha u}\sin 2\omega u\,du = \mathscr{K}_s$*.*

Remark 6.9. The $\cos\omega(\tau_i+\tau_j)$ and $\sin\omega(\tau_i+\tau_j)$ terms in C_{ij}^{ud} are absent in all other models. They produce swaption variances that oscillate as a function of payment-date sums — the underdamped signature in swaption space.

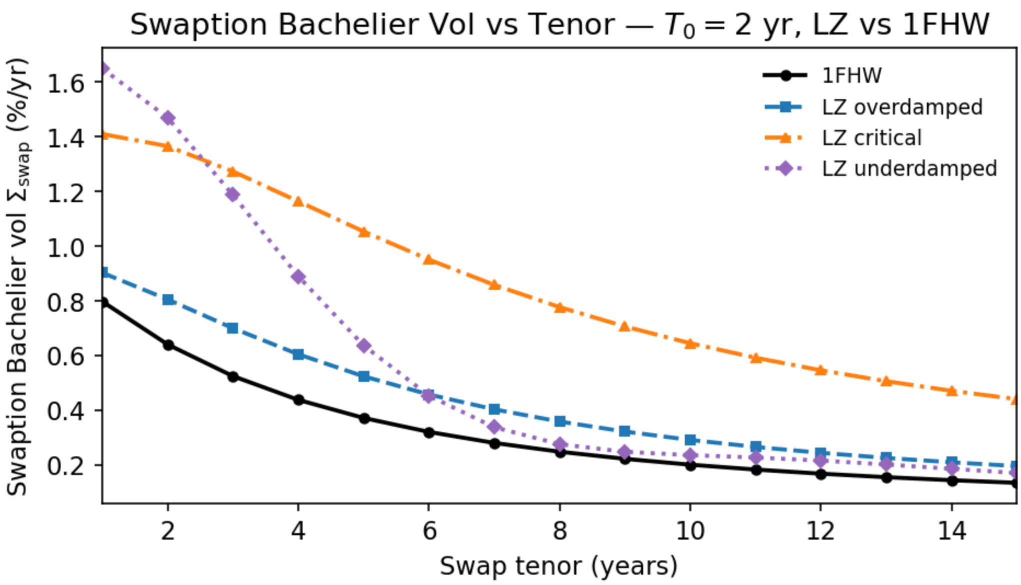

Figure 4: Swaption implied volatility Σ_{swap} as a function of swap tenor (annual payments, $T_0 = 2$ yr) for all four models. Equal-weight sensitivities. All LZ curves lie below 1FHW at short tenors (inertial delay); the LZ underdamped curve (purple dotted) is non-monotone due to oscillatory cross-covariances.

7. The Unified LZ-HJM Pricer

```
Input: Model type  {HW, LZ-od, LZ-cd, LZ-ud}
       Parameters (m,  ,  ,  ,  ) or (a,  ) for HW
       Option type  {caplet, bond-option, swaption}
       Tenors  (caplet) or  (,  _S) (bond) or  (,  T…,, T, weights  w) (swap)
       Strike K, current state ( r,  v)

Step 1: Compute regime parameters
   If LZ: compute Δ =  ² - 4 m
     If Δ > 0 (overdamped):   = √Δ(+)/(2m),   = √Δ(-)/(2m)
     If Δ = 0 (critically damped):   = /(2m)
     If Δ < 0 (underdamped):   = /(2m),   = √Δ(-)/(2m)

Step 2: Compute bond prices P(t, T) via affine formula
    B () ← closed form (Proposition 2.1)
    IB () =   ^{} B(s) ds  ← closed form or numerical
    B () = (/m) IB () -
   A ()  =  -(/m) IB () + (²/2m²) B ()²
   P(t, T) = exp{A () + B () r + B () v}

Step 3: Dispatch on option type
   If caplet:
     V () ← Theorem 3.2 / 3.3 / 3.4 / 3.5
     d± = (log(F/K) ± V/2) / √V   where F = f(t, t +)
     Price = P(t,T) [F Φ( d) - K Φ( d)]

```

```
    If bond-option:
      _P² (,  _S) ← Theorem 4.4 / 4.5 / 4.6 / 4.7
      d± = (log[P(t,S)/(K P(t,T))] ± _P²/2) / _P
      Price = P(t,S) Φ( d) - K P(t,T) Φ( d)

    If swaption:
      For all pairs (i, j):  C () ← Theorem 5.3 / 5.4 / 5.5 / 5.6
      Σ² =  (1/) Σ  w w  C
       S = (P(t, T) - P(t, T)) / Σ   P(t, T)
      Price = Annuity(t) × Bachelier( S, K, Σ√ )    ← normal approx

Output: Option price, implied vol, bond prices
```

Remark 7.1 (Complexity). All intermediate quantities are closed-form exponentials and trigonometric functions — no ODE integration, no numerical quadrature. For a swap with n payment dates the swaption computation requires $O(n^2)$ evaluations of the C_{ij} formula. Each evaluation is $O(1)$. The total complexity is $O(n^2)$.

8. Numerical Comparison

We fix $\sigma = 0.02$, $m = 1$, $\theta = 0.04$ across all models. Parameters are chosen so that the LZ decay rates match or are comparable to the 1FHW mean reversion $a = 0.5$. LZ overdamped: $\gamma = 1.5, \kappa = 0.5$, giving $\mu_1 = 1.0, \mu_2 = 0.5$ — the slow eigenvalue $\mu_2 = a_{\text{HW}}$ ensures identical long-run decay. Critically damped: $\gamma = 1.0, \kappa = 0.25$, $\mu = 0.5$. Underdamped: $\gamma = 0.5, \kappa = 0.5$, $\alpha = 0.25$, $\omega = \sqrt{1.75}/2 \approx 0.661$. The 1FHW benchmark uses $a = 0.5$, same σ.

Caplet vol (Figure 2). The 1FHW caplet vol starts at $\sigma = 2\%$ and decreases monotonically to zero. All LZ curves start at zero, rise to a hump (the inertial delay), then fall. The underdamped curve is non-monotone with oscillations decaying as $\tau \to \infty$.

Bond option vol (Figure 3). With $\tau_0 = 2$ years fixed and τ_S varying from 0 to 15 years, the 1FHW vol (13) is monotone increasing and saturates. The LZ overdamped and critical curves are also monotone increasing but grow more slowly at short tenors (inertial delay). The LZ underdamped curve is non-monotone: it rises initially, then dips near the first resonance period $2\pi/\omega \approx 4.75$ years, consistent with Corollary 4.8.

Swaption vol (Figure 4). With $T_0 = 2$ years and annual swap payments, the swaption vol as a function of swap tenor shows the inertial delay: all LZ models start below 1FHW at short tenors. The underdamped model produces non-monotone swaption vols, with the oscillations in C_{ij}^{ud} driving the swaption surface below the overdamped and critical cases at long tenors.

Remark 8.1 (Calibration Principle). A standard calibration would match both models to a target caplet vol at one reference maturity τ_0^*, then compare predictions everywhere else. The comparison shown here uses identical σ and different a vs (γ, κ). The qualitative features — humped caplet vol, non-monotone bond vol in underdamped regime, LZ below 1FHW at short tenors — are robust to any calibration that preserves the structural constraint $\sigma_{\text{LZ}}^{\text{HJM}}(0) = 0$.

9. Contributions

This paper establishes the following results.

Contribution 1 — Exact caplet/floorlet pricing under LZ. Market caplets are priced exactly as bond put options (Theorem 3.1): Caplet $= (1 + \delta K) \cdot \mathrm{Put}(P(T, T + \delta); K^*)$. Since $\log P(T, T + \delta)$ is Gaussian in the LZ model, this is an exact Black formula on the lognormal bond price, using $\sigma_P^2(\tau_0, \delta)$ from Section 4. The HJM variance profile $V(\tau)$ is given in closed form in all three regimes (Theorems 3.3–3.6) and characterises the vol term structure.

Contribution 2 — Closed-form bond option pricing under LZ. The master formula (12) reduces to a single integral of a known function. Full closed forms are given in Theorems 4.5–4.7 for all three regimes. The critically damped formula involves polynomials in τ_S multiplied by decaying exponentials; the underdamped formula involves the auxiliary quantities P_S, Q_S, A_S, B_S and three standard integrals $\mathcal{I}_0, \mathcal{I}_c, \mathcal{I}_s$.

Contribution 3 — Swaption pricing via Bachelier approximation. The exact swap rate is a ratio of sums of lognormal bond prices and has no closed-form density. Under the linear swap rate approximation (Assumption 5.1), the swap rate becomes approximately Gaussian as a linear combination of Gaussian forward rates. Swaptions are then priced by the Bachelier (normal) formula (Theorem 5.2), with normal vol Σ_{swap} computed from the model's cross-covariance matrix. This approximation is standard for all multi-factor Gaussian HJM models and is used uniformly here for all four models (including the 1FHW benchmark) to enable consistent comparison. The Fourier machinery required in jump or stochastic-vol models is not needed: the Gaussian distribution of (r_{T_0}, v_{T_0}) is fully explicit, making the approximation both natural and efficient. Closed-form cross-covariance integrals $C_{ij}(\tau_0)$ are given in Theorems 5.3–5.6 for all four models. The underdamped cross-covariance contains $\cos \omega(\tau_i \pm \tau_j)$ terms absent in all other models, producing non-monotone swaption vol surfaces.

Contribution 4 — Systematic comparison against 1FHW. Both models are Gaussian HJM models with one Brownian driver. All pricing differences reduce to the single structural gap: $\sigma_{\mathrm{LZ}}^{\mathrm{HJM}}(0) = 0$ versus $\sigma_{\mathrm{HW}}^{\mathrm{HJM}}(0) = \sigma$. The 1FHW bond variance factorises (rank-1 structure); the LZ overdamped variance does not.

Contribution 5 — The underdamped implied vol signature. The underdamped LZ model produces oscillatory caplet implied vols, non-monotone bond option vols (Corollary 4.8), and non-monotone swaption vol surfaces (Remark 5.7). These features are structurally impossible in any Gaussian HJM model with $\sigma^{\mathrm{HJM}}(\tau) > 0$, including 1FHW, 2FHW, and all Ho-Lee variants. They constitute a testable prediction: markets exhibiting oscillatory cap vol surfaces across maturities may require a second-order (inertial) term structure model.

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