Brownian Local Time, Tanaka's Formula, and the Quantum Delta Potential

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1. Abstract

We construct Brownian local time L_t^x as the density of the occupation measure of standard Brownian motion and establish three foundational results: the occupation time formula, Tanaka's formula extending Itô's lemma to $|B_t - a|$, and Lévy's representation theorem identifying L_t^0 in distribution with $|B_t|$. The entire development is motivated by a single problem in quantum mechanics: the Schrödinger operator $H = -\frac{1}{2}\partial_{xx} + \alpha\delta$ requires, via the Feynman–Kac formula, a rigorous interpretation of $\int_0^\tau \delta(B_s)\,ds$ — which is precisely the local time L_τ^0. The quantum consequences follow as direct corollaries: the Feynman–Kac weight for the delta potential is $e^{-\alpha L_\tau^0}$, and the bound state energy $E_0 = -\alpha^2/2$ (for $\alpha < 0$) is derived from the Laplace transform of L_t^0 established via Lévy's theorem.

2. Motivation: The Quantum Delta Potential

The Schrödinger operator with a point interaction at the origin,

$$H = -\frac{1}{2}\partial_{xx} + \alpha\delta(x), \qquad \alpha \in \mathbb{R}, \tag{2.1}$$

is the canonical one-dimensional model of a quantum particle coupled to an impurity. For $\alpha < 0$ (attractive coupling), H has exactly one negative eigenvalue $E_0 = -\alpha^2/2$ with L^2-normalised eigenfunction $\psi_0(x) = \sqrt{|\alpha|}\, e^{-|\alpha|\,|x|}$; the continuous spectrum is $[0, \infty)$ with no bound state for $\alpha \geq 0$.

The imaginary-time evolution $u(\cdot, \tau) = e^{-\tau H} f$ satisfies the parabolic problem

$$\partial_\tau u = \frac{1}{2}\partial_{xx} u - \alpha\delta(x)\, u, \qquad u(\cdot, 0) = f \in L^2(\mathbb{R}). \tag{2.2}$$

Applying the Feynman–Kac formula formally to (2.2) gives

$$u(x, \tau) = \mathbb{E}_x\left[\exp\left(-\alpha \int_0^\tau \delta(B_s)\, ds\right) f(B_\tau)\right]. \tag{2.3}$$

The expression $\int_0^\tau \delta(B_s)\, ds$ is pathwise ill-defined: Brownian motion visits any fixed point on a set of Lebesgue measure zero, so the integrand vanishes ds-almost everywhere along every path. The object needed to make (2.3) rigorous is a random field $\{L_t^x\}$ satisfying

$$\int_0^t \varphi(B_s)\,ds = \int_{-\infty}^{\infty} \varphi(x)\,L_t^x\,dx \quad \text{a.s.} \tag{2.4}$$

for all Borel functions φ. Setting $\varphi = \delta(\cdot - a)$ recovers L_t^a as the rigorous substitute for $\int_0^t \delta(B_s - a)\,ds$. The construction and properties of this field — Brownian local time — are the subject of Sections 3–5.

3. Setup and Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, and let B_t be a standard $\mathbb{R}$-valued $(\mathcal{F}_t)$-Brownian motion. Write $\mathbb{P}_x$ for the law of B started at x and $\mathbb{E}_x$ for the corresponding expectation. The quadratic variation satisfies $\langle B\rangle_t = t$.

Symbol	**Description**
B_t	Standard Brownian motion, $B_0 = x$
L_t^x	Local time of B at level x up to time t
$\operatorname{sgn}(x)$	$\mathbf{1}_{x>0} - \mathbf{1}_{x\leq 0}$
M_t	Running maximum $\sup_{s\leq t} B_s$
Φ	Standard normal CDF: $\Phi(z) = (2\pi)^{-1/2}\int_{-\infty}^{z} e^{-u^2/2}\,du$
H	Schrödinger operator $-\frac{1}{2}\partial_{xx} + \alpha\delta$
E_0	Ground state energy of H
ψ_0	Bound state eigenfunction of H

4. Brownian Local Time

Definition 4.1 (Occupation measure)**.** For $t \geq 0$ and Borel $A \subseteq \mathbb{R}$, the occupation measure of B up to time t is $\mu_t(A) = \int_0^t \mathbf{1}_{B_s\in A}\,ds$. The map $A \mapsto \mu_t(A)$ is a random Borel measure on $\mathbb{R}$, absolutely continuous with respect to Lebesgue measure almost surely.

Theorem 4.2 (Existence and occupation time formula)**.** *There exists a jointly measurable random field $\{L_t^x\}_{x\in\mathbb{R},\, t\geq 0}$ such that, a.s. for all $t \geq 0$: (i) $\mu_t(A) = \int_A L_t^x\,dx$ for every Borel set A; (ii) for every non-negative Borel $f : \mathbb{R} \to [0,\infty)$, equation (4.1) holds.*

$$\int_0^t f(B_s)\,ds = \int_{-\infty}^{\infty} f(x)\,L_t^x\,dx. \tag{4.1}$$

The occupation time formula (4.1) makes (2.4) precise. The field L_t^x measures, in units of time, how densely the path $s \mapsto B_s$ visits level x up to time t. Applying (4.1) with $f = \mathbf{1}_A$ gives $\mu_t(A) = \int_A L_t^x\,dx$, confirming (i).

Lemma 4.3 (Lévy approximation)**.** *For every $x \in \mathbb{R}$ and $t \geq 0$, almost surely, equation (4.2) holds, with convergence uniform in x on compact sets.*

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{|B_s - x| < \epsilon\}} \, ds. \tag{4.2}$$

Proof. The right side of (4.2) equals $\mu_t(x-\epsilon, x+\epsilon)/(2\epsilon)$. Since $\mu_t \ll \text{Leb}$ with density $L_t^x \in L^1_{\text{loc}}$, the Lebesgue differentiation theorem yields pointwise convergence for a.e. x. Uniformity in x follows from the $\frac{1}{2}$-Hölder continuity of L_t^x in x, established via Tanaka's formula (Theorem 4.1) and the Kolmogorov continuity criterion. □

Lemma 3.3 provides a constructive characterisation of L_t^x and a practical numerical scheme: approximate (4.2) along a discretised Brownian path with bandwidth $\epsilon = O(\sqrt{\Delta t})$ (see Algorithm 1 in Section 7).

5. Tanaka's Formula

Itô's formula requires $f \in C^2$. The function $g_a(x) = |x - a|$ has $g_a' = \text{sgn}(x - a)$ and distributional second derivative $g_a'' = 2\delta(x-a)$. Tanaka's formula is the correct Itô formula for g_a, with the local time appearing as the correction term that replaces $\frac{1}{2}\int_0^t g_a''(B_s)\, ds$.

Theorem 5.1 (Tanaka–Meyer)**.** *For every $a \in \mathbb{R}$ and $t \geq 0$, equation (5.1) holds.*

$$|B_t - a| = |B_0 - a| + \int_0^t \text{sgn}(B_s - a)\, dB_s + L_t^a. \tag{5.1}$$

Approximate $g_a(x) = |x - a|$ by $f_\epsilon(x) = \sqrt{(x-a)^2 + \epsilon^2} \in C^\infty$, with $f_\epsilon'(x) = (x - a)/\sqrt{(x-a)^2 + \epsilon^2}$ and $f_\epsilon''(x) = \epsilon^2/((x-a)^2 + \epsilon^2)^{3/2}$. Itô's formula gives:

$$f_\epsilon(B_t) = f_\epsilon(B_0) + \int_0^t f_\epsilon'(B_s)\, dB_s + \frac{1}{2}\int_0^t f_\epsilon''(B_s)\, ds. \tag{5.2}$$

Proof. Each term in (5.2) has a limit as $\epsilon \downarrow 0$. (a) $f_\epsilon(B_t) \to |B_t - a|$ uniformly in t. (b) Since $|f_\epsilon'| \leq 1$ and $f_\epsilon'(B_s) \to \text{sgn}(B_s - a)$ pointwise, dominated convergence in the Itô isometry gives L^2-convergence of the stochastic integral to $\int_0^t \text{sgn}(B_s - a)\, dB_s$. (c) For the final term, the occupation time formula (4.1) with $f = f_\epsilon''/2$ gives $\frac{1}{2}\int_0^t f_\epsilon''(B_s)\, ds = \int_{-\infty}^\infty k_\epsilon(x-a)\, L_t^x\, dx$, where $k_\epsilon(y) = \epsilon^2/(2(y^2+\epsilon^2)^{3/2})$ is a positive approximate identity with $\int k_\epsilon = 1$ and $k_\epsilon \to \delta$ weakly. Since $x \mapsto L_t^x$ is continuous at a, the convolution converges to L_t^a. Taking $\epsilon \downarrow 0$ in (5.2) yields (5.1). □

Corollary 5.2 (Properties of local time)**.** *For each $a \in \mathbb{R}$: (i) $t \mapsto L_t^a$ is continuous, non-decreasing, and $L_0^a = 0$ a.s.; (ii) L_t^a increases only on $\{s : B_s = a\}$, i.e., $\int_0^\infty \mathbf{1}_{\{B_s \neq a\}}\, dL_s^a = 0$ a.s.; (iii) $x \mapsto L_t^x$ is $\frac{1}{2}$-Hölder continuous, a.s. for each t.*

Proof. follows from the path-continuity of the right side of (5.1). (ii) The process $t \mapsto |B_t - a|$ is flat precisely when $B_t \neq a$, so L_t^a can only increase when $B_t = a$. (iii) For $a < b$, apply Tanaka's formula to both $|B_t - a|$ and $|B_t - b|$ and subtract; the difference $L_t^b - L_t^a$ is bounded by a stochastic integral and $|b-a|$-type terms; the BDG inequality and a standard Kolmogorov argument give Hölder-$\frac{1}{2}$ continuity. ☐

6. Lévy's Representation and the Distribution of Local Time

Theorem 6.1 (Lévy)**.** *Under $\mathbb{P}_0$, the processes $(|B_t|,\, L_t^0)_{t\geq 0}$ and $(M_t - B_t,\, M_t)_{t\geq 0}$ have the same distribution, where $M_t = \sup_{s\leq t} B_s$.*

Proof. Set $R_t = M_t - B_t$. Since M_t is non-decreasing and flat off $\{R_t = 0\}$, the pair (R_t, M_t) solves the Skorokhod reflection equation: $R_t \geq 0$, M_t non-decreasing, $\int_0^\infty \mathbf{1}_{\{R_s > 0\}}\, dM_s = 0$. By Tanaka's formula (5.1), $|B_t|$ satisfies an identical equation with L_t^0 in place of M_t, since $|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t^0$. The Skorokhod reflection problem has a pathwise unique solution, so the two pairs coincide in law. ☐

Corollary 6.2. *Under $\mathbb{P}_0$, $L_t^0 \overset{d}{=} M_t \overset{d}{=} |B_t|$ for each $t > 0$.*

Proof. From Theorem 5.1, $L_t^0 \overset{d}{=} M_t$. The reflection principle gives $M_t \overset{d}{=} |B_t|$. ☐

Lemma 6.3 (Laplace transform)**.** *For $\lambda \geq 0$ and $t > 0$, equation* (6.1) *holds.*

$$\mathbb{E}_0\left[e^{-\lambda L_t^0}\right] = 2\, e^{\lambda^2 t/2}\, \Phi\left(-\lambda\sqrt{t}\right). \tag{6.1}$$

Proof. By Corollary 5.2, $L_t^0 \overset{d}{=} |B_t| \sim |N(0,t)|$, so $\mathbb{E}_0[e^{-\lambda L_t^0}] = \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^{-\lambda x - x^2/(2t)}\, dx$. Completing the square, $-\lambda x - x^2/(2t) = -(x+\lambda t)^2/(2t) + \lambda^2 t/2$, which gives: ☐

$$\mathbb{E}_0\left[e^{-\lambda L_t^0}\right] = \frac{2\, e^{\lambda^2 t/2}}{\sqrt{2\pi t}} \int_0^\infty e^{-(x+\lambda t)^2/(2t)}\, dx = \frac{2\, e^{\lambda^2 t/2}}{\sqrt{2\pi}} \int_{\lambda\sqrt{t}}^\infty e^{-u^2/2}\, du = 2\, e^{\lambda^2 t/2}\, \Phi(-\lambda\sqrt{t}). \tag{6.2}$$

Remark 6.4. For attractive coupling $\alpha < 0$, set $\kappa = |\alpha|$ and replace λ by $-\kappa$ in (6.1) to obtain equation (6.3).

$$\mathbb{E}_0\left[e^{\kappa L_t^0}\right] = 2\, e^{\kappa^2 t/2}\, \Phi\left(\kappa\sqrt{t}\right). \tag{6.3}$$

As $t \to \infty$, $\Phi(\kappa\sqrt{t}) \to 1$, so the right side of (6.3) grows as $2e^{\kappa^2 t/2}$. The exponential growth rate $\kappa^2/2$ identifies the ground state energy in Section 6.

7. The Delta Potential: Corollaries

We now derive the quantum mechanical consequences directly from the probabilistic results above.

Corollary 7.1 (Feynman–Kac for the delta potential). *Let $f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The solution to (2.2) is given by equation (7.1).*

$$u(x,\tau) = \mathbb{E}_x\left[e^{-\alpha L_\tau^0} f(B_\tau)\right]. \tag{7.1}$$

Proof. Apply the occupation time formula (4.1) with $f = \alpha\delta(\cdot)$:

$$\alpha \int_0^\tau \delta(B_s)\, ds = \alpha \int_{-\infty}^{\infty} \delta(x)\, L_\tau^x\, dx = \alpha L_\tau^0 \quad \text{a.s.} \tag{7.2}$$

Substituting into (2.3) makes it rigorous: the previously ill-defined integral is replaced by αL_τ^0, a well-defined non-negative random variable by Corollary 4.2.

Theorem 7.2 (Bound state energy). *For $\alpha < 0$, the ground state energy of H defined in (2.1) is $E_0 = -\alpha^2/2$.*

Proof. Away from the origin, $-\frac{1}{2}f'' = E_0 f$ with $E_0 < 0$ gives $f(x) = Ce^{-\kappa|x|}$ with $\kappa = \sqrt{-2E_0} > 0$. The domain condition for the self-adjoint realisation of H requires $f'(0^+) - f'(0^-) = 2\alpha f(0)$, i.e., $-2\kappa C = 2\alpha C$, hence $\kappa = -\alpha = |\alpha|$, establishing (7.3). The probabilistic confirmation: from (6.3), $\mathbb{E}_0[e^{\kappa L_\tau^0}] \sim 2e^{\kappa^2\tau/2}$ as $\tau \to \infty$, so the exponential growth rate is $\kappa^2/2 = |E_0|$. □

$$E_0 = -\frac{\kappa^2}{2} = -\frac{\alpha^2}{2}. \tag{7.3}$$

Corollary 7.3 (Convergence to bound state). *Let $\alpha < 0$, $\kappa = |\alpha|$, and $f \in L^2(\mathbb{R})$ with $\langle f, \psi_0 \rangle \neq 0$. Then (7.4) holds in $L^2(\mathbb{R})$, and in particular $u(x,\tau)/u(0,\tau) \to e^{-\kappa|x|}$ for each x.*

$$e^{E_0\tau}\, \mathbb{E}_x\left[e^{\kappa L_\tau^0} f(B_\tau)\right] \xrightarrow{\tau\to\infty} \langle f, \psi_0 \rangle\, \psi_0(x). \tag{7.4}$$

Proof. By the spectral theorem, $e^{-\tau H} = e^{-E_0\tau}\langle \cdot, \psi_0 \rangle \psi_0 + \int_0^\infty e^{-\lambda\tau}\, dE_\lambda$, where E_λ is the spectral projection onto $[0,\infty)$. Multiplying by $e^{E_0\tau}$ and using $E_0 < 0 \le \lambda$, the continuous-spectrum term is $O(e^{E_0\tau}) \to 0$. □

8. Numerical Algorithm and Results

8.1 Algorithm

The Monte Carlo approximation of $u(x,\tau) = \mathbb{E}_x[e^{-\alpha L_\tau^0} f(B_\tau)]$ combines the Euler–Maruyama scheme for B_t with the Lévy approximation (4.2) for L_τ^0. The bandwidth $\epsilon = O(\sqrt{\Delta t})$ balances the δ-function approximation error against the path discretisation error.

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Algorithm 1: Monte Carlo Feynman-Kac for the delta potential

Input:  x0     (starting position)
        alpha  (coupling; alpha < 0 for attractive potential)
        tau    (imaginary time)
        N      (number of paths)
        M      (time steps)

dt  <- tau / M
eps <- 3 * sqrt(dt)         # Levy bandwidth: O(sqrt(dt))

for i = 1 to N:
    B <- x0
    L <- 0
    for k = 1 to M:
        B <- B + sqrt(dt) * Normal(0,1)
        if |B| < eps:
            L <- L + dt / (2 * eps)
    weight[i] <- exp(-alpha * L) * f(B)

u_hat <- mean(weight)
se    <- std(weight) / sqrt(N)
```

8.2 Results

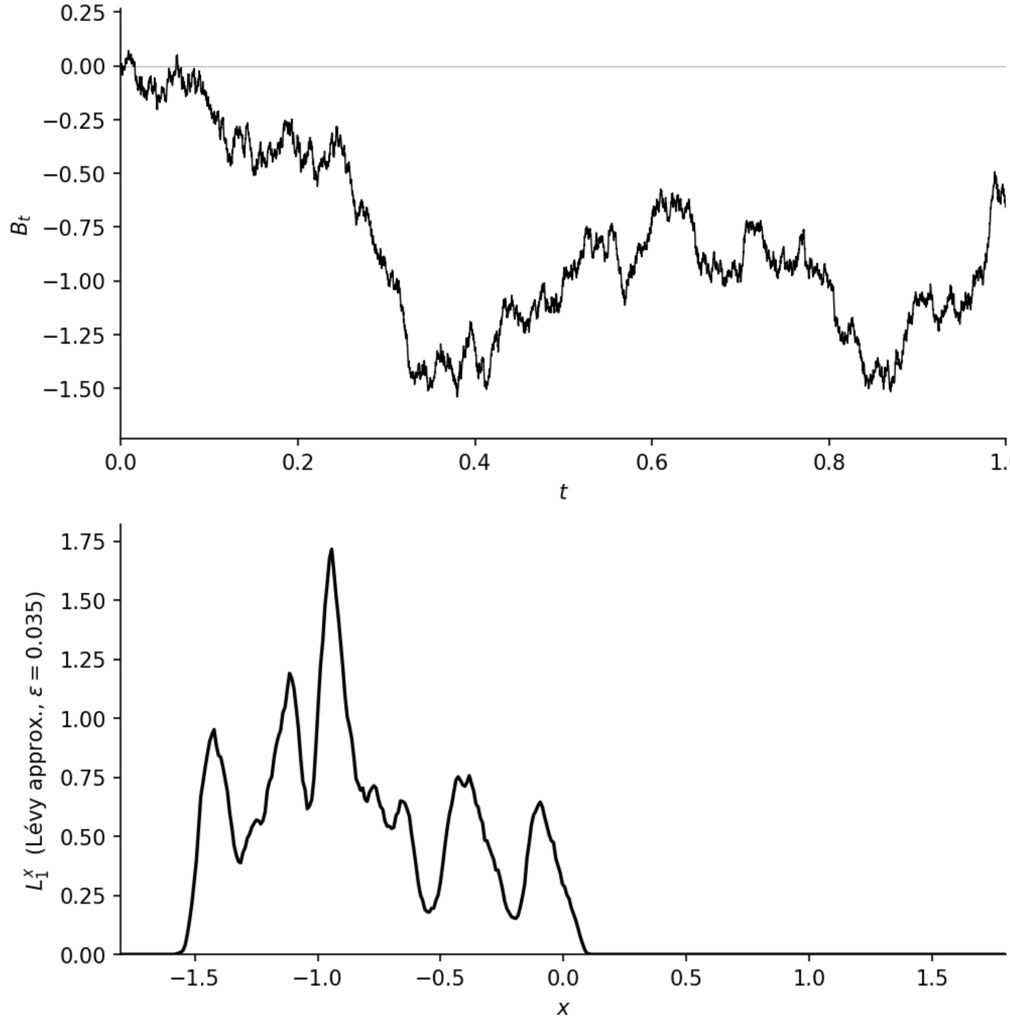

Figure 1: Brownian path $t \mapsto B_t$ (top) and local time profile $x \mapsto L_1^x$ (bottom) for a single trajectory with $M = 6000$ steps. The Lévy approximation (4.2) with $\epsilon = 0.035$. The profile concentrates near the range visited by the path and vanishes on intervals not traversed.

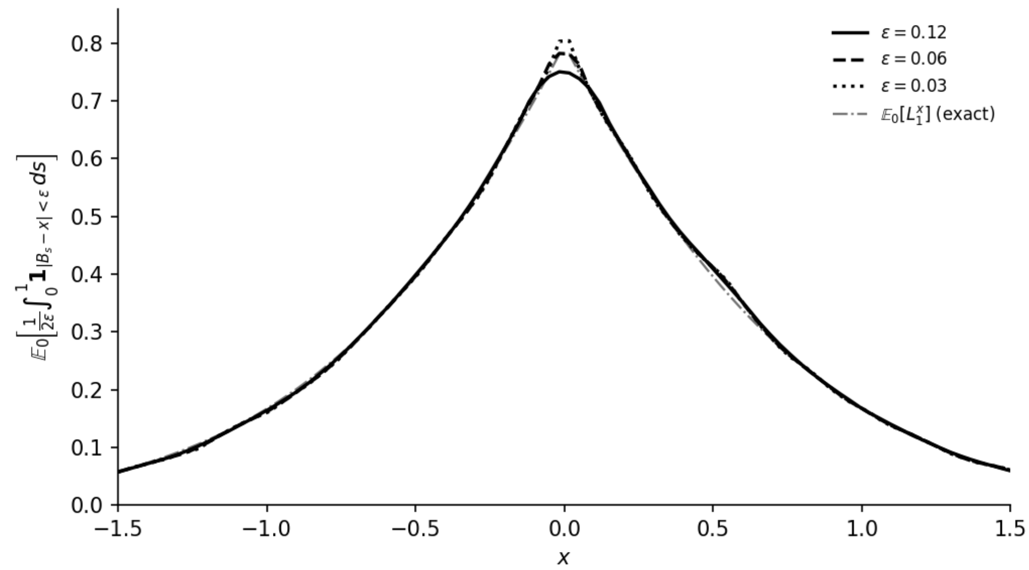

Figure 2: Convergence of the Lévy approximation to $\mathbb{E}_0[L_1^x]$ as $\epsilon \downarrow 0$, for $\epsilon \in \{0.12, 0.06, 0.03\}$ averaged over 3×10^3 paths with $M = 800$ steps. The theoretical curve $\mathbb{E}_0[L_1^x] = \int_0^1 (2\pi s)^{-1/2} e^{-x^2/(2s)}\, ds$ is shown for reference. Convergence is monotone and uniform on the plotted range.

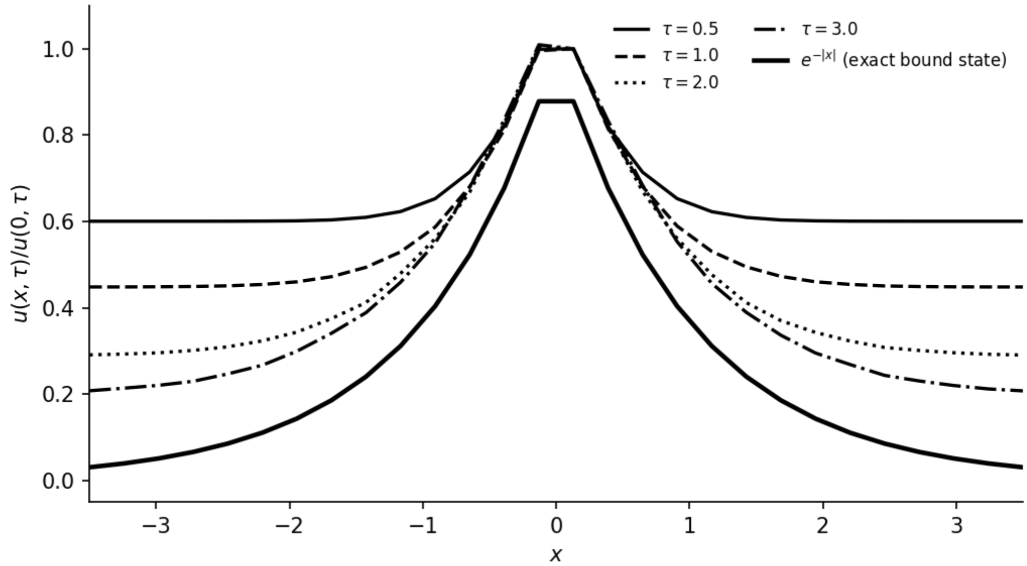

Figure 3: Monte Carlo estimates of $u(x,\tau)/u(0,\tau)$ for $\alpha = -1$ ($\kappa = 1$) at $\tau \in \{0.5, 1.0, 2.0, 3.0\}$, with $N = 25000$ paths. The curves converge to the exact bound state $e^{-|x|}$ (thick solid line), consistent with Corollary 6.3.

9. References

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