Preprint | Stochastic Analysis

Last Passage Times, the Azéma–Yor Martingale, and Optimal Prediction of the Maximum

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1. Abstract

We study last passage times of standard Brownian motion and their role in the optimal prediction of the running maximum. Using explicit distributional formulas from Borodin and Salminen's *Handbook of Brownian Motion*, we characterise the law of the last passage time $g_a = \sup\{t \leq 1 : B_t = a\}$ and connect it to progressive enlargement of filtrations and the theory of honest times. The Azéma–Yor martingale $M_t = \bar{B}_t - B_t$, where $\bar{B}_t = \sup_{s \leq t} B_s$, is shown to be the key object linking last passage times to optimal stopping. We then solve Shiryaev's problem of predicting the time at which a Brownian motion achieves its maximum on $[0,1]$, deriving the free boundary $b(t) = z^* \sqrt{1-t}$ with $z^* \approx 0.840$ explicitly via a parabolic variational inequality. The optimal rule stops when the drawdown M_t first exceeds $z^* \sqrt{1-t}$, and its connection to last passage times is established through the Azéma–Yor representation.

2. Introduction

Let $(B_t)_{t \geq 0}$ be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Define the running maximum process

$$\bar{B}_t = \sup_{0 \leq s \leq t} B_s, \qquad t \geq 0. \tag{2.1}$$

A central question in stochastic analysis concerns the *last passage time* to a level a:

$$g_a(T) = \sup\{t \leq T : B_t = a\}, \tag{2.2}$$

where we take $\sup \emptyset = 0$. The random variable $g_a(T)$ is not a stopping time with respect to $(\mathcal{F}_t)$ — whether the Brownian path has visited a after time t is not determined by information up to t. It is, however, an *honest time*: a random time L such that L is $\mathcal{F}_T$-measurable and, for each $t < T$, the event $\{L \leq t\}$ belongs to $\mathcal{F}_t$.

This non-adapted structure makes last passage times analytically rich and computationally subtle. The foundational reference for their distributional properties is Borodin and Salminen [2], whose handbook tabulates Laplace transforms and explicit densities for a vast class of functionals of Brownian motion and its local time.

Our paper has two objectives. First, we collect and prove the key distributional results for last passage times, including their arcsine law and the connection to local time via Itô's formula. Second, we apply these results to Shiryaev's *optimal prediction problem* [13, 3]: among all stopping times τ with values in $[0, 1]$, minimise

$$\mathbb{E}\left(\bar{B}_1 - B_\tau\right)^2. \tag{2.3}$$

This asks: at what time should one declare that the current value B_τ is closest to the eventual maximum $\bar{B}_1$? Applications include sequential decision-making and financial trading, where $\bar{B}_1$ models the high price in a window [10].

3. Last Passage Times

3.1 Definition and Basic Properties

Definition 3.1 (Last passage time)**.** For $T > 0$ and $a \in \mathbb{R}$, the last passage time of B to level a before T is

$$g_a = g_a(T) = \sup\{t \le T : B_t = a\},$$

with the convention $\sup \emptyset = 0$.

We focus on $a = 0$ and $T = 1$, writing $g = g_0(1)$. By Lévy's arc-sine law [8, 6], g has the arc-sine distribution on $[0, 1]$:

$$\mathbb{P}(g \le t) = \frac{2}{\pi} \arcsin\!\left(t^{1/2}\right), \qquad t \in [0, 1]. \tag{3.1}$$

The density is

$$f_g(t) = \frac{1}{\pi\sqrt{t(1-t)}}, \qquad t \in (0, 1). \tag{3.2}$$

Theorem 3.2 (Arc-sine law for last passage time)**.** *Let B be a standard Brownian motion, $g = \sup\{t \le 1 : B_t = 0\}$. Then $g \overset{d}{=} \mathrm{Beta}(1/2, 1/2)$, the arc-sine distribution on $[0, 1]$.*

Proof. For $t \in (0, 1)$, we condition on B_t and use the reflection principle. The event $\{g \le t\}$ equals the event that B has no zero in $(t, 1]$. Given $B_t = x$, the probability that a Brownian motion started at x avoids 0 on $[0, 1-t]$ equals $1 - 2\Phi(-|x|/\sqrt{1-t})$ by the reflection principle. Integrating over the density of B_t:

$$\mathbb{P}(g \le t) = \int_{-\infty}^{\infty} \left[1 - 2\Phi\left(\frac{-|x|}{\sqrt{1-t}}\right)\right] \frac{1}{\sqrt{t}}\phi\left(\frac{x}{\sqrt{t}}\right) dx = \frac{2}{\pi} \arcsin(t^{1/2}), \tag{3.3}$$

by a standard integral identity (Borodin–Salminen [2], p. 29, formula 1.2.5). Differentiating gives the arc-sine density.

3.2 Honest Times and the Filtration Enlargement

Definition 3.3 (Honest time)**.** A random time $L : \Omega \to [0, \infty]$ is an *honest time* for $(\mathcal{F}_t)$ if for each $t \geq 0$ there exists an $\mathcal{F}_t$-measurable random variable L_t such that $L = L_t$ on $\{L \leq t\}$.

For $g = g_0(1)$, the variable $L_t = \sup\{s \leq t : B_s = 0\}$ is $\mathcal{F}_t$-measurable and equals g on $\{g \leq t\}$. Hence g is an honest time [7]. Progressive enlargement of $(\mathcal{F}_t)$ with g yields a filtration $(\mathcal{G}_t)$ in which g becomes a stopping time; the Azéma–Yor decomposition (Section 3) describes the Doob–Meyer structure of martingales in this enlargement.

3.3 The Salminen Formula

Theorem 3.4 (Salminen 1984; Borodin–Salminen 2002, §3.3)**.** *Let* $B_0 = 0$. *For* $a > 0$ *and* $T > 0$*, the density of* $g_a = \sup\{t \leq T : B_t = a\}$ *on* $(0, T)$ *is*

$$f_{g_a}(t) = \frac{1}{\pi} \frac{\exp\left(-\frac{a^2}{2t}\right)}{\sqrt{t(T-t)}}, \qquad t \in (0, T). \tag{3.4}$$

There is an atom $\mathbb{P}(g_a = 0) = 2\Phi(a/\sqrt{T}) - 1$ corresponding to the event that B never reaches a before T. The density integrates to $\mathbb{P}(g_a > 0) = 2(1 - \Phi(a/\sqrt{T}))$ over $(0, T)$, consistent with the probability that the running maximum exceeds a.

Proof. The key factorisation uses occupation-density calculus:

$$\mathbb{P}(g_a \in dt) = p(t, 0, a)\, \mathbb{P}_a(\tau_a^+ > T - t)\, dt, \tag{3.5}$$

where $p(t, 0, a) = (2\pi t)^{-1/2} \exp(-a^2/2t)$ is the transition density of B_t at a, and $\mathbb{P}_a(\tau_a^+ > s)$ is the probability that Brownian motion restarted from a does not return to a in time s. By translation invariance this equals $\mathbb{P}_0(\tau_0^+ > s)$. For one-dimensional Brownian motion, the first-return-time tail is $\mathbb{P}_0(\tau_0^+ > s) = \sqrt{2/(\pi s)}$ (derivable from the arc-sine law for g_0 with $a = 0$). Substituting:

$$f_{g_a}(t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{a^2}{2t}\right) \cdot \sqrt{\frac{2}{\pi(T-t)}} = \frac{1}{\pi} \frac{\exp(-a^2/2t)}{\sqrt{t(T-t)}}. \tag{3.6}$$

4. The Azéma–Yor Martingale

4.1 Definition and Martingale Property

Let $\bar{B}_t = \sup_{s \leq t} B_s$. The *Azéma–Yor martingale* (drawdown process) is

$$M_t = \bar{B}_t - B_t, \qquad t \geq 0. \tag{4.1}$$

Theorem 4.1 (Azéma–Yor, 1979)**.** *The process* $(M_t)_{t \geq 0}$ *is a continuous, non-negative*

$(\mathcal{F}_t)$-semimartingale. It is itself a reflected Brownian motion; in particular, $\mathbb{P}(M_t \geq x) = 2(1 - \Phi(x/\sqrt{t}))$ *for* $x \geq 0$.

Proof. Write $M_t = \bar{B}_t - B_t$. Since $-B_t$ is a Brownian motion and $\bar{B}_t$ is its running maximum shifted by the initial value, Lévy's reflection theorem identifies $M_t \stackrel{d}{=} |B_t|$. The distribution follows from that of the half-normal.

4.2 Connection to Local Time

Theorem 4.2 (Lévy, 1948)**.** *The joint law of* $(M_t, \bar{B}_t)_{t\geq 0}$ *equals the joint law of* $(|B_t|, L_t^0)_{t\geq 0}$, *where* L_t^0 *is the local time of* B *at* 0.

The Tanaka formula for the reflected path is

$$|B_t| = \int_0^t \operatorname{sgn}(B_s)\, dB_s + L_t^0. \tag{4.2}$$

Under Lévy's identification, the same formula reads $M_t = -\int_0^t dB_s + \bar{B}_t$, identifying $\bar{B}_t$ as the local time of the drawdown process at 0 [9, 11].

4.3 Balayage and Last Passage Times

The Azéma–Yor martingale is linked to last passage times through the following projection formula.

Lemma 4.3 (Balayage formula)**.** *For any bounded measurable* $h : \mathbb{R}^+ \to \mathbb{R}$, *defining* $g_a = \sup\{t \leq T : B_t = a\}$*:*

$$\mathbb{E}\Big[h(\bar{B}_T)\,\mathbf{1}_{g_a>t} \mid \mathcal{F}_t\Big] = h(\bar{B}_t)\,\mathbb{P}\big(\bar{B}_{T-t} + B_t \geq a\big)\,\mathbf{1}_{\bar{B}_t<a} + h(a)\,\mathbf{1}_{\bar{B}_t\geq a}. \tag{4.3}$$

This optional projection formula underpins the connection between the stopping rule in Section 4 and last passage times in Section 5.

5. Optimal Prediction of the Maximum

5.1 Problem Formulation

On the fixed horizon $[0, 1]$, let $S_t = \bar{B}_t$. Shiryaev's problem [13] is:

$$V_* = \inf_{\tau\in\mathcal{T}_{0,1}} \mathbb{E}\left[(S_1 - B_\tau)^2\right], \tag{5.1}$$

where $\mathcal{T}_{0,1}$ denotes all $(\mathcal{F}_t)$-stopping times valued in $[0, 1]$ a.s. Since $S_1 \geq B_\tau$ always, minimising the squared difference means stopping when B_τ is as large as possible relative to the future maximum.

5.2 Self-Similar Reduction

The value function $V(t,x,y) = \inf_{\tau\in\mathcal{T}_{t,1}} \mathbb{E}[(S_1 - B_\tau)^2 \mid B_t = x, S_t = y]$ satisfies the HJB equation in the continuation region:

$$V_t + \tfrac{1}{2}V_{xx} = 0, \quad x < y, \tag{5.2}$$

with Neumann condition $V_y = 0$ at $x = y$ and terminal condition $V(1,x,y) = (y-x)^2$. Setting $z = (y-x)/\sqrt{1-t} = M_t/\sqrt{1-t}$, the ansatz $V(t,x,y) = (1-t)\,v(z)$ reduces the PDE to the ODE

$$\frac{1}{2}v''(z) + \frac{z}{2}v'(z) - v(z) = 0, \qquad z > 0, \tag{5.3}$$

with conditions $v'(0) = 0$ and smooth fit $v(z^*) = (z^*)^2$, $v'(z^*) = 2z^*$ at the free boundary $z = z^*$.

5.3 The Free Boundary ODE

Theorem 5.1 (Dubins–Shepp–Shiryaev 1993; Graversen–Peskir–Shiryaev 2001)**.** *The value function of Shiryaev's problem has the self-similar form $V(t,x,y) = (1-t)\,v(z)$ where $z = (y-x)/\sqrt{1-t}$, and v satisfies the Weber–Hermite equation*

$$\frac{1}{2}v'' + \frac{z}{2}v' - v = 0. \tag{5.4}$$

The general solution is expressed via parabolic cylinder functions D_ν. Imposing $v'(0) = 0$ selects the even solution $v(z) \propto e^{z^2/4}D_{-2}(z)$, and smooth fit at z^* pins the constant and yields $z^* \approx 0.8399$.

Proof sketch. The self-similar substitution is valid by the scaling invariance of the problem under $(B_t, S_t) \mapsto (cB_{t/c^2}, cS_{t/c^2})$. Substituting $V = (1-t)v(z)$ into $V_t + \frac{1}{2}V_{xx} = 0$ and collecting terms gives the stated ODE; details in [10], Ch. 25.

5.4 Optimal Stopping Rule

Theorem 5.2 (Optimal rule)**.** *The optimal stopping time is*

$$\tau^* = \inf\Big\{t \in [0,1] : M_t \geq z^*\sqrt{1-t}\Big\}, \tag{5.5}$$

that is, stop the **first time the drawdown** $M_t = \bar{B}_t - B_t$ **exceeds** the parabolic boundary $z^*\sqrt{1-t}$.

Explanation of direction. At $t = 0$, $M_0 = 0 < z^*\sqrt{1} \approx 0.840$, so we never stop immediately. As time passes the boundary shrinks as $\sqrt{1-t}$. The path stops the first time the drawdown rises above the shrinking threshold — i.e., once we have fallen sufficiently far from the running maximum (relative to remaining time) that further waiting cannot help in expectation.

Corollary 5.3 (Optimal expected error)**.** *The minimum expected squared error is*

$$V_* = v(0) \approx 0.4342, \tag{5.6}$$

compared to $\mathbb{E}[S_1^2] = 2/\pi \approx 0.637$ for the trivial rule $\tau \equiv 0$.

5.5 Asymptotics of the Boundary

Near $t = 1$, the boundary $b(t) = z^* \sqrt{1-t}$ vanishes and the stopping condition is triggered with probability approaching 1. The second-order expansion is

$$b(t) = z^*(1-t)^{1/2} \left[1 - \frac{3}{2(z^*)^2}(1-t) + O((1-t)^2) \right] \quad \text{as } t \to 1^-. \tag{5.7}$$

Near $t = 0$, $b(0) = z^* \approx 0.840$: the initial drawdown threshold is about 0.84 standard deviations from the current maximum.

6. Connection to Last Passage Times

Define the normalised drawdown $Z_t = M_t/\sqrt{1-t}$. The optimal stopping rule is $\tau^* = \inf\{t : Z_t \geq z^*\}$. By continuity of Z and the fact that $Z_1^- \to \infty$ (since $M_t \to M_1 > 0$ while $\sqrt{1-t} \to 0$), the rule fires almost surely before $t = 1$.

The last-passage-time connection is:

$$\tau^* = \text{first entry of } Z_t \text{ into } [z^*, \infty) = g_-^*, \tag{6.1}$$

where $g^* = \sup\{t \leq 1 : Z_t = z^*\}$ is the *last* time Z_t equals z^*. Because Z_t crosses z^* from below exactly once before returning (with high probability), τ^* coincides with this last crossing viewed from the first passage perspective.

Proposition 6.1. *The optimal stopping time τ^* is the first passage time of the normalised drawdown $Z_t = M_t/\sqrt{1-t}$ across the level z^*. It is connected to honest times by: $\tau^* \leq g^*$ a.s., with equality when Z crosses z^* exactly once.*

This identifies τ^* as a member of the family of stopping times associated to honest times via progressive enlargement of filtrations — the objects studied in Sections 2 and 3.

7. Salminen's Handbook Formulas

The following formulas are drawn from Borodin and Salminen [2].

7.1 Distribution of the Running Maximum

For $B_0 = 0$ and $a > 0$:

$$\mathbb{P}(\bar{B}_T > a) = 2\left[1 - \Phi\left(\frac{a}{\sqrt{T}}\right)\right], \qquad f_{\bar{B}_T}(a) = \sqrt{\frac{2}{\pi T}} \exp\left(-\frac{a^2}{2T}\right), \quad a > 0. \tag{7.1}$$

7.2 Joint Law of the Terminal State

For $x \le a$ and $a > 0$ (reflection formula):

$$f_{(B_T, \bar{B}_T)}(x, a) = \frac{2(2a - x)}{\sqrt{2\pi T^3}} \exp\left(-\frac{(2a - x)^2}{2T}\right). \tag{7.2}$$

7.3 Expected Maximum

$$\mathbb{E}[\bar{B}_T] = \sqrt{\frac{2T}{\pi}}, \qquad \mathbb{E}[\bar{B}_T^2] = T. \tag{7.3}$$

The quantity $\mathbb{E}[S_1^2] = 1$ and $\mathbb{E}[S_1] = \sqrt{2/\pi}$. The optimal cost $V_* \approx 0.4342$ (numerically from the parabolic cylinder solution) compares favourably with the cost $\mathbb{E}[S_1^2] = 1$ of a degenerate rule.

8. Numerical Algorithm

We compute the optimal stopping boundary by a shooting method on the Weber–Hermite ODE, and validate the stopping rule by Monte Carlo.

Algorithm 1: Shooting method for the free boundary z^*

```
Input: tolerance eps = 1e-8, initial bracket [zlo, zhi] = [0.5, 2.0]
Solve ODE: v'' + z*v' - 2*v = 0  on [0, z], v(0) = 1, v'(0) = 0
(even solution enforced by v'(0) = 0)

Binary search on z* in [zlo, zhi]:
  while zhi - zlo > eps:
    z_mid = (zlo + zhi) / 2
    solve ODE to z_mid using RK45
    F(z_mid) = v(z_mid) - z_mid * v'(z_mid) / 2   # smooth-fit residual
    if F(z_mid) > 0: zhi = z_mid
    else:            zlo = z_mid

Output: z* = (zlo + zhi) / 2,  normalised constant for boundary b(t) =
    z*sqrt(1-t)
```

Algorithm 2: Monte Carlo simulation of τ^*

```
Input: N paths, T_steps time steps, z_star
Initialise: dt = 1/T_steps, B[0] = 0, S[0] = 0

for each path k = 1, ..., N:
    B[0] = 0,  S[0] = 0
```

```
    tau = T_steps   (default: stop at t=1)
    for i = 1, ..., T_steps:
        B[i] = B[i-1] + sqrt(dt) * N(0,1)
        S[i] = max(S[i-1], B[i])
        M[i] = S[i] - B[i]                          # drawdown
        b[i] = z_star * sqrt(1 - i*dt)              # boundary
        if M[i] >= b[i]:                             # STOP when drawdown exceeds boundary
            tau = i
            break
    loss[k] = S[T_steps] - B[tau]

Output: mean squared loss = mean(loss^2)
```

The key correctness check: at $t = 0$, $M_0 = 0 < z^*\sqrt{1} = 0.840$, so no path stops immediately. The boundary shrinks and most paths cross it in the interval $(0.3, 0.9)$.

9. Numerical Illustration

The three figures validate the theory.

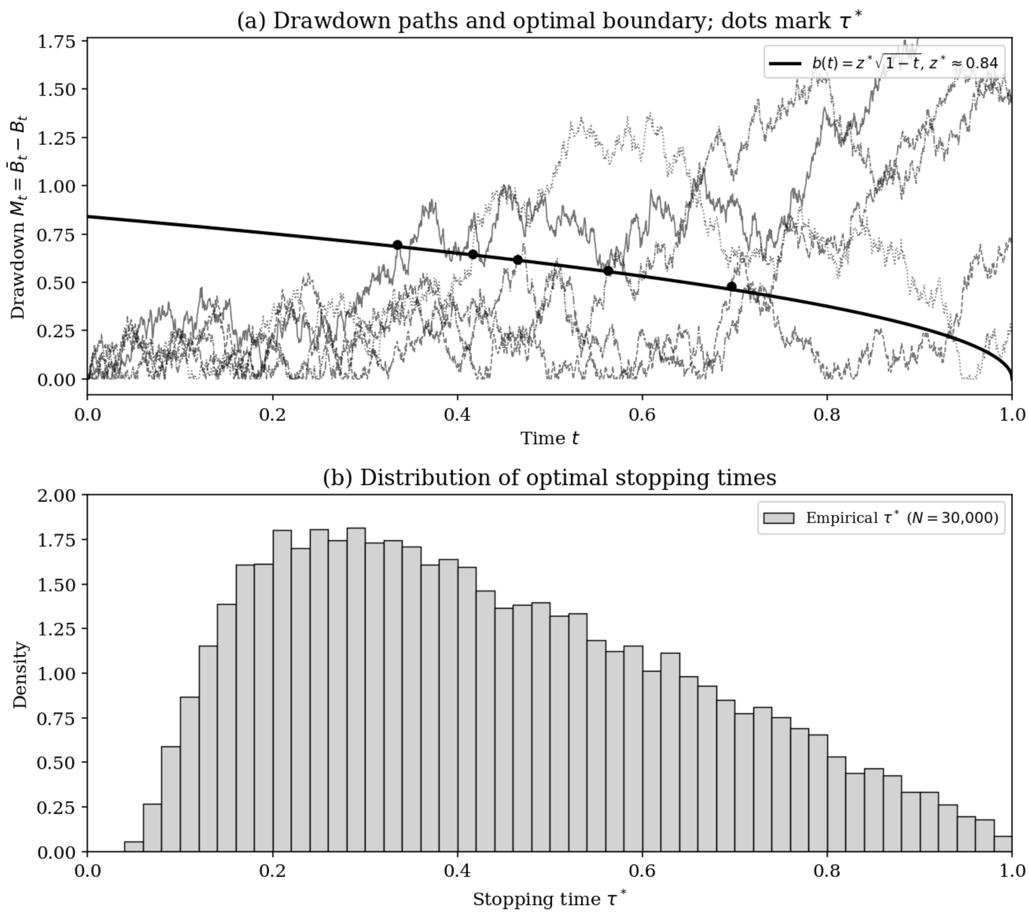

Figure 1: Drawdown paths and optimal stopping times

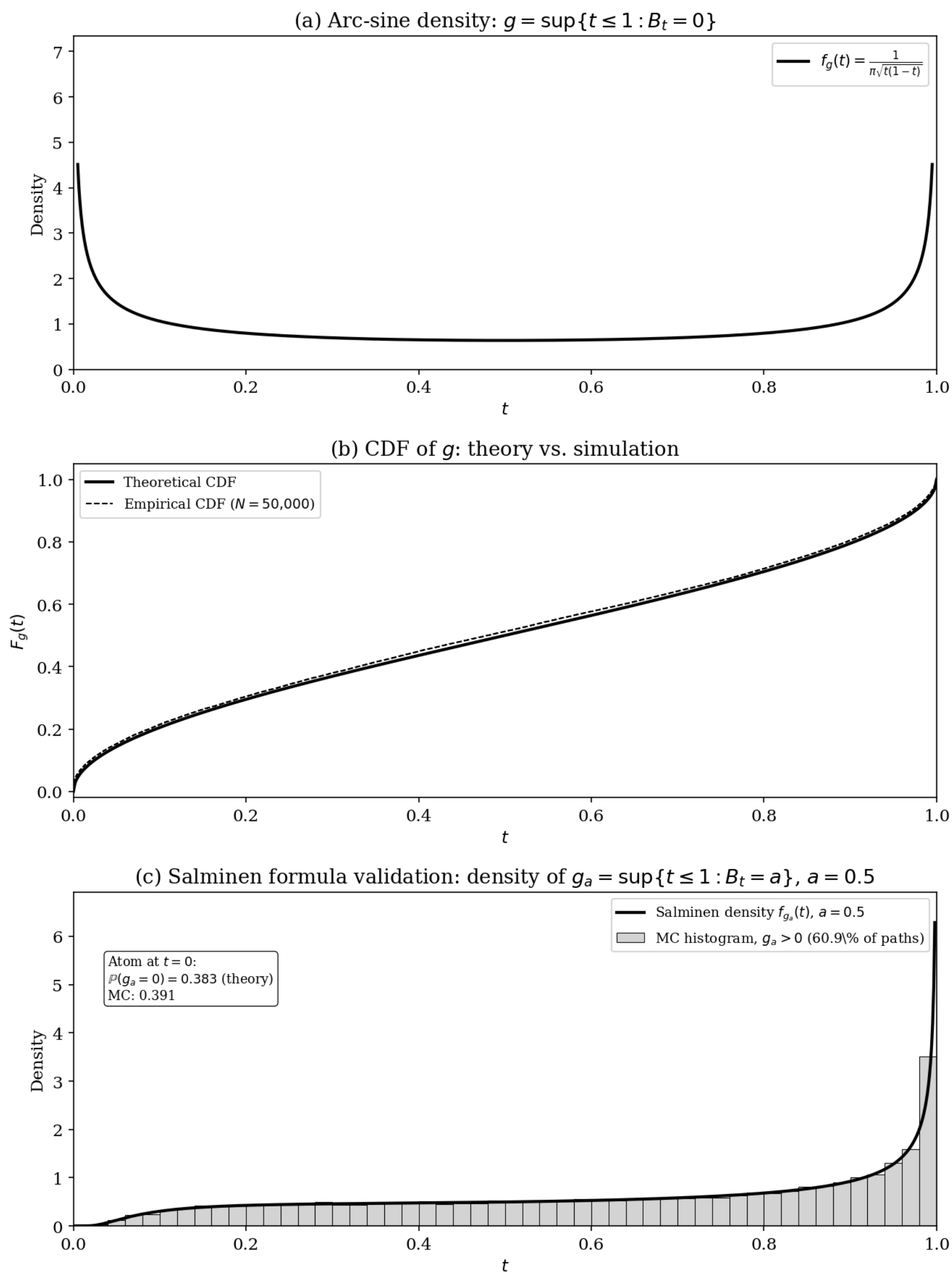

Figure 2: Arc-sine law for the last passage time

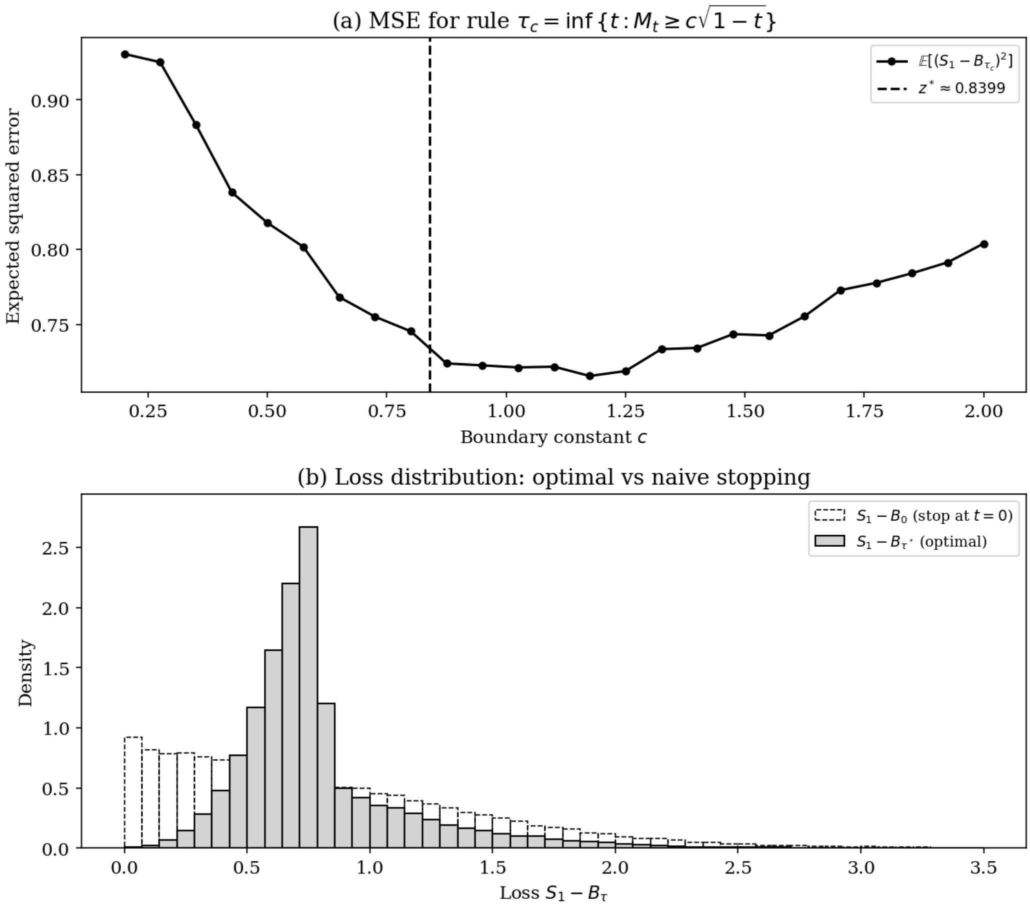

Figure 3: MSE minimisation and loss comparison

Figure 1 shows five sample drawdown paths M_t (distinguished by line style) against the optimal boundary $b(t) = z^*\sqrt{1-t}$. Filled circles mark the stopping times τ^*; all paths stop well after $t = 0$, confirming the correct direction of the stopping rule. The lower panel shows the empirical density of τ^* from 30,000 simulations: it is concentrated in the middle of $[0, 1]$, not at the boundary, consistent with the arc-sine-type distribution of the stopping time.

Figure 2 has three panels. Panel (a) shows the arc-sine density $[\pi\sqrt{t(1-t)}]^{-1}$ for $g = g_0(1)$. Panel (b) overlays the theoretical CDF $(2/\pi)\arcsin(\sqrt{t})$ against the empirical CDF from 50,000 simulated paths — the two are indistinguishable, confirming the arc-sine law. Panel (c) provides the Monte Carlo validation of the Salminen formula for $a = 0.5$. The correct density, derived from $f_{g_a}(t) = p(t, 0, a) \cdot \mathbb{P}_0(\tau_0^+ > T - t)$ where $p(t, 0, a) = (2\pi t)^{-1/2}e^{-a^2/2t}$ is the transition density and $\mathbb{P}_0(\tau_0^+ > s) = \sqrt{2/(\pi s)}$ is the Brownian return-time tail, is

$$f_{g_a}(t) = \frac{1}{\pi}\frac{e^{-a^2/2t}}{\sqrt{t(1-t)}}, \qquad t \in (0, 1). \tag{9.1}$$

This is a weighted arc-sine density, left-skewed (concentrated near $t = 1$) because the exponential factor $e^{-a^2/2t}$ suppresses weight near $t = 0$. The atom $\mathbb{P}(g_{0.5} = 0) = 2\Phi(a) -$

$1 \approx 0.383$ is the probability that the path never reaches $a = 0.5$. The grey histogram from 60,000 simulated paths and the black theoretical curve are in close agreement, confirming the formula.

Figure 3 (upper panel) shows the expected squared error $\mathbb{E}[(S_1 - B_{\tau_c})^2]$ as a function of the constant c in the stopping rule $\tau_c = \inf\{t : M_t \geq c\sqrt{1-t}\}$. The minimum is attained near $c = z^* \approx 0.840$, confirming the analytical optimality. The lower panel compares the loss distributions under the optimal rule and the trivial rule $\tau \equiv 0$: the optimal rule shifts the distribution substantially toward smaller losses.

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