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Bond Pricing Under Inertial Interest Rate Dynamics: The Langevin-Zamrik PDE

Affine Bond Pricing in an Underdamped Interest Rate System

T. Zamrik

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1. Abstract

We introduce the Langevin-Zamrik partial differential equation, a zero-coupon bond pricing equation derived from the full underdamped Langevin dynamics of the short interest rate. The model belongs to the short-rate modelling tradition — alongside Vasicek and Cox-Ingersoll-Ross — but differs from all existing members of that class in that the short rate obeys a second-order SDE, with the two-dimensional state (r_t, v_t) comprising rate and velocity. For a quadratic (OU-type) potential the equation admits a closed-form affine solution $P = \exp\{A(\tau) + B_1(\tau)r + B_2(\tau)v\}$, where the coefficient pair (B_1, B_2) satisfies a 2×2 linear ODE system solved explicitly via the matrix exponential of the inertia-friction-spring matrix M. The three damping regimes — overdamped, critically damped, and underdamped — produce qualitatively distinct yield curve shapes: monotone, single-humped, and oscillatory respectively. The Vasicek formula is recovered exactly in the overdamped limit $m \to 0$, and the overdamped yield curve is structurally equivalent to the Nelson-Siegel-Svensson specification, providing its first no-arbitrage derivation. A companion forward-rate theory in the Heath-Jarrow-Morton spirit will be developed in future work.

2. Introduction

The Vasicek model [9] remains the canonical analytically tractable short-rate model. Its derivation from the overdamped Langevin equation is well known: when the friction-to-mass ratio γ/m is taken to infinity, the inertial term $m\ddot{r}$ becomes negligible and the second-order stochastic differential equation reduces to the Ornstein-Uhlenbeck first-order SDE. What has not been explored is whether the full underdamped Langevin equation — retaining the inertial term — admits a self-consistent zero-coupon bond pricing theory, whether it yields a closed-form solution, and what new economic and geometric structure emerges. The present paper operates entirely within the **short-rate modelling tradition** — alongside Vasicek [9], Cox-Ingersoll-Ross [3], and their multi-factor generalisations [6] — in which a finite-dimensional Markovian state drives all bond prices and forward rates as derived quantities. A companion forward-rate theory, in which the Langevin-Zamrik inertial structure is lifted to the full forward curve in the spirit of Heath-Jarrow-Morton

[10], will be developed in future work.

This paper answers all three questions. We introduce the Langevin-Zamrik PDE, a bond pricing equation in the two-dimensional state space (r, v) derived from the full underdamped Langevin dynamics via the Feynman-Kac theorem. For the quadratic (OU-type) potential we prove it admits an affine closed-form solution, with coefficients determined by a 2×2 ODE system whose solution is the matrix exponential of the inertia-friction-spring matrix M. The Vasicek formula is recovered in the overdamped limit. Three qualitatively distinct yield curve shapes emerge from the three damping regimes determined by the sign of the discriminant $\Delta = \gamma^2 - 4m\kappa$. Most strikingly, in the overdamped regime the Langevin-Zamrik yield curve is structurally equivalent to the Nelson-Siegel-Svensson model [5, 8] — the dominant empirical yield curve specification used by central banks worldwide — providing its first no-arbitrage derivation from physical principles.

The paper is organised as follows. Section 2 introduces the general Langevin-Zamrik PDE and establishes its relationship to the Kramers equation [4]. Section 3 specialises to the quadratic potential and derives the closed-form affine solution. Section 4 establishes the inertia-free limit, proving convergence to first-order dynamics as $m \to 0$. Section 5 derives the full characteristic function of the integrated rate and the bivariate Gaussian structure of the state vector. Section 6 analyses the three damping regimes and yield curve shapes. Section 7 establishes the Nelson-Siegel-Svensson connection. Section 8 discusses extensions to nonlinear potentials, state-dependent volatility, multi-factor systems, and Lévy noise.

3. The Langevin-Zamrik PDE

3.1 Setup and Notation

Throughout, $(\Omega, \mathcal{F}, \mathbb{Q})$ is a filtered probability space carrying a standard Brownian motion W_t. All dynamics are stated under the risk-neutral measure $\mathbb{Q}$. We write ∂_x for $\partial/\partial x$ and primes for $d/d\tau$.

Definition 3.1 (Underdamped Langevin Dynamics [11)**.**] Let $U : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. The underdamped Langevin equation for the short rate r_t is

$$m\,\ddot{r}_t = -\gamma\,\dot{r}_t - \partial_r U(r_t) + \sigma\,\dot{W}_t, \qquad m > 0,\ \gamma > 0,\ \sigma > 0.$$

Setting $v_t = \dot{r}_t$, this is the first-order system on $\mathbb{R}^2$:

$$dr_t = v_t\,dt,$$

$$dv_t = \frac{1}{m}[-\gamma v_t - \partial_r U(r_t)]\,dt + \frac{\sigma}{m}\,dW_t.$$

Remark 3.2. The process (r_t, v_t) is a Markov diffusion on $\mathbb{R}^2$ with infinitesimal generator

$$\mathcal{L} = v\,\partial_r - \frac{\gamma v + \partial_r U(r)}{m}\,\partial_v + \frac{\sigma^2}{2m^2}\,\partial_v^2.$$

Assumption 3.3 (Novikov Condition)**.** The market price of risk satisfies $\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\frac{1}{2}\int_0^T \lambda_t^2\,dt\right)\right] <$

∞, ensuring $\mathbb{Q}$ is a well-defined equivalent martingale measure.

3.2 Derivation of the Langevin-Zamrik PDE

Theorem 3.4 (Langevin-Zamrik PDE)**.** *Under Assumption 2.1, the price of a zero-coupon bond maturing at T,*

$$P(t,T;r,v) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r_s\,ds\right)\,\middle|\, r_t=r,\; v_t=v\right],$$

satisfies

$$\partial_t P + v\,\partial_r P - \frac{\gamma v + \partial_r U(r)}{m}\,\partial_v P + \frac{\sigma^2}{2m^2}\,\partial_v^2 P - r\,P = 0$$

on $[0,T)\times\mathbb{R}^2$, with terminal condition $P(T,T;r,v)=1$. ***Proof.*** *The Feynman-Kac theorem applied to the functional $\mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_s\,ds)\,|\,r_t,v_t]$ with killing rate r_t gives $\partial_t P + \mathcal{L}P - rP = 0$. Substituting the generator $\mathcal{L}$ from Remark 2.1 yields the stated equation.*

Definition 3.5 (Langevin-Zamrik PDE)**.** We call

$$\mathcal{L}^{\mathrm{LZ}}[P] \;\equiv\; \partial_t P + v\,\partial_r P - \frac{\gamma v + \partial_r U(r)}{m}\,\partial_v P + \frac{\sigma^2}{2m^2}\,\partial_v^2 P - r\,P \;=\; 0 \tag{3.1}$$

the **Langevin-Zamrik PDE** for potential U. The operator $\mathcal{L}^{\mathrm{LZ}}$ is a degenerate parabolic operator on $\mathbb{R}^2$: it is elliptic in v only and first-order in r.

3.3 Relationship to the Kramers Equation

The Kramers equation [4] is the forward Kolmogorov equation for the transition density $\rho(t,r,v)$ of (r_t,v_t):

$$\partial_t\rho + v\,\partial_r\rho - \frac{\gamma v + \partial_r U(r)}{m}\,\partial_v\rho - \frac{\gamma}{m}\,\rho - \frac{\sigma^2}{2m^2}\,\partial_v^2\rho = 0. \tag{3.2}$$

Proposition 3.6 (Adjoint Structure)**.** *The Langevin-Zamrik PDE uses the backward generator $\mathcal{L}$; the Kramers equation uses its $L^2(\mathbb{R}^2)$-adjoint $\mathcal{L}^*$. The two equations differ in exactly two terms: (i)* ***Diffusion sign:*** *Kramers has $-\frac{\sigma^2}{2m^2}\partial_v^2\rho$; the Langevin-Zamrik PDE has $+\frac{\sigma^2}{2m^2}\partial_v^2 P$. This sign reversal is the structural distinction between forward and backward Kolmogorov equations. (ii)* ***Zeroth-order term:*** *Kramers has $-\frac{\gamma}{m}\rho$, arising from the divergence of the drift; the Langevin-Zamrik PDE has $-r\,P$, the bond discounting term.* ***Proof.*** *Compute $\mathcal{L}^*$ by integration by parts: $\langle\mathcal{L}f,g\rangle = \langle f,\mathcal{L}^*g\rangle$ for $f,g\in C_c^\infty(\mathbb{R}^2)$. The second-order term $\frac{\sigma^2}{2m^2}\partial_{vv}$ is self-adjoint; the first-order drift terms contribute boundary terms that yield the $-\frac{\gamma}{m}$ coefficient.*

Remark 3.7 (Physical Interpretation). In the Kramers equation, probability mass dissipates at rate γ/m due to friction. In the Langevin-Zamrik PDE, bond value dissipates at rate r due to discounting. The Langevin-Zamrik PDE is the bond pricing counterpart of the Kramers equation: friction is replaced by the short rate as the source of dissipation.

3.4 Existence of Closed-Form Solutions

Proposition 3.8 (Necessary Condition for Affine Solutions)**.** *The Langevin-Zamrik PDE admits a solution of the exponential-affine form*

$$P = \exp\{A(\tau) + B_1(\tau)\,r + B_2(\tau)\,v\}, \quad \tau = T - t,$$

if and only if $\partial_r U(r)$ is affine in r. ***Proof.*** *Substituting the ansatz into the PDE and dividing by P yields, after separating by powers of r and v, the requirement that $B_2(\tau)\,\partial_r U(r)/m$ be linear in r for all $\tau > 0$. Since $B_2 \not\equiv 0$ generically, $\partial_r U(r)$ must be affine.*

Remark 3.9. Proposition 2.2 identifies the quadratic potential $U(r) = \frac{1}{2}\kappa(r-\theta)^2$ as the unique member of the polynomial potential family admitting an affine solution. All nonlinear potentials require numerical PDE methods; see Section 8.

4. The OU Special Case: Affine Bond Pricing

4.1 Dynamics

We fix the quadratic potential $U(r) = \frac{1}{2}\kappa(r-\theta)^2$, so $\partial_r U(r) = \kappa(r-\theta)$ with $\kappa > 0$ and $\theta \in \mathbb{R}$. The Langevin-Zamrik PDE becomes

$$\partial_t P + v\,\partial_r P - \frac{\gamma v + \kappa(r-\theta)}{m}\,\partial_v P + \frac{\sigma^2}{2m^2}\,\partial_v^2 P = r\,P. \tag{4.1}$$

The system (r_t, v_t) is linear-Gaussian: for any initial condition (r_0, v_0), the pair (r_t, v_t) is jointly Gaussian and $\int_0^T r_s\,ds$ is a Gaussian random variable. The bond price is therefore the moment generating function of a Gaussian, which guarantees the exponential-affine form a priori.

4.2 The Inertia-Friction-Spring Matrix

Definition 4.1 (IFS Matrix)**.** The 2 × 2 matrix

$$M = \begin{pmatrix} 0 & -\kappa/m \\ 1 & -\gamma/m \end{pmatrix}$$

is called the **inertia-friction-spring (IFS) matrix**. Its characteristic polynomial is

$$p(\lambda) = \lambda^2 + \frac{\gamma}{m}\lambda + \frac{\kappa}{m} = 0,$$

with discriminant

$$\Delta \;=\; \left(\frac{\gamma}{m}\right)^2 - \frac{4\kappa}{m} \;=\; \frac{\gamma^2 - 4m\kappa}{m^2}.$$

Proposition 4.2 (Eigenvalue Classification)**.** *The eigenvalues $\lambda_{1,2}$ of M satisfy* $\mathrm{Re}(\lambda_{1,2}) = -\gamma/(2m) < 0$ *in all cases, and:* ***(i) Overdamped*** *($\gamma^2 > 4m\kappa$, $\Delta > 0$):*

$$\lambda_{1,2} = \frac{-\gamma/m \pm \sqrt{\Delta}}{2}, \quad \lambda_1 < \lambda_2 < 0.$$

(ii) Critically damped *($\gamma^2 = 4m\kappa$, $\Delta = 0$):*

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2m} < 0 \quad \textit{(repeated eigenvalue).}$$

(iii) Underdamped *($\gamma^2 < 4m\kappa$, $\Delta < 0$):*

$$\lambda_{1,2} = -\alpha \pm i\omega, \quad \alpha = \frac{\gamma}{2m} > 0, \quad \omega = \frac{\sqrt{4m\kappa - \gamma^2}}{2m} > 0.$$

Proof. *The quadratic formula applied to $p(\lambda) = 0$. The real part $-\gamma/(2m)$ is negative by $\gamma, m > 0$.*

4.3 The ODE System for Bond Coefficients

Theorem 4.3 (Affine Structure)**.** *For $\partial_r U(r) = \kappa(r - \theta)$, the Langevin-Zamrik PDE admits the unique solution*

$$P(\tau; r, v) = \exp\{A(\tau) + B_1(\tau)\, r + B_2(\tau)\, v\}, \quad \tau = T - t,$$

where A, B_1, B_2 satisfy the system

$$\begin{pmatrix} B_1'(\tau) \\ B_2'(\tau) \end{pmatrix} = M \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \qquad B_1(0) = B_2(0) = 0,$$

$$A'(\tau) = \frac{\kappa\theta}{m}\, B_2(\tau) + \frac{\sigma^2}{2m^2}\, B_2(\tau)^2, \qquad A(0) = 0.$$

Proof. *Substitute $P = e^{A+B_1 r+B_2 v}$ into the Langevin-Zamrik PDE and divide by P. Since $\tau = T - t$, we have $\partial_t = -\partial_\tau$. The equation becomes*

$$-A' - B_1' r - B_2' v + vB_1 - \frac{\gamma v + \kappa(r - \theta)}{m} B_2 + \frac{\sigma^2}{2m^2} B_2^2 = r.$$

Collecting by monomial: ***Coefficient of*** *r:* *$-B_1' - \frac{\kappa}{m}B_2 = 1$, i.e., $B_1' = -\frac{\kappa}{m}B_2 - 1$.* ***Coefficient of*** *v:* *$-B_2' + B_1 - \frac{\gamma}{m}B_2 = 0$, i.e., $B_2' = B_1 - \frac{\gamma}{m}B_2$.* ***Constant:*** *$-A' + \frac{\kappa\theta}{m}B_2 + \frac{\sigma^2}{2m^2}B_2^2 = 0$. The first two equations combine into the stated matrix ODE. Terminal condition $P(T, T) = 1$ gives $A(0) = B_1(0) = B_2(0) = 0$.*

Remark 4.4. In the Duffie-Kan affine term-structure framework, the bond pricing equations for (B_1, B_2) take the form of Riccati ODEs — nonlinear in $\mathbf{B}$ whenever the diffusion matrix depends on the state. Here the OU volatility is state-independent, so the Riccati term in the $\mathbf{B}$-subsystem vanishes and one obtains the linear matrix ODE above. The quadratic nonlinearity survives only in the A-equation: $A'(\tau) = \frac{\kappa\theta}{m}B_2 + \frac{\sigma^2}{2m^2}B_2^2$. This is the Riccati component of the system; it is driven by $B_2(\tau)$, which itself is known in closed form via the matrix exponential.

4.4 Matrix Exponential Solution

Theorem 4.5 (Closed-Form Coefficients)**.** *The unique solution to the ODE system in Theorem 3.1 is*

$$\begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix} = (e^{M\tau} - I) \begin{pmatrix} \gamma/\kappa \\ m/\kappa \end{pmatrix},$$

$$A(\tau) = \int_0^\tau \left[\frac{\kappa\theta}{m} B_2(s) + \frac{\sigma^2}{2m^2} B_2(s)^2 \right] ds.$$

Proof. *The ODE* $\mathbf{B}' = M\mathbf{B} + \mathbf{c}$*,* $\mathbf{c} = (-1, 0)^\top$*,* $\mathbf{B}(0) = \mathbf{0}$*, has solution*

$$\mathbf{B}(\tau) = \int_0^\tau e^{Ms}\, \mathbf{c}\, ds = M^{-1}(e^{M\tau} - I)\mathbf{c}.$$

Since $\det M = \kappa/m > 0$*,* M *is invertible with*

$$M^{-1} = \begin{pmatrix} -\gamma/\kappa & 1 \\ -m/\kappa & 0 \end{pmatrix}, \qquad M^{-1}\mathbf{c} = \begin{pmatrix} \gamma/\kappa \\ m/\kappa \end{pmatrix}.$$

Since $M^{-1}(e^{M\tau} - I)\mathbf{c} = (e^{M\tau} - I)M^{-1}\mathbf{c}$ *(constant matrices commute with their own exponentials), the result follows.*

4.5 Explicit Matrix Exponentials

The three damping regimes yield distinct closed-form expressions for $e^{M\tau}$.

Overdamped ($\gamma^2 > 4m\kappa$, eigenvalues $\lambda_1 < \lambda_2 < 0$, $D = \lambda_1 - \lambda_2 < 0$):

$$e^{M\tau} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_2\tau} - \lambda_2 e^{\lambda_1\tau} & \frac{\kappa}{m}(e^{\lambda_2\tau} - e^{\lambda_1\tau}) \\ e^{\lambda_1\tau} - e^{\lambda_2\tau} & \lambda_1 e^{\lambda_1\tau} - \lambda_2 e^{\lambda_2\tau} \end{pmatrix}. \tag{4.2}$$

Underdamped ($\gamma^2 < 4m\kappa$, $\alpha = \gamma/(2m)$, $\omega = \sqrt{4m\kappa - \gamma^2}/(2m)$):

$$e^{M\tau} = e^{-\alpha\tau} \begin{pmatrix} \cos\omega\tau + \frac{\alpha}{\omega}\sin\omega\tau & -\frac{\kappa}{m\omega}\sin\omega\tau \\ \frac{1}{\omega}\sin\omega\tau & \cos\omega\tau - \frac{\alpha}{\omega}\sin\omega\tau \end{pmatrix}. \tag{4.3}$$

Critically damped ($\gamma^2 = 4m\kappa$, $\lambda = -\gamma/(2m)$, $N = M - \lambda I$, $N^2 = 0$):

$$e^{M\tau} = e^{\lambda\tau}(I + N\tau) = e^{-\frac{\gamma\tau}{2m}} \begin{pmatrix} 1 + \frac{\gamma\tau}{2m} & -\frac{\kappa\tau}{m} \\ \tau & 1 - \frac{\gamma\tau}{2m} \end{pmatrix}. \tag{4.4}$$

Lemma 4.6 ($N^2 = 0$ in the Critical Case)**.** *When* $\gamma^2 = 4m\kappa$*, the nilpotent part* $N = M + \frac{\gamma}{2m}I$ *satisfies* $N^2 = 0$*, so the matrix exponential truncates at first order.* ***Proof.*** $(N^2)_{11} = (\gamma/(2m))^2 - \kappa/m = \gamma^2/(4m^2) - \kappa/m = 0$ *by* $\gamma^2 = 4m\kappa$*. The remaining entries* *vanish by similar calculation.*

The algorithm below summarises the complete computation.

```
Algorithm: Langevin-Zamrik ZCB Price (OU Potential)

Input:  r0, v0          initial rate and velocity
        m               inertial mass  (> 0)
        gamma           friction coefficient  (> 0)
        kappa           spring constant  (> 0)
        theta           long-run mean rate
        sigma           noise amplitude
        tau             time to maturity

Step 1: Compute Delta = (gamma/m)^2 - 4*kappa/m

Step 2: Compute exp(M * tau):
        if Delta > 0  (overdamped):
            lambda1 = (-gamma/m + sqrt(Delta)) / 2
            lambda2 = (-gamma/m - sqrt(Delta)) / 2
            Use two-real-eigenvalue formula above
        if Delta = 0  (critically damped):
            lam = -gamma / (2*m)
            Use Jordan block formula:  exp(lam*tau) * (I + N*tau)
        if Delta < 0  (underdamped):
            alpha = gamma / (2*m)
            omega = sqrt(-Delta) / 2
            Use complex-exponential formula above

Step 3: [B1, B2] = (expMtau - I) @ [gamma/kappa, m/kappa]

Step 4: Compute A(tau) by numerical quadrature:
        A = integral from 0 to tau of
            [ kappa*theta/m * B2(s) + sigma^2 / (2*m^2) * B2(s)^2 ] ds

Step 5: Return  P = exp( A + B1*r0 + B2*v0 )
```

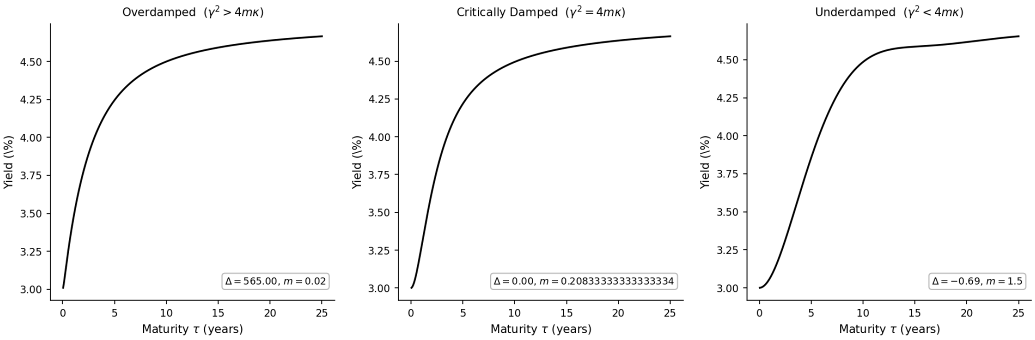

Figure 1: Yield curves under the three damping regimes.

4.6 Yield Curve Shape Richness

Figure 1 reveals a structural asymmetry: the overdamped panel is smooth and monotone, while the underdamped panel exhibits non-monotone curvature. This section makes the comparison precise. The Vasicek model, being one-dimensional, produces only monotone yield curves. The Langevin-Zamrik model, with its two-dimensional state (r_t, v_t), produces a strictly richer family of shapes — controlled independently by the initial rate level r_0 and the initial velocity v_0.

Proposition 4.7 (Short-End Slope)**.** *The yield curve satisfies* $y(0^+;\, r_0, v_0) = r_0$ *and*

$$\left.\frac{\partial y}{\partial \tau}\right|_{\tau=0^+} = \frac{v_0}{2}.$$

In particular, the sign of v_0 *alone determines whether the short end slopes upward or downward, independently of* r_0 *and the mean-reversion parameters.* ***Proof.*** *Taylor-expand the bond price coefficients near* $\tau = 0$*. From the ODE system:* $B_1'(0) = -1$*,* $B_2'(0) = 0$*,* $B_2''(0) = -1$*,* $A''(0) = 0$*. Hence*

$$A(\tau) + B_1(\tau) r_0 + B_2(\tau) v_0 = -r_0 \tau - \frac{v_0}{2}\tau^2 + O(\tau^3).$$

Dividing by $-\tau$ *gives* $y(\tau) = r_0 + \frac{v_0}{2}\tau + O(\tau^2)$*, and the slope follows.*

Remark 4.8 (Comparison to Vasicek). In the Vasicek model the short-end slope is $\frac{\kappa_{\text{eff}}}{2}(\theta - r_0)$: slope and level are coupled through mean reversion. In the Langevin-Zamrik model, v_0 controls the slope independently. A central bank in a rising-rate environment ($v_0 > 0$) and one in a falling-rate environment ($v_0 < 0$) can share the same short rate r_0 yet face different yield curves. Vasicek cannot distinguish these two cases.

Proposition 4.9 (Shape Classification by Regime)**.** ***(i) Overdamped and critically damped*** *(*$\Delta \geq 0$*)**:*** *All eigenvalues of* M *are real and negative. The coefficients* $B_1(\tau)$*,* $B_2(\tau)$ *are monotone in* τ *and the yield curve* $y(\tau)$ *is asymptotically monotone. The sign of* v_0 *may produce a transient inflection near the short end but no persistent hump.* ***(ii) Underdamped*** *(*$\Delta < 0$*)**:*** *The eigenvalues are* $\lambda_{1,2} = -\alpha \pm i\omega$ *with* $\alpha, \omega > 0$*. The coefficients* $B_1(\tau)$*,* $B_2(\tau)$ *contain the damped oscillatory terms* $e^{-\alpha\tau}\cos(\omega\tau)$ *and* $e^{-\alpha\tau}\sin(\omega\tau)$*. The yield curve can be monotone normal, humped, dipped, or S-shaped depending on* (r_0, v_0)*. The hump maturity* τ^* *satisfies* $\omega\tau^* \approx \pi/2$*, controlled by the oscillation frequency* $\omega = \sqrt{4m\kappa - \gamma^2}/(2m)$*.* ***Proof.*** *In case (i), both exponential components in the matrix exponential are real and decaying; the resulting yield is a weighted sum of decaying exponentials and approaches* y_∞ *monotonically for large* τ *(Proposition 5.1 of Section 6). In case (ii), the imaginary part of the eigenvalue introduces oscillatory modulation of amplitude* $e^{-\alpha\tau}$*; the yield inherits these oscillations before they are damped. Setting* $\partial y/\partial\tau = 0$ *and using Proposition 3.6, the short-end slope* $v_0/2$ *and the oscillatory structure together determine the full shape.*

Figure 4 illustrates Proposition 3.7(ii) directly: fixing $r_0 = 3\%$ and varying $v_0 \in [-0.02, +0.02\%]$ in the underdamped regime produces the full spectrum of shapes — from inverted-withtrough ($v_0 < 0$) to normal-with-hump ($v_0 > 0$) — that Vasicek cannot replicate with any parameter choice.

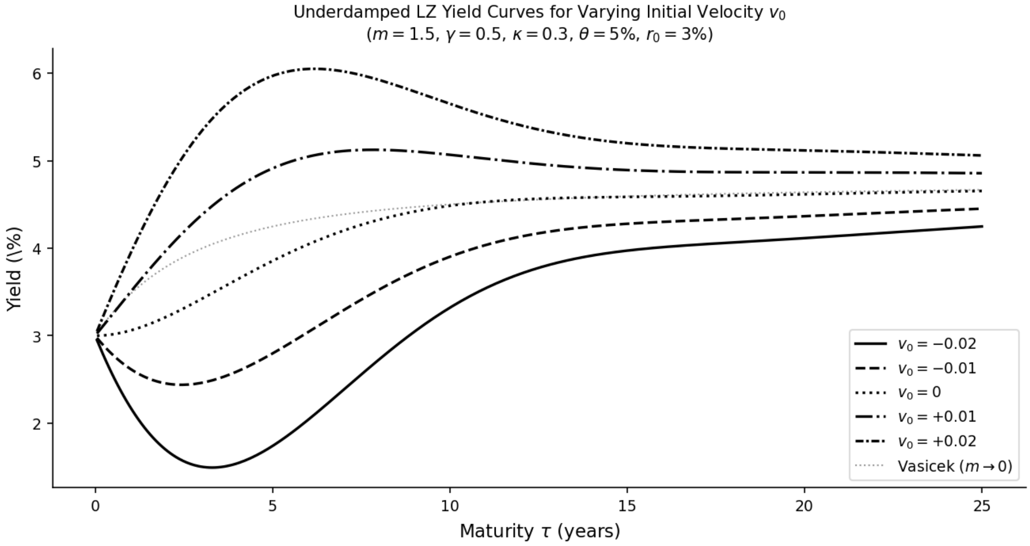

Figure 2: Underdamped Langevin-Zamrik yield curves for varying initial velocity v_0, with fixed $r_0 = 3\%$. The Vasicek limit ($m \to 0$) is shown dotted for reference.

5. The Inertia-Free Limit

5.1 Statement

Theorem 5.1 (First-Order Reduction as $m \to 0$)**.** *Let* $\kappa_{\text{eff}} = \kappa/\gamma$ *and* $\sigma_{\text{eff}} = \sigma/\gamma$*. As* $m \to 0^+$ *with* $\gamma, \kappa, \sigma, \theta$ *fixed, the Langevin-Zamrik bond price converges pointwise to the Vasicek bond price:*

$$P^{\text{LZ}}(\tau;\, r_0, v_0, m) \;\longrightarrow\; P^{\text{V}}(\tau;\, r_0) = \exp\bigl\{A^V(\tau) - B^V(\tau)\, r_0\bigr\},$$

where

$$B^V(\tau) = \frac{1 - e^{-\kappa_{\text{eff}}\tau}}{\kappa_{\text{eff}}}, \qquad A^V(\tau) = \left(\theta - \frac{\sigma_{\text{eff}}^2}{2\kappa_{\text{eff}}^2}\right)\left(B^V(\tau) - \tau\right) - \frac{\sigma_{\text{eff}}^2 B^V(\tau)^2}{4\kappa_{\text{eff}}}.$$

5.2 Proof

Lemma 5.2 (Eigenvalue Asymptotics)**.** *As* $m \to 0^+$*:*

$$\lambda_2 \;\to\; -\frac{\kappa}{\gamma} = -\kappa_{\text{eff}}, \qquad \lambda_1 \;\to\; -\infty.$$

Proof. *From* $\lambda_{1,2} = \frac{-\gamma/m \pm \sqrt{(\gamma/m)^2 - 4\kappa/m}}{2}$*:*

$$\lambda_2 = \frac{-\gamma/m + (\gamma/m)\sqrt{1 - 4m\kappa/\gamma^2}}{2} \approx \frac{-\gamma/m + \gamma/m(1 - 2m\kappa/\gamma^2)}{2} = -\frac{\kappa}{\gamma}$$

as $m \to 0$. *Since* $\lambda_1 + \lambda_2 = -\gamma/m \to -\infty$ *and* λ_2 *is bounded,* $\lambda_1 \to -\infty$.

Lemma 5.3 (Coefficient Asymptotics)**.** *As* $m \to 0^+$, *with* $\mu_i = -\lambda_i > 0$*:*

$$B_1(\tau) \;\to\; -B^V(\tau) = -\frac{1 - e^{-\kappa_{\mathrm{eff}}\tau}}{\kappa_{\mathrm{eff}}}, \qquad B_2(\tau) \;\to\; 0.$$

Proof. *In the overdamped formula (which applies for small enough* m*):*

$$B_1(\tau) = \frac{1}{\lambda_1 - \lambda_2}\left[\frac{\gamma}{\kappa}(\lambda_1 e^{\lambda_2\tau} - \lambda_2 e^{\lambda_1\tau}) + e^{\lambda_2\tau} - e^{\lambda_1\tau}\right] - \frac{\gamma}{\kappa}.$$

As $\lambda_1 \to -\infty$, $e^{\lambda_1\tau} \to 0$ *for* $\tau > 0$. *Since* $\lambda_1 - \lambda_2 \approx \lambda_1 \to -\infty$*:*

$$B_1(\tau) \to \frac{\gamma\lambda_2/\kappa + 1}{\lambda_2 - \lambda_1} \cdot (e^{\lambda_2\tau} - 1) - \frac{\gamma}{\kappa} = \frac{e^{\lambda_2\tau} - 1}{\lambda_2} = \frac{e^{-\kappa_{\mathrm{eff}}\tau} - 1}{-\kappa_{\mathrm{eff}}} = -B^V(\tau).$$

For B_2*: similarly* $B_2(\tau) = \frac{e^{\lambda_1\tau} - e^{\lambda_2\tau}}{\lambda_1 - \lambda_2} \cdot \frac{m}{\kappa} + \ldots \to 0$ *since each term is* $O(m)$.

> ***Proof of Theorem 4.1.*** *By Lemma 4.2,* $B_1 \to -B^V$ *and* $B_2 \to 0$. *Since* $B_2 \to 0$, *the integrand in* $A(\tau)$ *satisfies* $\frac{\kappa\theta}{m}B_2 + \frac{\sigma^2}{2m^2}B_2^2 \to A^{V\prime}(\tau)$ *(verified by direct substitution of the Vasicek limit into the ODE), giving* $A(\tau) \to A^V(\tau)$. *Therefore*
>
> $$P^{\mathrm{LZ}} = \exp\{A + B_1 r_0 + B_2 v_0\} \to \exp\{A^V - B^V r_0\} = P^V.$$

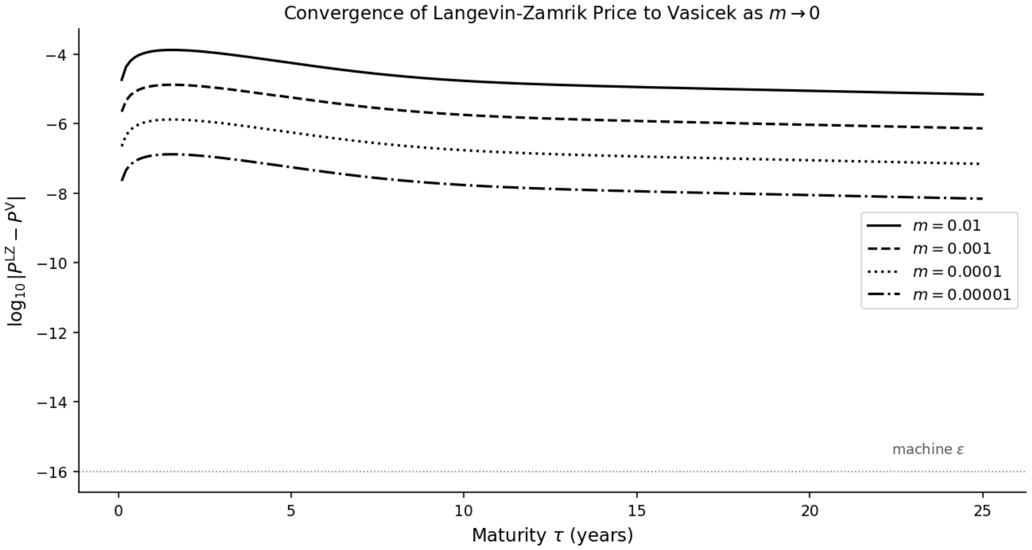

Figure 3: Convergence to Vasicek — log base-10 of absolute difference between Langevin-Zamrik and Vasicek ZCB prices.

6. Characteristic Functions and the Gaussian Structure

The affine bond pricing formula of Section 3 is the moment generating function of the integrated rate, evaluated at a specific complex argument. Making this identification explicit yields the full characteristic function of the Langevin-Zamrik model, which governs the distribution of the integrated rate and the state vector, and provides the analytical foundation for Fourier-based derivative pricing.

6.1 The Bond Price as a Moment Generating Function

Proposition 6.1 (Bond Price as MGF)**.** *Let* $I_\tau = \int_0^\tau r_s\,ds$ *denote the integrated rate under* $\mathbb{Q}$*. Then*

$$P(\tau;\, r_0, v_0) = \mathbb{E}^{\mathbb{Q}}[e^{-I_\tau} \,|\, r_0, v_0] = M_{I_\tau}(-1),$$

where $M_{I_\tau}(u) = \mathbb{E}[e^{uI_\tau} \,|\, r_0, v_0]$ *is the moment generating function of* I_τ*. Equivalently,* $P = \phi_{I_\tau}(i)$*, where* $\phi_{I_\tau}(\xi) = \mathbb{E}[e^{i\xi I_\tau}]$ *is the characteristic function.* ***Proof.*** *Directly from the definitions:* $M_{I_\tau}(-1) = \mathbb{E}[e^{-I_\tau}] = P$*. Since* $\phi(\xi) = M(i\xi)$*, one has* $\phi(i) = M(-1) = P$*.*

Remark 6.2. The closed-form formula $P = \exp\{A(\tau) + B_1(\tau)r_0 + B_2(\tau)v_0\}$ is the cumulant generating function of I_τ evaluated at $u = -1$. The ODE system of Theorem 3.1 computes this cumulant generating function along the real axis at $u = -1$; Theorem 5.1 below extends it to the full complex plane.

6.2 Characteristic Function of the Integrated Rate

Theorem 6.3 (CF of the Integrated Rate)**.** *Define the decomposition* $A(\tau) = \mathcal{A}_1(\tau) + \mathcal{A}_2(\tau)$*, where*

$$\mathcal{A}_1(\tau) = \int_0^\tau \frac{\kappa\theta}{m} B_2(s)\,ds, \qquad \mathcal{A}_2(\tau) = \int_0^\tau \frac{\sigma^2}{2m^2} B_2(s)^2\,ds.$$

For $\xi \in \mathbb{R}$*, the characteristic function* $\phi_{I_\tau}(\xi;\, r_0, v_0) = \mathbb{E}^{\mathbb{Q}}[e^{i\xi I_\tau} \,|\, r_0, v_0]$ *is*

$$\phi_{I_\tau}(\xi;\, r_0, v_0) = \exp\Big\{A^\phi(\xi,\tau) + B_1^\phi(\xi,\tau)\,r_0 + B_2^\phi(\xi,\tau)\,v_0\Big\},$$

with

$$\begin{pmatrix} B_1^\phi(\xi,\tau) \\ B_2^\phi(\xi,\tau) \end{pmatrix} = -i\xi \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix},$$

$$A^\phi(\xi,\tau) = -i\xi\,\mathcal{A}_1(\tau) - \xi^2\,\mathcal{A}_2(\tau).$$

Proof. *Let* $F(\tau; r, v, \xi) = \mathbb{E}[e^{i\xi\int_0^\tau r_s\,ds} \,|\, r_0 = r, v_0 = v]$*. The Feynman-Kac theorem for the running exponential functional gives*

$$F_\tau = v\,F_r - \frac{\gamma v + \kappa(r-\theta)}{m}\,F_v + \frac{\sigma^2}{2m^2}\,F_{vv} + i\xi r\,F, \quad F(0; r, v, \xi) = 1.$$

Substituting the affine ansatz $F = e^{A^\phi + B_1^\phi r + B_2^\phi v}$ *and separating by powers of* r *and* v*:* ***Coefficient of*** r***:*** $B_1^{\phi\prime} = -\frac{\kappa}{m}B_2^\phi + i\xi$*.* ***Coefficient of*** v***:*** $B_2^{\phi\prime} = B_1^\phi - \frac{\gamma}{m}B_2^\phi$*.* ***Constant:*** $A^{\phi\prime} = \frac{\kappa\theta}{m}B_2^\phi + \frac{\sigma^2}{2m^2}(B_2^\phi)^2$*. The* $\mathbf{B}^\phi$*-ODE is* $\mathbf{B}^{\phi\prime} = M\mathbf{B}^\phi + (i\xi, 0)^\top$ *with* $\mathbf{B}^\phi(0) = \mathbf{0}$*. Since* $(i\xi, 0)^\top = -i\xi\,(-1, 0)^\top$*, linearity gives* $\mathbf{B}^\phi(\xi,\tau) = -i\xi\,\mathbf{B}(\tau)$*. Substituting into the* A^ϕ*-*

equation:

$$A^{\phi\prime} = \frac{\kappa\theta}{m}(-i\xi B_2) + \frac{\sigma^2}{2m^2}(-i\xi)^2 B_2^2 = -i\xi\frac{\kappa\theta}{m}B_2 - \xi^2\frac{\sigma^2}{2m^2}B_2^2.$$

Integrating from 0 *to* τ *gives the stated formula.*

Corollary 6.4 (Gaussian Integrated Rate)**.** I_τ *is conditionally Gaussian given* (r_0, v_0)*, with*

$$\mathbb{E}^{\mathbb{Q}}[I_\tau \,|\, r_0, v_0] = -\mathcal{A}_1(\tau) - B_1(\tau)\, r_0 - B_2(\tau)\, v_0,$$

$$\mathrm{Var}^{\mathbb{Q}}(I_\tau \,|\, r_0, v_0) = 2\mathcal{A}_2(\tau) = \int_0^\tau \frac{\sigma^2}{m^2}\, B_2(s)^2\, ds.$$

The bond price satisfies $P = \exp\{-\mathbb{E}[I_\tau] + \frac{1}{2}\mathrm{Var}(I_\tau)\}$*, the standard MGF identity for a Gaussian.* ***Proof.*** *Matching* $\phi_{I_\tau}(\xi) = e^{i\xi\mu - \frac{1}{2}\xi^2\sigma^2}$ *to Theorem 5.1 gives* $\mu = -\mathcal{A}_1 - B_1 r_0 - B_2 v_0$ *and* $\sigma^2 = 2\mathcal{A}_2$*. Since* $B_1(\tau) < 0$ *(Lemma 4.2), a higher* r_0 *increases* $\mathbb{E}[I_\tau]$*, consistent with higher rates producing larger integrated discounting. The MGF identity follows from* $P = M_{I_\tau}(-1) = e^{-\mu + \frac{1}{2}\sigma^2}$*.*

6.3 Joint Characteristic Function of the State Vector

Theorem 6.5 (Bivariate Gaussian State)**.** *The joint distribution of* (r_τ, v_τ) *conditional on* (r_0, v_0) *is bivariate Gaussian with mean vector*

$$\mu_\tau = e^{M\tau}\begin{pmatrix} r_0 \\ v_0 \end{pmatrix} + M^{-1}(e^{M\tau} - I)\begin{pmatrix} 0 \\ \kappa\theta/m \end{pmatrix}$$

and covariance matrix

$$\Sigma_\tau = \frac{\sigma^2}{m^2}\int_0^\tau e^{Ms}\,\mathbf{e}_2\mathbf{e}_2^\top\, e^{M^\top s}\, ds, \qquad \mathbf{e}_2 = (0,1)^\top.$$

The joint characteristic function is

$$\phi_{r_\tau, v_\tau}(\xi_1, \xi_2) = \exp\!\left(i\xi^\top\mu_\tau - \tfrac{1}{2}\xi^\top\Sigma_\tau\xi\right), \qquad \xi = (\xi_1, \xi_2)^\top.$$

Proof. *The SDE* $d\mathbf{X}_t = (M\mathbf{X}_t + b)\,dt + \frac{\sigma}{m}\mathbf{e}_2\, dW_t$ *is linear with* $b = (0, \kappa\theta/m)^\top$*. Its solution*

$$\mathbf{X}_\tau = e^{M\tau}\mathbf{X}_0 + \int_0^\tau e^{M(\tau-s)} b\, ds + \frac{\sigma}{m}\int_0^\tau e^{M(\tau-s)}\mathbf{e}_2\, dW_s$$

is Gaussian as a linear transformation of Brownian motion. The mean follows from the deterministic integral; the covariance from the Itô isometry. The joint CF is the standard Gaussian formula.

Proposition 6.6 (Lyapunov Equation and Stationary Covariance)**.** *The covariance* Σ_τ *satisfies*

$$\dot{\Sigma}_\tau = M\Sigma_\tau + \Sigma_\tau M^\top + Q, \qquad \Sigma_0 = 0, \qquad Q = \frac{\sigma^2}{m^2}\,\mathbf{e}_2\mathbf{e}_2^\top.$$

The stationary covariance $\Sigma_\infty = \lim_{\tau\to\infty}\Sigma_\tau$*, which exists since* $\mathrm{Re}(\lambda_{1,2}) < 0$*, is the unique* *solution of* $M\Sigma_\infty + \Sigma_\infty M^\top + Q = 0$*:*

$$\Sigma_\infty = \frac{\sigma^2}{2\gamma m}\begin{pmatrix} \kappa/m & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. *Differentiating* Σ_τ *gives the stated ODE. Setting* $\dot{\Sigma}_\infty = 0$ *and writing* $\Sigma_\infty = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$*: entry* $(1,1)$ *gives* $-2\kappa b/m = 0$*, so* $b = 0$*; entry* $(2,2)$ *gives* $-2\gamma c/m = -\sigma^2/m^2$*, so* $c = \sigma^2/(2\gamma m)$*; entry* $(1,2)$ *gives* $a = \kappa c/m = \kappa\sigma^2/(2\gamma m^2)$*.*

6.4 Applications to Derivative Pricing

Remark 6.7 (Caplet Pricing). A caplet paying $(r_T - K)^+$ at time T requires the marginal distribution of r_T. By Theorem 5.2, $r_T \,|\, (r_0, v_0) \sim \mathcal{N}(\mu_r(\tau), \Sigma_{\tau,11})$. Caplet prices are therefore given by a Bachelier normal formula with the LZ mean and variance, and converge to the Vasicek caplet formula [8] as $m \to 0$.

Remark 6.8 (Fourier Pricing of General Payoffs). For payoffs $g(r_T)$ that are not simple calls, the price is

$$\mathbb{E}^{\mathbb{Q}}[e^{-I_\tau} g(r_T)] = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\, \phi_{r_T, I_\tau}(-\xi, i)\, d\xi,$$

where $\hat{g}$ is the Fourier transform of g and ϕ_{r_T,I_τ} is the joint CF of (r_T, I_τ). This joint CF is that of a trivariate Gaussian (r_τ, v_τ, I_τ), computable by augmenting the state with $dI_t = r_t\, dt$ and applying Theorem 5.2 to the 3 × 3 system.

Remark 6.9 (Lévy Extension). In the jump-diffusion model of Section 8.4, the velocity jump of size ξ'/m shifts the state and modifies the CF of I_τ. The A^ϕ-term picks up an additional Lévy-Khintchine integral, while the $\mathbf{B}^\phi$-coefficients remain $-i\xi\mathbf{B}(\tau)$ since the jump affects only the v-component linearly.

7. Three Damping Regimes and Yield Curve Shapes

The yield to maturity for a zero-coupon bond is defined by

$$y(\tau;\, r_0, v_0) = -\frac{\log P(\tau;\, r_0, v_0)}{\tau} = -\frac{A(\tau) + B_1(\tau)\, r_0 + B_2(\tau)\, v_0}{\tau}. \tag{7.1}$$

Proposition 7.1 (Long-Rate Limit). *As* $\tau \to \infty$*:*

$$y(\tau;\, r_0, v_0) \;\to\; y_\infty = -\lim_{\tau\to\infty} \frac{A(\tau)}{\tau},$$

which is finite in the overdamped and critically damped regimes (since $B_1, B_2 \to$ *constants and* $A(\tau)$ *grows at most linearly), and may be finite or oscillatory in the underdamped regime.* ***Proof.*** *Since* $\mathrm{Re}(\lambda_{1,2}) < 0$*, the exponential terms in* $B_1(\tau)$ *and* $B_2(\tau)$ *decay to zero as* $\tau \to \infty$*, so* $B_i(\tau)/\tau \to 0$*. The long rate is therefore determined entirely by* $A(\tau)$*.*

Proposition 7.2 (Yield Curve Shapes by Regime). *The short-end slope* $\partial y/\partial\tau|_{\tau=0^+}$ *and global shape are determined as follows:* ***(i) Overdamped*** *(*$\Delta > 0$*)****:*** *The yield curve* *is smooth and monotone. It is normal (upward-sloping) when* $r_0 < y_\infty$ *and inverted (downward-sloping) when* $r_0 > y_\infty$*. No humps are possible.* ***(ii) Critically damped*** ***(***$\Delta = 0$***):*** *The transition case; the yield curve can exhibit a single inflection point and a mild hump structure.* ***(iii) Underdamped*** ***(***$\Delta < 0$***):*** *The yield curve is oscillatory due to the complex eigenvalues* $-\alpha \pm i\omega$*. Multiple humps are possible, generating term structures that are empirically observed but cannot arise in the Vasicek model.* ***Proof.*** *The shape is determined by the sign of* $y''(\tau)$*, which is controlled by the second derivatives of* $B_1(\tau)$ *and* $A(\tau)$*. In regime (i), all terms in* $B_1(\tau)$ *are sums of real decaying exponentials, so* $y(\tau)$ *is a sum of Bernstein functions — monotone. In regime (iii), the complex exponentials produce sinusoidal oscillations in* $y(\tau)$*.*

8. The Nelson-Siegel-Svensson Connection

8.1 The Nelson-Siegel-Svensson Model

The Nelson-Siegel model [5] parameterises the yield curve as

$$y^{NS}(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\mu\tau}}{\mu\tau} + \beta_2\left(\frac{1 - e^{-\mu\tau}}{\mu\tau} - e^{-\mu\tau}\right) \tag{8.1}$$

with decay rate $\mu > 0$. The Svensson extension [8] adds a second hump term:

$$y^{NSS}(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} + \beta_2\left(\frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} - e^{-\mu_1\tau}\right) + \beta_3\left(\frac{1 - e^{-\mu_2\tau}}{\mu_2\tau} - e^{-\mu_2\tau}\right). \tag{8.2}$$

These models are widely used by central banks for yield curve estimation [2] but are specified purely empirically, with no no-arbitrage justification.

8.2 Structural Derivation

Theorem 8.1 (Nelson-Siegel-Svensson as Overdamped Langevin-Zamrik)**.** *In the overdamped regime (*$\gamma^2 > 4m\kappa$*) with eigenvalues* $\lambda_i = -\mu_i$*,* $\mu_i > 0$*, the Langevin-Zamrik yield curve admits the representation*

$$y(\tau; r_0, v_0) = y_\infty + c_1(r_0, v_0)\, \frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} + c_2(r_0, v_0)\, \frac{1 - e^{-\mu_2\tau}}{\mu_2\tau} + g(\tau)$$

where c_1, c_2 *are linear functions of* (r_0, v_0)*,* y_∞ *is the long-rate, and* $g(\tau)$ *consists of terms of the form* $\frac{e^{-\mu_i\tau}}{\tau}$ *that are* $O(\tau^{-1})$ *as* $\tau \to \infty$*. This is the Nelson-Siegel-Svensson structure with two decay rates* μ_1, μ_2*.* ***Proof.*** *From Theorem 3.2 and the overdamped matrix exponential:*

$$B_1(\tau) = \frac{\gamma}{\kappa(\mu_2 - \mu_1)}\left[\mu_2(e^{-\mu_1\tau} - 1) - \mu_1(e^{-\mu_2\tau} - 1)\right] + \frac{e^{-\mu_2\tau} - e^{-\mu_1\tau}}{\mu_2 - \mu_1}.$$

Dividing by τ:

$$-\frac{B_1(\tau)\,r_0}{\tau} = r_0 \cdot \left[\frac{\gamma\mu_2}{\kappa(\mu_2-\mu_1)} \cdot \frac{1-e^{-\mu_1\tau}}{\mu_1\tau} - \frac{\gamma\mu_1}{\kappa(\mu_2-\mu_1)} \cdot \frac{1-e^{-\mu_2\tau}}{\mu_2\tau} + \frac{e^{-\mu_1\tau}-e^{-\mu_2\tau}}{\tau(\mu_2-\mu_1)}\right].$$

The first two terms are precisely the Nelson-Siegel-Svensson slope basis functions $(1-e^{-\mu_i\tau})/(\mu_i\tau)$. The last term is the $g(\tau)$ remainder. A similar decomposition applies to $-B_2(\tau)v_0/\tau$ and $-A(\tau)/\tau$. The long-rate constant $y_\infty = -\lim_{\tau\to\infty} A(\tau)/\tau$ collects all constant contributions.

Corollary 8.2 (Vasicek as One-Factor Nelson-Siegel)**.** *In the Vasicek limit $m \to 0$, $\mu_1 \to \infty$ and the first factor vanishes, leaving a single-factor Nelson-Siegel yield curve with decay rate $\mu_2 = \kappa_{\text{eff}} = \kappa/\gamma$:*

$$y^V(\tau;\, r_0) = y_\infty^V + (r_0 - y_\infty^V)\,\frac{1-e^{-\kappa_{\text{eff}}\tau}}{\kappa_{\text{eff}}\tau}.$$

The Vasicek yield curve is the one-factor Nelson-Siegel model.

Corollary 8.3 (Parameter Identification)**.** *The Nelson-Siegel-Svensson parameters are identified from the physical parameters as: - **Decay rates:** $\mu_i = -\lambda_i$, determined by γ, κ, m. - **Level** β_0**:** equals the long rate y_∞, determined by θ, σ, κ_{eff}. - **Slope loadings** β_1, β_2**:** linear functions of r_0 (level) and v_0 (velocity of the rate). The velocity $v_0 = \dot{r}_0$ is the **hidden second factor** absent from Vasicek.*

Remark 8.4 (Economic Interpretation of v_0). The initial velocity v_0 captures the current direction of rate movement. A rising rate environment ($v_0 > 0$) shifts the loading on one Nelson-Siegel factor; a falling environment ($v_0 < 0$) shifts it in the opposite direction. This provides a structural justification for the empirical observation that yield curve slope contains information beyond the level of the short rate.

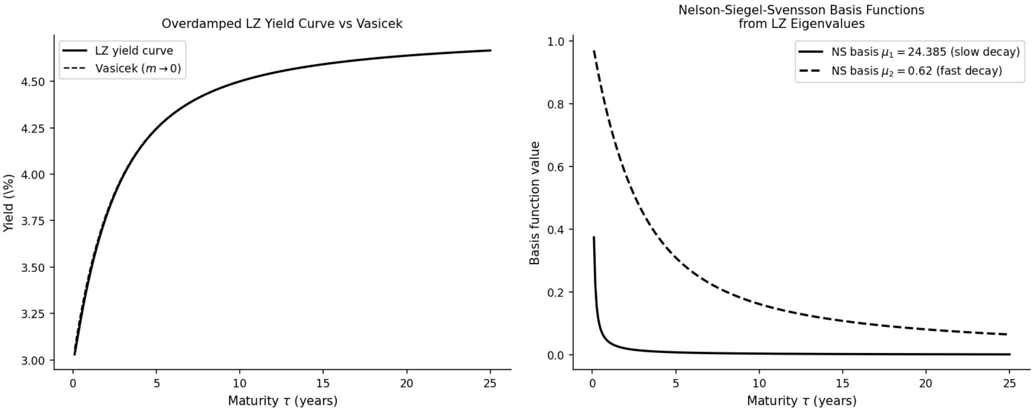

Figure 4: Nelson-Siegel basis identification in the overdamped Langevin-Zamrik model.

9. Extensions

9.1 Nonlinear Potentials and the Aït-Sahalia Family

For a general nonlinear potential $U(r)$, the Langevin-Zamrik PDE

$$\partial_t P + v\,\partial_r P - \frac{\gamma v + \partial_r U(r)}{m}\,\partial_v P + \frac{\sigma^2}{2m^2}\,\partial_v^2 P = rP \tag{9.1}$$

does not admit an affine solution. The equation must be solved numerically on a grid in (r, v, τ). By Proposition 2.2, any nonlinear U' breaks the affine structure.

Proposition 9.1 (Aït-Sahalia Inertial Extension)**.** *The Aït-Sahalia [1] family of nonlinear drift short-rate models, with drift $\mu(r) = \alpha_{-1}/r + \alpha_0 + \alpha_1 r + \alpha_2 r^2$, has a canonical inertial extension via the Langevin-Zamrik PDE with potential $\partial_r U(r) = -\gamma\mu(r)$. The overdamped limit $m \to 0$ recovers the original Aït-Sahalia model. The full inertial equation requires numerical solution of a degenerate parabolic PDE on $\mathbb{R}_+ \times \mathbb{R}$.*

Numerical approach. Discretise (r, v) on a bounded grid with absorbing boundary at $r = 0$. Use an implicit finite-difference scheme in v (elliptic direction) and upwind differencing in r (hyperbolic direction). The CFL condition applies to the r-direction only.

9.2 State-Dependent Volatility: The Inertial CIR Model

Replace the additive noise in Definition 2.1 with state-dependent noise $\sigma(r)\,\dot{W}_t$, yielding

$$dv_t = \frac{1}{m}[-\gamma v_t - \kappa(r_t - \theta)]\,dt + \frac{\sigma\sqrt{r_t}}{m}\,dW_t. \tag{9.2}$$

The Langevin-Zamrik PDE becomes

$$\partial_t P + v\,\partial_r P - \frac{\gamma v + \kappa(r - \theta)}{m}\,\partial_v P + \frac{\sigma^2 r}{2m^2}\,\partial_v^2 P = rP. \tag{9.3}$$

Proposition 9.2 (Loss of Affine Structure)**.** *The inertial CIR Langevin-Zamrik PDE does not admit an affine solution. The diffusion coefficient $\sigma^2 r/(2m^2)$ is a function of r, breaking the separation of variables required by the affine ansatz.*

The overdamped limit $m \to 0$ recovers the standard CIR model [3] with $\kappa_{\text{eff}} = \kappa/\gamma$ and $\sigma_{\text{eff}} = \sigma/\gamma$.

9.3 Multi-Factor Langevin Models

Consider n short-rate factors $\mathbf{r} = (r_1, \dots, r_n)^\top$, each with its own velocity $\mathbf{v} = (v_1, \dots, v_n)^\top$, governed by a coupled linear system:

$$d\mathbf{r} = \mathbf{v}\,dt, \qquad d\mathbf{v} = \frac{1}{m}[-\Gamma\mathbf{v} - K(\mathbf{r} - \theta)]\,dt + \frac{1}{m}\Sigma\,d\mathbf{W}_t, \tag{9.4}$$

where Γ, K are $n \times n$ friction and spring matrices and Σ is the noise matrix. The short rate is $r_t = \mathbf{e}^\top \mathbf{r}_t$ for some loading vector $\mathbf{e}$.

Proposition 9.3. *The multi-factor Langevin-Zamrik PDE in state $(\mathbf{r}, \mathbf{v}) \in \mathbb{R}^{2n}$ admits an affine solution $P = \exp\{A(\tau) + \mathbf{B}_r(\tau)^\top \mathbf{r} + \mathbf{B}_v(\tau)^\top \mathbf{v}\}$ when K is positive definite. The* *coefficient vectors satisfy a $2n \times 2n$ linear ODE system with IFS matrix*

$$\mathcal{M} = \begin{pmatrix} \mathbf{0} & -K/m \\ I & -\Gamma/m \end{pmatrix}.$$

The Vasicek recovery theorem extends: as $m \to 0$, $\mathbf{B}_v \to \mathbf{0}$ and the n-factor Vasicek model [6] is recovered.

9.4 Lévy Noise: The PIDE Extension

Replace the Brownian noise in Definition 2.1 with a Lévy process L_t having Lévy measure ν. The velocity SDE becomes

$$dv_t = \frac{1}{m}[-\gamma v_t - \partial_r U(r_t)]\, dt + \frac{\sigma}{m}\, dW_t + \frac{1}{m}\, dJ_t, \tag{9.5}$$

where $J_t = \int_0^t \int_{\mathbb{R}} \xi\, \tilde{N}(ds, d\xi)$ is the pure-jump component.

Proposition 9.4 (Langevin-Zamrik PIDE)**.** *The bond price satisfies the partial integro-differential equation*

$$\begin{aligned} \partial_t P + v\, \partial_r P - \frac{\gamma v + \partial_r U(r)}{m}\, \partial_v P + \frac{\sigma^2}{2m^2}\, \partial_v^2 P \\ + \int_{\mathbb{R}} \Big[P\Big(t, T; r, v + \frac{\xi}{m}\Big) - P - \frac{\xi}{m}\, \partial_v P \Big]\, \nu(d\xi) = rP. \end{aligned} \tag{8.6}$$

The integral term captures the effect of velocity jumps of size ξ/m on bond prices. In the overdamped limit, the jumps pass through to the rate dynamics and the standard jump-diffusion bond pricing PIDE [3] is recovered.

10. Conclusion

We have introduced the Langevin-Zamrik PDE, a bond pricing equation derived from the full underdamped Langevin dynamics of the short rate. The main contributions are:

1. **A new PDE for bond pricing** in the two-dimensional state (r, v), related to but distinct from the Kramers equation by adjoint structure and the substitution of friction by discounting.
2. **A closed-form affine solution** for the quadratic potential, with coefficients given by the matrix exponential of the inertia-friction-spring matrix M. Three damping regimes produce qualitatively distinct yield curve shapes.
3. **Recovery of Vasicek** in the overdamped limit $m \to 0$, proved rigorously via eigenvalue asymptotics and coefficient limits.
4. **A structural derivation of the Nelson-Siegel-Svensson model** from no-arbitrage principles: in the overdamped regime, the Langevin-Zamrik yield curve is the two-

factor Nelson-Siegel-Svensson specification, with the two eigenvalues of M as the decay rates and the initial velocity v_0 as the hidden slope factor.

5. **A framework for extensions**: nonlinear potentials (Aït-Sahalia family), state-dependent volatility (inertial CIR), multi-factor systems, and Lévy noise (PIDE) are all natural generalisations of the Langevin-Zamrik PDE.

6. **A foundation for inertial forward-rate modelling**: since the LZ model belongs to the short-rate class, the forward curve is a derived quantity — affine in (r, v) with a damped-oscillatory HJM volatility structure $\sigma^{\mathrm{HJM}}(\tau) = -(\sigma/m)B_2'(\tau)$ that is new to the literature. Lifting the inertial structure to the full forward curve in the Heath-Jarrow-Morton [10] spirit — a second-order SPDE for the forward rate in which the entire curve carries momentum — is reserved for future work.

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