

Bond Pricing under Inertial Interest Rate Dynamics: The Langevin-Zamrik PDE

Affine Bond Pricing in an Underdamped Interest Rate System

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1. Abstract

We introduce the Langevin-Zamrik partial differential equation, a zero-coupon bond pricing equation derived from the full underdamped Langevin dynamics of the short interest rate. The model belongs to the short-rate modelling tradition — alongside Vasicek and Cox-Ingersoll-Ross — but differs from all existing members of that class in that the short rate obeys a second-order SDE, with the two-dimensional state (r_t, v_t) comprising rate and velocity. For a quadratic (OU-type) potential the equation admits a closed-form affine solution $P = \exp\{A(\tau) + B_1(\tau)r + B_2(\tau)v\}$, where the coefficient pair (B_1, B_2) satisfies a 2×2 linear ODE system solved explicitly via the matrix exponential of the inertia-friction-spring matrix M . The three damping regimes — overdamped, critically damped, and underdamped — produce qualitatively distinct yield curve shapes: monotone, single-humped, and oscillatory respectively. The Vasicek formula is recovered exactly in the overdamped limit $m \rightarrow 0$, and the overdamped yield curve is structurally equivalent to the Nelson-Siegel-Svensson specification, providing its first no-arbitrage derivation. A companion forward-rate theory in the Heath-Jarrow-Morton spirit will be developed in future work.

2. Introduction

The Vasicek model [9] remains the canonical analytically tractable short-rate model. Its derivation from the overdamped Langevin equation is well known: when the friction-to-mass ratio γ/m is taken to infinity, the inertial term $m\ddot{r}$ becomes negligible and the second-order stochastic differential equation reduces to the Ornstein-Uhlenbeck first-order SDE. What has not been explored is whether the full underdamped Langevin equation — retaining the inertial term — admits a self-consistent zero-coupon bond pricing theory, whether it yields a closed-form solution, and what new economic and geometric structure emerges. The present paper operates entirely within the **short-rate modelling tradition** — alongside Vasicek [9], Cox-Ingersoll-Ross [3], and their multi-factor generalisations [6] — in which a finite-dimensional Markovian state drives all bond prices and forward rates as derived quantities. A companion forward-rate theory, in which the Langevin-Zamrik inertial structure is lifted to the full forward curve in the spirit of Heath-Jarrow-Morton [10], will be developed in future work.

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This paper answers all three questions. We introduce the Langevin-Zamrik PDE, a bond pricing equation in the two-dimensional state space (r, v) derived from the full underdamped Langevin dynamics via the Feynman-Kac theorem. For the quadratic (OU-type) potential we prove it admits an affine closed-form solution, with coefficients determined by a 2×2 ODE system whose solution is the matrix exponential of the inertia-friction-spring matrix M . The Vasicek formula is recovered in the overdamped limit. Three qualitatively distinct yield curve shapes emerge from the three damping regimes determined by the sign of the discriminant $\Delta = \gamma^2 - 4m\kappa$. Most strikingly, in the overdamped regime the Langevin-Zamrik yield curve is structurally equivalent to the Nelson-Siegel-Svensson model [5, 8] — the dominant empirical yield curve specification used by central banks worldwide — providing its first no-arbitrage derivation from physical principles.

The paper is organised as follows. Section 2 introduces the general Langevin-Zamrik PDE and establishes its relationship to the Kramers equation [4]. Section 3 specialises to the quadratic potential and derives the closed-form affine solution. Section 4 establishes the inertia-free limit, proving convergence to first-order dynamics as $m \rightarrow 0$. Section 5 derives the full characteristic function of the integrated rate and the bivariate Gaussian structure of the state vector. Section 6 analyses the three damping regimes and yield curve shapes. Section 7 establishes the Nelson-Siegel-Svensson connection. Section 8 discusses extensions to nonlinear potentials, state-dependent volatility, multi-factor systems, and Lévy noise.

3. The Langevin-Zamrik PDE

3.1 Setup and Notation

Throughout, $(\Omega, \mathcal{F}, \mathbb{Q})$ is a filtered probability space carrying a standard Brownian motion W_t . All dynamics are stated under the risk-neutral measure \mathbb{Q} . We write ∂_x for $\partial/\partial x$ and primes for $d/d\tau$.

Definition 3.1 (Underdamped Langevin Dynamics [11].) Let $U : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable. The underdamped Langevin equation for the short rate r_t is

$$m \ddot{r}_t = -\gamma \dot{r}_t - \partial_r U(r_t) + \sigma \dot{W}_t, \quad m > 0, \gamma > 0, \sigma > 0.$$

Setting $v_t = \dot{r}_t$, this is the first-order system on \mathbb{R}^2 :

$$dr_t = v_t dt,$$

$$dv_t = \frac{1}{m} [-\gamma v_t - \partial_r U(r_t)] dt + \frac{\sigma}{m} dW_t.$$

Remark 3.2. The process (r_t, v_t) is a Markov diffusion on \mathbb{R}^2 with infinitesimal generator

$$\mathcal{L} = v \partial_r - \frac{\gamma v + \partial_r U(r)}{m} \partial_v + \frac{\sigma^2}{2m^2} \partial_v^2.$$

Assumption 3.3 (Novikov Condition). The market price of risk satisfies $\mathbb{E}^{\mathbb{Q}} \left[\exp \left(\frac{1}{2} \int_0^T \lambda_t^2 dt \right) \right] <$

∞ , ensuring \mathbb{Q} is a well-defined equivalent martingale measure.

3.2 Derivation of the Langevin-Zamrik PDE

Theorem 3.4 (Langevin-Zamrik PDE). *Under Assumption 2.1, the price of a zero-coupon bond maturing at T ,*

$$P(t, T; r, v) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_s ds \right) \middle| r_t = r, v_t = v \right],$$

satisfies

$$\partial_t P + v \partial_r P - \frac{\gamma v + \partial_r U(r)}{m} \partial_v P + \frac{\sigma^2}{2m^2} \partial_v^2 P - r P = 0$$

on $[0, T) \times \mathbb{R}^2$, with terminal condition $P(T, T; r, v) = 1$. **Proof.** The Feynman-Kac theorem applied to the functional $\mathbb{E}^{\mathbb{Q}}[\exp(-\int_t^T r_s ds) | r_t, v_t]$ with killing rate r_t gives $\partial_t P + \mathcal{L}P - rP = 0$. Substituting the generator \mathcal{L} from Remark 2.1 yields the stated equation. \square

Definition 3.5 (Langevin-Zamrik PDE). We call

$$\mathcal{L}^{\text{LZ}}[P] \equiv \partial_t P + v \partial_r P - \frac{\gamma v + \partial_r U(r)}{m} \partial_v P + \frac{\sigma^2}{2m^2} \partial_v^2 P - r P = 0 \quad (3.1)$$

the **Langevin-Zamrik PDE** for potential U . The operator \mathcal{L}^{LZ} is a degenerate parabolic operator on \mathbb{R}^2 : it is elliptic in v only and first-order in r .

3.3 Relationship to the Kramers Equation

The Kramers equation [4] is the forward Kolmogorov equation for the transition density $\rho(t, r, v)$ of (r_t, v_t) :

$$\partial_t \rho + v \partial_r \rho - \frac{\gamma v + \partial_r U(r)}{m} \partial_v \rho - \frac{\gamma}{m} \rho - \frac{\sigma^2}{2m^2} \partial_v^2 \rho = 0. \quad (3.2)$$

Proposition 3.6 (Adjoint Structure). *The Langevin-Zamrik PDE uses the backward generator \mathcal{L} ; the Kramers equation uses its $L^2(\mathbb{R}^2)$ -adjoint \mathcal{L}^* . The two equations differ in exactly two terms: (i) **Diffusion sign:** Kramers has $-\frac{\sigma^2}{2m^2} \partial_v^2 \rho$; the Langevin-Zamrik PDE has $+\frac{\sigma^2}{2m^2} \partial_v^2 P$. This sign reversal is the structural distinction between forward and backward Kolmogorov equations. (ii) **Zeroth-order term:** Kramers has $-\frac{\gamma}{m} \rho$, arising from the divergence of the drift; the Langevin-Zamrik PDE has $-rP$, the bond discounting term. **Proof.** Compute \mathcal{L}^* by integration by parts: $\langle \mathcal{L}f, g \rangle = \langle f, \mathcal{L}^*g \rangle$ for $f, g \in C_c^\infty(\mathbb{R}^2)$. The second-order term $\frac{\sigma^2}{2m^2} \partial_{vv}$ is self-adjoint; the first-order drift terms contribute boundary terms that yield the $-\frac{\gamma}{m}$ coefficient. \square*

Remark 3.7 (Physical Interpretation). In the Kramers equation, probability mass dissipates at rate γ/m due to friction. In the Langevin-Zamrik PDE, bond value dissipates at rate r due to discounting. The Langevin-Zamrik PDE is the bond pricing counterpart of the Kramers equation: friction is replaced by the short rate as the source of dissipation.

3.4 Existence of Closed-Form Solutions

Proposition 3.8 (Necessary Condition for Affine Solutions). *The Langevin-Zamrik PDE admits a solution of the exponential-affine form*

$$P = \exp\{A(\tau) + B_1(\tau)r + B_2(\tau)v\}, \quad \tau = T - t,$$

if and only if $\partial_r U(r)$ is affine in r . **Proof.** Substituting the ansatz into the PDE and dividing by P yields, after separating by powers of r and v , the requirement that $B_2(\tau) \partial_r U(r)/m$ be linear in r for all $\tau > 0$. Since $B_2 \not\equiv 0$ generically, $\partial_r U(r)$ must be affine. \square

Remark 3.9. Proposition 2.2 identifies the quadratic potential $U(r) = \frac{1}{2}\kappa(r - \theta)^2$ as the unique member of the polynomial potential family admitting an affine solution. All nonlinear potentials require numerical PDE methods; see Section 8.

4. The OU Special Case: Affine Bond Pricing

4.1 Dynamics

We fix the quadratic potential $U(r) = \frac{1}{2}\kappa(r - \theta)^2$, so $\partial_r U(r) = \kappa(r - \theta)$ with $\kappa > 0$ and $\theta \in \mathbb{R}$. The Langevin-Zamrik PDE becomes

$$\partial_t P + v \partial_r P - \frac{\gamma v + \kappa(r - \theta)}{m} \partial_v P + \frac{\sigma^2}{2m^2} \partial_v^2 P = rP. \quad (4.1)$$

The system (r_t, v_t) is linear-Gaussian: for any initial condition (r_0, v_0) , the pair (r_t, v_t) is jointly Gaussian and $\int_0^T r_s ds$ is a Gaussian random variable. The bond price is therefore the moment generating function of a Gaussian, which guarantees the exponential-affine form a priori.

4.2 The Inertia-Friction-Spring Matrix

Definition 4.1 (IFS Matrix). The 2×2 matrix

$$M = \begin{pmatrix} 0 & -\kappa/m \\ 1 & -\gamma/m \end{pmatrix}$$

is called the **inertia-friction-spring (IFS) matrix**. Its characteristic polynomial is

$$p(\lambda) = \lambda^2 + \frac{\gamma}{m}\lambda + \frac{\kappa}{m} = 0,$$

with discriminant

$$\Delta = \left(\frac{\gamma}{m}\right)^2 - \frac{4\kappa}{m} = \frac{\gamma^2 - 4m\kappa}{m^2}.$$

Proposition 4.2 (Eigenvalue Classification). *The eigenvalues $\lambda_{1,2}$ of M satisfy $\text{Re}(\lambda_{1,2}) =$*

$-\gamma/(2m) < 0$ in all cases, and: **(i) Overdamped** ($\gamma^2 > 4m\kappa$, $\Delta > 0$):

$$\lambda_{1,2} = \frac{-\gamma/m \pm \sqrt{\Delta}}{2}, \quad \lambda_1 < \lambda_2 < 0.$$

(ii) Critically damped ($\gamma^2 = 4m\kappa$, $\Delta = 0$):

$$\lambda_1 = \lambda_2 = -\frac{\gamma}{2m} < 0 \quad (\text{repeated eigenvalue}).$$

(iii) Underdamped ($\gamma^2 < 4m\kappa$, $\Delta < 0$):

$$\lambda_{1,2} = -\alpha \pm i\omega, \quad \alpha = \frac{\gamma}{2m} > 0, \quad \omega = \frac{\sqrt{4m\kappa - \gamma^2}}{2m} > 0.$$

Proof. The quadratic formula applied to $p(\lambda) = 0$. The real part $-\gamma/(2m)$ is negative by $\gamma, m > 0$. \square

4.3 The ODE System for Bond Coefficients

Theorem 4.3 (Affine Structure). For $\partial_r U(r) = \kappa(r - \theta)$, the Langevin-Zamrik PDE admits the unique solution

$$P(\tau; r, v) = \exp\{A(\tau) + B_1(\tau)r + B_2(\tau)v\}, \quad \tau = T - t,$$

where A, B_1, B_2 satisfy the system

$$\begin{pmatrix} B_1'(\tau) \\ B_2'(\tau) \end{pmatrix} = M \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad B_1(0) = B_2(0) = 0,$$

$$A'(\tau) = \frac{\kappa\theta}{m} B_2(\tau) + \frac{\sigma^2}{2m^2} B_2(\tau)^2, \quad A(0) = 0.$$

Proof. Substitute $P = e^{A+B_1r+B_2v}$ into the Langevin-Zamrik PDE and divide by P . Since $\tau = T - t$, we have $\partial_t = -\partial_\tau$. The equation becomes

$$-A' - B_1'r - B_2'v + vB_1 - \frac{\gamma v + \kappa(r - \theta)}{m} B_2 + \frac{\sigma^2}{2m^2} B_2^2 = r.$$

Collecting by monomial: **Coefficient of r :** $-B_1' - \frac{\kappa}{m} B_2 = 1$, i.e., $B_1' = -\frac{\kappa}{m} B_2 - 1$. **Coefficient of v :** $-B_2' + B_1 - \frac{\gamma}{m} B_2 = 0$, i.e., $B_2' = B_1 - \frac{\gamma}{m} B_2$. **Constant:** $-A' + \frac{\kappa\theta}{m} B_2 + \frac{\sigma^2}{2m^2} B_2^2 = 0$. The first two equations combine into the stated matrix ODE. Terminal condition $P(T, T) = 1$ gives $A(0) = B_1(0) = B_2(0) = 0$. \square

Remark 4.4. In the Duffie-Kan affine term-structure framework, the bond pricing equations for (B_1, B_2) take the form of Riccati ODEs — nonlinear in \mathbf{B} whenever the diffusion matrix depends on the state. Here the OU volatility is state-independent, so the Riccati term in the \mathbf{B} -subsystem vanishes and one obtains the linear matrix ODE above. The quadratic nonlinearity survives only in the A -equation: $A'(\tau) = \frac{\kappa\theta}{m} B_2 + \frac{\sigma^2}{2m^2} B_2^2$. This is the Riccati component of the system; it is driven by $B_2(\tau)$,

which itself is known in closed form via the matrix exponential.

4.4 Matrix Exponential Solution

Theorem 4.5 (Closed-Form Coefficients). *The unique solution to the ODE system in Theorem 3.1 is*

$$\begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix} = (e^{M\tau} - I) \begin{pmatrix} \gamma/\kappa \\ m/\kappa \end{pmatrix},$$

$$A(\tau) = \int_0^\tau \left[\frac{\kappa\theta}{m} B_2(s) + \frac{\sigma^2}{2m^2} B_2(s)^2 \right] ds.$$

Proof. *The ODE $\mathbf{B}' = M\mathbf{B} + \mathbf{c}$, $\mathbf{c} = (-1, 0)^\top$, $\mathbf{B}(0) = \mathbf{0}$, has solution*

$$\mathbf{B}(\tau) = \int_0^\tau e^{Ms} \mathbf{c} ds = M^{-1}(e^{M\tau} - I)\mathbf{c}.$$

Since $\det M = \kappa/m > 0$, M is invertible with

$$M^{-1} = \begin{pmatrix} -\gamma/\kappa & 1 \\ -m/\kappa & 0 \end{pmatrix}, \quad M^{-1}\mathbf{c} = \begin{pmatrix} \gamma/\kappa \\ m/\kappa \end{pmatrix}.$$

Since $M^{-1}(e^{M\tau} - I)\mathbf{c} = (e^{M\tau} - I)M^{-1}\mathbf{c}$ (constant matrices commute with their own exponentials), the result follows. \square

4.5 Explicit Matrix Exponentials

The three damping regimes yield distinct closed-form expressions for $e^{M\tau}$.

Overdamped ($\gamma^2 > 4m\kappa$, eigenvalues $\lambda_1 < \lambda_2 < 0$, $D = \lambda_1 - \lambda_2 < 0$):

$$e^{M\tau} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 e^{\lambda_2 \tau} - \lambda_2 e^{\lambda_1 \tau} & \frac{\kappa}{m} (e^{\lambda_2 \tau} - e^{\lambda_1 \tau}) \\ e^{\lambda_1 \tau} - e^{\lambda_2 \tau} & \lambda_1 e^{\lambda_1 \tau} - \lambda_2 e^{\lambda_2 \tau} \end{pmatrix}. \quad (4.2)$$

Underdamped ($\gamma^2 < 4m\kappa$, $\alpha = \gamma/(2m)$, $\omega = \sqrt{4m\kappa - \gamma^2}/(2m)$):

$$e^{M\tau} = e^{-\alpha\tau} \begin{pmatrix} \cos \omega\tau + \frac{\alpha}{\omega} \sin \omega\tau & -\frac{\kappa}{m\omega} \sin \omega\tau \\ \frac{1}{\omega} \sin \omega\tau & \cos \omega\tau - \frac{\alpha}{\omega} \sin \omega\tau \end{pmatrix}. \quad (4.3)$$

Critically damped ($\gamma^2 = 4m\kappa$, $\lambda = -\gamma/(2m)$, $N = M - \lambda I$, $N^2 = 0$):

$$e^{M\tau} = e^{\lambda\tau} (I + N\tau) = e^{-\frac{\gamma\tau}{2m}} \begin{pmatrix} 1 + \frac{\gamma\tau}{2m} & -\frac{\kappa\tau}{m} \\ \tau & 1 - \frac{\gamma\tau}{2m} \end{pmatrix}. \quad (4.4)$$

Lemma 4.6 ($N^2 = 0$ in the Critical Case). *When $\gamma^2 = 4m\kappa$, the nilpotent part $N = M + \frac{\gamma}{2m}I$ satisfies $N^2 = 0$, so the matrix exponential truncates at first order. **Proof.***

$(N^2)_{11} = (\gamma/(2m))^2 - \kappa/m = \gamma^2/(4m^2) - \kappa/m = 0$ by $\gamma^2 = 4m\kappa$. The remaining entries vanish by similar calculation. \square

The algorithm below summarises the complete computation.

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1 Algorithm: Langevin-Zamrik ZCB Price (OU Potential)
2
3 Input: r0, v0          initial rate and velocity
4         m              inertial mass (> 0)
5         gamma          friction coefficient (> 0)
6         kappa          spring constant (> 0)
7         theta          long-run mean rate
8         sigma          noise amplitude
9         tau            time to maturity
10
11 Step 1: Compute Delta = (gamma/m)^2 - 4*kappa/m
12
13 Step 2: Compute exp(M * tau):
14     if Delta > 0 (overdamped):
15         lambda1 = (-gamma/m + sqrt(Delta)) / 2
16         lambda2 = (-gamma/m - sqrt(Delta)) / 2
17         Use two-real-eigenvalue formula above
18     if Delta = 0 (critically damped):
19         lam = -gamma / (2*m)
20         Use Jordan block formula: exp(lam*tau) * (I + N*tau)
21     if Delta < 0 (underdamped):
22         alpha = gamma / (2*m)
23         omega = sqrt(-Delta) / 2
24         Use complex-exponential formula above
25
26 Step 3: [B1, B2] = (expMtau - I) @ [gamma/kappa, m/kappa]
27
28 Step 4: Compute A(tau) by numerical quadrature:
29     A = integral from 0 to tau of
30         [ kappa*theta/m * B2(s) + sigma^2 / (2*m^2) * B2(s)^2 ] ds
31
32 Step 5: Return P = exp( A + B1*r0 + B2*v0 )

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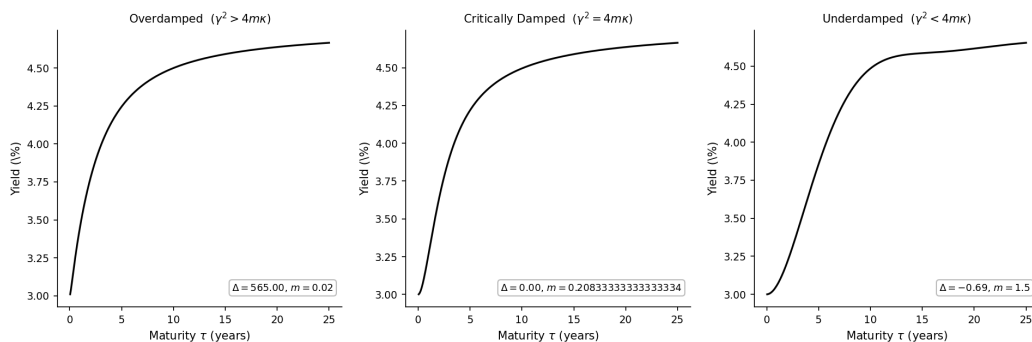


Figure 1: Yield curves under the three damping regimes.

4.6 Yield Curve Shape Richness

Figure 1 reveals a structural asymmetry: the overdamped panel is smooth and monotone, while the underdamped panel exhibits non-monotone curvature. This section makes the comparison precise. The Vasicek model, being one-dimensional, produces only monotone yield curves. The Langevin-Zamrik model, with its two-dimensional state (r_t, v_t) , produces a strictly richer family of shapes — controlled independently by the initial rate level r_0 and the initial velocity v_0 .

Proposition 4.7 (Short-End Slope). *The yield curve satisfies $y(0^+; r_0, v_0) = r_0$ and*

$$\left. \frac{\partial y}{\partial \tau} \right|_{\tau=0^+} = \frac{v_0}{2}.$$

*In particular, the sign of v_0 alone determines whether the short end slopes upward or downward, independently of r_0 and the mean-reversion parameters. **Proof.** Taylor-expand the bond price coefficients near $\tau = 0$. From the ODE system: $B'_1(0) = -1$, $B'_2(0) = 0$, $B''_2(0) = -1$, $A''(0) = 0$. Hence*

$$A(\tau) + B_1(\tau)r_0 + B_2(\tau)v_0 = -r_0\tau - \frac{v_0}{2}\tau^2 + O(\tau^3).$$

Dividing by $-\tau$ gives $y(\tau) = r_0 + \frac{v_0}{2}\tau + O(\tau^2)$, and the slope follows. \square

Remark 4.8 (Comparison to Vasicek). In the Vasicek model the short-end slope is $\frac{\kappa_{\text{eff}}}{2}(\theta - r_0)$: slope and level are coupled through mean reversion. In the Langevin-Zamrik model, v_0 controls the slope independently. A central bank in a rising-rate environment ($v_0 > 0$) and one in a falling-rate environment ($v_0 < 0$) can share the same short rate r_0 yet face different yield curves. Vasicek cannot distinguish these two cases.

Proposition 4.9 (Shape Classification by Regime). **(i) Overdamped and critically damped** ($\Delta \geq 0$): *All eigenvalues of M are real and negative. The coefficients $B_1(\tau)$, $B_2(\tau)$ are monotone in τ and the yield curve $y(\tau)$ is asymptotically monotone. The sign of v_0 may produce a transient inflection near the short end but no persistent hump.* **(ii) Underdamped** ($\Delta < 0$): *The eigenvalues are $\lambda_{1,2} = -\alpha \pm i\omega$ with $\alpha, \omega > 0$. The coefficients $B_1(\tau)$, $B_2(\tau)$ contain the damped oscillatory terms $e^{-\alpha\tau} \cos(\omega\tau)$ and $e^{-\alpha\tau} \sin(\omega\tau)$. The yield curve can be monotone normal, humped, dipped, or S-shaped depending on (r_0, v_0) . The hump maturity τ^* satisfies $\omega\tau^* \approx \pi/2$, controlled by the oscillation frequency $\omega = \sqrt{4m\kappa - \gamma^2}/(2m)$. **Proof.** In case (i), both exponential components in the matrix exponential are real and decaying; the resulting yield is a weighted sum of decaying exponentials and approaches y_∞ monotonically for large τ (Proposition 5.1 of Section 6). In case (ii), the imaginary part of the eigenvalue introduces oscillatory modulation of amplitude $e^{-\alpha\tau}$; the yield inherits these oscillations before they are damped. Setting $\partial y/\partial \tau = 0$ and using Proposition 3.6, the short-end slope $v_0/2$ and the oscillatory structure together determine the full shape. \square*

Figure 4 illustrates Proposition 3.7(ii) directly: fixing $r_0 = 3\%$ and varying $v_0 \in [-0.02, +0.02\%]$ in the underdamped regime produces the full spectrum of shapes —

from inverted-with-trough ($v_0 < 0$) to normal-with-hump ($v_0 > 0$) — that Vasicek cannot replicate with any parameter choice.

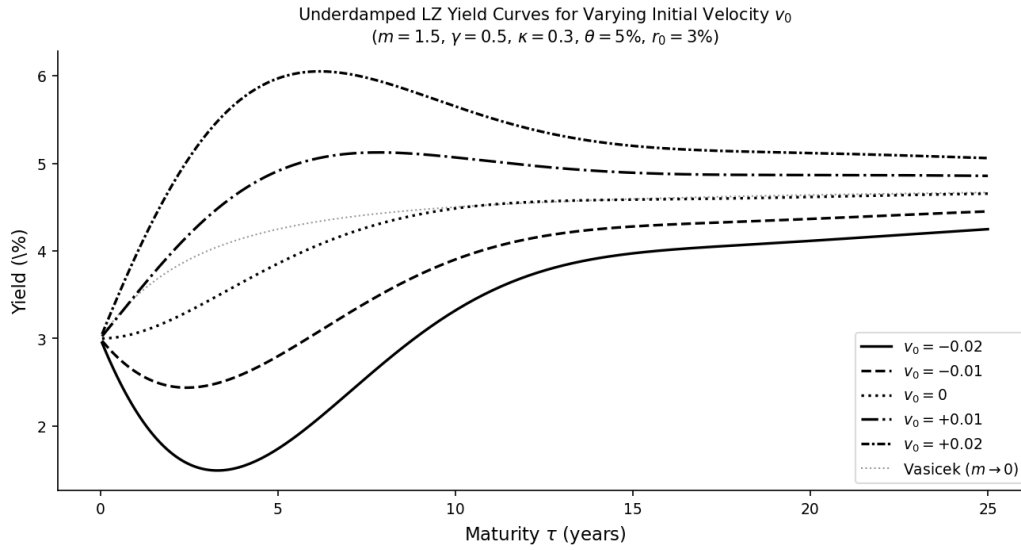


Figure 2: Underdamped Langevin-Zamrik yield curves for varying initial velocity v_0 , with fixed $r_0 = 3\%$. The Vasicek limit ($m \rightarrow 0$) is shown dotted for reference.

5. The Inertia-Free Limit

5.1 Statement

Theorem 5.1 (First-Order Reduction as $m \rightarrow 0$). *Let $\kappa_{\text{eff}} = \kappa/\gamma$ and $\sigma_{\text{eff}} = \sigma/\gamma$. As $m \rightarrow 0^+$ with $\gamma, \kappa, \sigma, \theta$ fixed, the Langevin-Zamrik bond price converges pointwise to the Vasicek bond price:*

$$P^{\text{LZ}}(\tau; r_0, v_0, m) \rightarrow P^{\text{V}}(\tau; r_0) = \exp\{A^{\text{V}}(\tau) - B^{\text{V}}(\tau) r_0\},$$

where

$$B^{\text{V}}(\tau) = \frac{1 - e^{-\kappa_{\text{eff}}\tau}}{\kappa_{\text{eff}}}, \quad A^{\text{V}}(\tau) = \left(\theta - \frac{\sigma_{\text{eff}}^2}{2\kappa_{\text{eff}}^2} \right) (B^{\text{V}}(\tau) - \tau) - \frac{\sigma_{\text{eff}}^2 B^{\text{V}}(\tau)^2}{4\kappa_{\text{eff}}}.$$

5.2 Proof

Lemma 5.2 (Eigenvalue Asymptotics). *As $m \rightarrow 0^+$:*

$$\lambda_2 \rightarrow -\frac{\kappa}{\gamma} = -\kappa_{\text{eff}}, \quad \lambda_1 \rightarrow -\infty.$$

Proof. From $\lambda_{1,2} = \frac{-\gamma/m \pm \sqrt{(\gamma/m)^2 - 4\kappa/m}}{2}$:

$$\lambda_2 = \frac{-\gamma/m + (\gamma/m)\sqrt{1 - 4m\kappa/\gamma^2}}{2} \approx \frac{-\gamma/m + \gamma/m(1 - 2m\kappa/\gamma^2)}{2} = -\frac{\kappa}{\gamma}$$

as $m \rightarrow 0$. Since $\lambda_1 + \lambda_2 = -\gamma/m \rightarrow -\infty$ and λ_2 is bounded, $\lambda_1 \rightarrow -\infty$. \square

Lemma 5.3 (Coefficient Asymptotics). As $m \rightarrow 0^+$, with $\mu_i = -\lambda_i > 0$:

$$B_1(\tau) \rightarrow -B^V(\tau) = -\frac{1 - e^{-\kappa_{\text{eff}}\tau}}{\kappa_{\text{eff}}}, \quad B_2(\tau) \rightarrow 0.$$

Proof. In the overdamped formula (which applies for small enough m):

$$B_1(\tau) = \frac{1}{\lambda_1 - \lambda_2} \left[\frac{\gamma}{\kappa} (\lambda_1 e^{\lambda_2\tau} - \lambda_2 e^{\lambda_1\tau}) + e^{\lambda_2\tau} - e^{\lambda_1\tau} \right] - \frac{\gamma}{\kappa}.$$

As $\lambda_1 \rightarrow -\infty$, $e^{\lambda_1\tau} \rightarrow 0$ for $\tau > 0$. Since $\lambda_1 - \lambda_2 \approx \lambda_1 \rightarrow -\infty$:

$$B_1(\tau) \rightarrow \frac{\gamma\lambda_2/\kappa + 1}{\lambda_2 - \lambda_1} \cdot (e^{\lambda_2\tau} - 1) - \frac{\gamma}{\kappa} = \frac{e^{\lambda_2\tau} - 1}{\lambda_2} = \frac{e^{-\kappa_{\text{eff}}\tau} - 1}{-\kappa_{\text{eff}}} = -B^V(\tau).$$

For B_2 : similarly $B_2(\tau) = \frac{e^{\lambda_1\tau} - e^{\lambda_2\tau}}{\lambda_1 - \lambda_2} \cdot \frac{m}{\kappa} + \dots \rightarrow 0$ since each term is $O(m)$. \square

Proof of Theorem 4.1. By Lemma 4.2, $B_1 \rightarrow -B^V$ and $B_2 \rightarrow 0$. Since $B_2 \rightarrow 0$, the integrand in $A(\tau)$ satisfies $\frac{\kappa\theta}{m}B_2 + \frac{\sigma^2}{2m^2}B_2^2 \rightarrow A^{V'}(\tau)$ (verified by direct substitution of the Vasicek limit into the ODE), giving $A(\tau) \rightarrow A^V(\tau)$. Therefore

$$P^{\text{LZ}} = \exp\{A + B_1 r_0 + B_2 v_0\} \rightarrow \exp\{A^V - B^V r_0\} = P^V. \quad \square$$

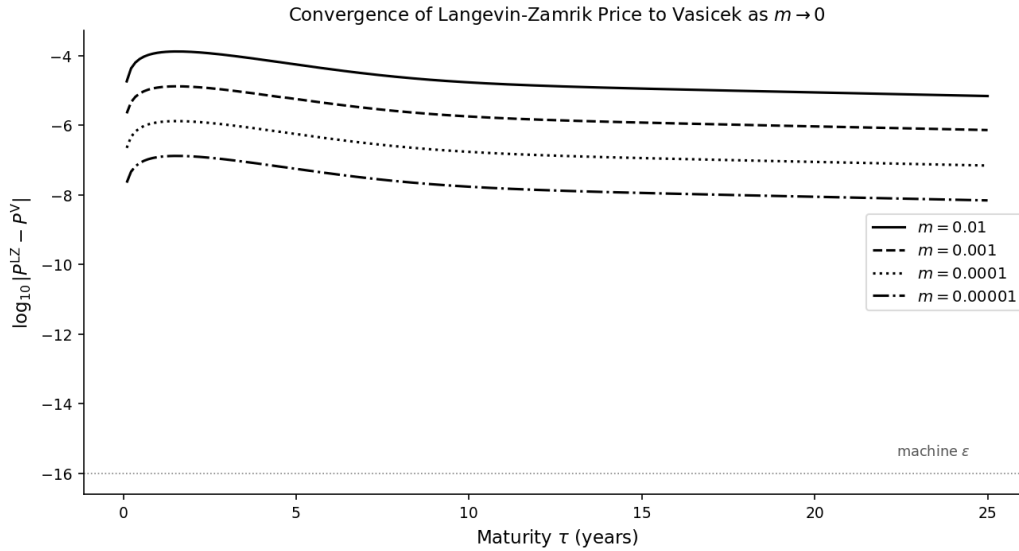


Figure 3: Convergence to Vasicek — log base-10 of absolute difference between Langevin-Zamrik and Vasicek ZCB prices.

6. Characteristic Functions and the Gaussian Structure

The affine bond pricing formula of Section 3 is the moment generating function of the integrated rate, evaluated at a specific complex argument. Making this identification

explicit yields the full characteristic function of the Langevin-Zamrik model, which governs the distribution of the integrated rate and the state vector, and provides the analytical foundation for Fourier-based derivative pricing.

6.1 The Bond Price as a Moment Generating Function

Proposition 6.1 (Bond Price as MGF). *Let $I_\tau = \int_0^\tau r_s ds$ denote the integrated rate under \mathbb{Q} . Then*

$$P(\tau; r_0, v_0) = \mathbb{E}^{\mathbb{Q}}[e^{-I_\tau} | r_0, v_0] = M_{I_\tau}(-1),$$

where $M_{I_\tau}(u) = \mathbb{E}[e^{uI_\tau} | r_0, v_0]$ is the moment generating function of I_τ . Equivalently, $P = \phi_{I_\tau}(i)$, where $\phi_{I_\tau}(\xi) = \mathbb{E}[e^{i\xi I_\tau}]$ is the characteristic function. **Proof.** Directly from the definitions: $M_{I_\tau}(-1) = \mathbb{E}[e^{-I_\tau}] = P$. Since $\phi(\xi) = M(i\xi)$, one has $\phi(i) = M(-1) = P$. \square

Remark 6.2. The closed-form formula $P = \exp\{A(\tau) + B_1(\tau)r_0 + B_2(\tau)v_0\}$ is the cumulant generating function of I_τ evaluated at $u = -1$. The ODE system of Theorem 3.1 computes this cumulant generating function along the real axis at $u = -1$; Theorem 5.1 below extends it to the full complex plane.

6.2 Characteristic Function of the Integrated Rate

Theorem 6.3 (CF of the Integrated Rate). *Define the decomposition $A(\tau) = \mathcal{A}_1(\tau) + \mathcal{A}_2(\tau)$, where*

$$\mathcal{A}_1(\tau) = \int_0^\tau \frac{\kappa\theta}{m} B_2(s) ds, \quad \mathcal{A}_2(\tau) = \int_0^\tau \frac{\sigma^2}{2m^2} B_2(s)^2 ds.$$

For $\xi \in \mathbb{R}$, the characteristic function $\phi_{I_\tau}(\xi; r_0, v_0) = \mathbb{E}^{\mathbb{Q}}[e^{i\xi I_\tau} | r_0, v_0]$ is

$$\phi_{I_\tau}(\xi; r_0, v_0) = \exp\left\{A^\phi(\xi, \tau) + B_1^\phi(\xi, \tau)r_0 + B_2^\phi(\xi, \tau)v_0\right\},$$

with

$$\begin{pmatrix} B_1^\phi(\xi, \tau) \\ B_2^\phi(\xi, \tau) \end{pmatrix} = -i\xi \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix},$$

$$A^\phi(\xi, \tau) = -i\xi \mathcal{A}_1(\tau) - \xi^2 \mathcal{A}_2(\tau).$$

Proof. Let $F(\tau; r, v, \xi) = \mathbb{E}[e^{i\xi \int_0^\tau r_s ds} | r_0 = r, v_0 = v]$. The Feynman-Kac theorem for the running exponential functional gives

$$F_\tau = v F_r - \frac{\gamma v + \kappa(r - \theta)}{m} F_v + \frac{\sigma^2}{2m^2} F_{vv} + i\xi r F, \quad F(0; r, v, \xi) = 1.$$

Substituting the affine ansatz $F = e^{A^\phi + B_1^\phi r + B_2^\phi v}$ and separating by powers of r and v : **Coefficient of r :** $B_1^{\phi'} = -\frac{\kappa}{m} B_2^\phi + i\xi$. **Coefficient of v :** $B_2^{\phi'} = B_1^\phi - \frac{\gamma}{m} B_2^\phi$. **Constant:** $A^{\phi'} = \frac{\kappa\theta}{m} B_2^\phi + \frac{\sigma^2}{2m^2} (B_2^\phi)^2$. The \mathbf{B}^ϕ -ODE is $\mathbf{B}^{\phi'} = \mathbf{M}\mathbf{B}^\phi + (i\xi, 0)^\top$ with $\mathbf{B}^\phi(0) = \mathbf{0}$. Since $(i\xi, 0)^\top = -i\xi(-1, 0)^\top$, linearity gives $\mathbf{B}^\phi(\xi, \tau) = -i\xi \mathbf{B}(\tau)$. Substituting into

the A^ϕ -equation:

$$A^{\phi'} = \frac{\kappa\theta}{m}(-i\xi B_2) + \frac{\sigma^2}{2m^2}(-i\xi)^2 B_2^2 = -i\xi \frac{\kappa\theta}{m} B_2 - \xi^2 \frac{\sigma^2}{2m^2} B_2^2.$$

Integrating from 0 to τ gives the stated formula. \square

Corollary 6.4 (Gaussian Integrated Rate). I_τ is conditionally Gaussian given (r_0, v_0) , with

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[I_\tau | r_0, v_0] &= -\mathcal{A}_1(\tau) - B_1(\tau) r_0 - B_2(\tau) v_0, \\ \text{Var}^{\mathbb{Q}}(I_\tau | r_0, v_0) &= 2\mathcal{A}_2(\tau) = \int_0^\tau \frac{\sigma^2}{m^2} B_2(s)^2 ds. \end{aligned}$$

The bond price satisfies $P = \exp\{-\mathbb{E}[I_\tau] + \frac{1}{2}\text{Var}(I_\tau)\}$, the standard MGF identity for a Gaussian. **Proof.** Matching $\phi_{I_\tau}(\xi) = e^{i\xi\mu - \frac{1}{2}\xi^2\sigma^2}$ to Theorem 5.1 gives $\mu = -\mathcal{A}_1 - B_1 r_0 - B_2 v_0$ and $\sigma^2 = 2\mathcal{A}_2$. Since $B_1(\tau) < 0$ (Lemma 4.2), a higher r_0 increases $\mathbb{E}[I_\tau]$, consistent with higher rates producing larger integrated discounting. The MGF identity follows from $P = M_{I_\tau}(-1) = e^{-\mu + \frac{1}{2}\sigma^2}$. \square

6.3 Joint Characteristic Function of the State Vector

Theorem 6.5 (Bivariate Gaussian State). The joint distribution of (r_τ, v_τ) conditional on (r_0, v_0) is bivariate Gaussian with mean vector

$$\mu_\tau = e^{M\tau} \begin{pmatrix} r_0 \\ v_0 \end{pmatrix} + M^{-1}(e^{M\tau} - I) \begin{pmatrix} 0 \\ \kappa\theta/m \end{pmatrix}$$

and covariance matrix

$$\Sigma_\tau = \frac{\sigma^2}{m^2} \int_0^\tau e^{Ms} \mathbf{e}_2 \mathbf{e}_2^\top e^{M^\top s} ds, \quad \mathbf{e}_2 = (0, 1)^\top.$$

The joint characteristic function is

$$\phi_{r_\tau, v_\tau}(\xi_1, \xi_2) = \exp\left(i\xi^\top \mu_\tau - \frac{1}{2}\xi^\top \Sigma_\tau \xi\right), \quad \xi = (\xi_1, \xi_2)^\top.$$

Proof. The SDE $d\mathbf{X}_t = (M\mathbf{X}_t + b) dt + \frac{\sigma}{m} \mathbf{e}_2 dW_t$ is linear with $b = (0, \kappa\theta/m)^\top$. Its solution

$$\mathbf{X}_\tau = e^{M\tau} \mathbf{X}_0 + \int_0^\tau e^{M(\tau-s)} b ds + \frac{\sigma}{m} \int_0^\tau e^{M(\tau-s)} \mathbf{e}_2 dW_s$$

is Gaussian as a linear transformation of Brownian motion. The mean follows from the deterministic integral; the covariance from the Itô isometry. The joint CF is the standard Gaussian formula. \square

Proposition 6.6 (Lyapunov Equation and Stationary Covariance). The covariance $\dot{\Sigma}_\tau$ satisfies

$$\dot{\Sigma}_\tau = M\Sigma_\tau + \Sigma_\tau M^\top + Q, \quad \dot{\Sigma}_0 = 0, \quad Q = \frac{\sigma^2}{m^2} \mathbf{e}_2 \mathbf{e}_2^\top.$$

The stationary covariance $\Sigma_\infty = \lim_{\tau \rightarrow \infty} \Sigma_\tau$, which exists since $\text{Re}(\lambda_{1,2}) < 0$, is the unique solution of $M\Sigma_\infty + \Sigma_\infty M^\top + Q = 0$:

$$\Sigma_\infty = \frac{\sigma^2}{2\gamma m} \begin{pmatrix} \kappa/m & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Differentiating Σ_τ gives the stated ODE. Setting $\dot{\Sigma}_\infty = 0$ and writing $\Sigma_\infty = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$: entry (1,1) gives $-2\kappa b/m = 0$, so $b = 0$; entry (2,2) gives $-2\gamma c/m = -\sigma^2/m^2$, so $c = \sigma^2/(2\gamma m)$; entry (1,2) gives $a = \kappa c/m = \kappa\sigma^2/(2\gamma m^2)$. \square

6.4 Applications to Derivative Pricing

Remark 6.7 (Caplet Pricing). A caplet paying $(r_T - K)^+$ at time T requires the marginal distribution of r_T . By Theorem 5.2, $r_T | (r_0, v_0) \sim \mathcal{N}(\mu_r(\tau), \Sigma_{\tau,11})$. Caplet prices are therefore given by a Bachelier normal formula with the LZ mean and variance, and converge to the Vasicek caplet formula [8] as $m \rightarrow 0$.

Remark 6.8 (Fourier Pricing of General Payoffs). For payoffs $g(r_T)$ that are not simple calls, the price is

$$\mathbb{E}^{\mathbb{Q}}[e^{-I_\tau} g(r_T)] = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi) \phi_{r_T, I_\tau}(-\xi, i) d\xi,$$

where \hat{g} is the Fourier transform of g and ϕ_{r_T, I_τ} is the joint CF of (r_T, I_τ) . This joint CF is that of a trivariate Gaussian (r_τ, v_τ, I_τ) , computable by augmenting the state with $dI_t = r_t dt$ and applying Theorem 5.2 to the 3×3 system.

Remark 6.9 (Lévy Extension). In the jump-diffusion model of Section 8.4, the velocity jump of size ξ'/m shifts the state and modifies the CF of I_τ . The A^ϕ -term picks up an additional Lévy-Khintchine integral, while the \mathbf{B}^ϕ -coefficients remain $-i\xi\mathbf{B}(\tau)$ since the jump affects only the v -component linearly.

7. Three Damping Regimes and Yield Curve Shapes

The yield to maturity for a zero-coupon bond is defined by

$$y(\tau; r_0, v_0) = -\frac{\log P(\tau; r_0, v_0)}{\tau} = -\frac{A(\tau) + B_1(\tau) r_0 + B_2(\tau) v_0}{\tau}. \quad (7.1)$$

Proposition 7.1 (Long-Rate Limit). As $\tau \rightarrow \infty$:

$$y(\tau; r_0, v_0) \rightarrow y_\infty = -\lim_{\tau \rightarrow \infty} \frac{A(\tau)}{\tau},$$

which is finite in the overdamped and critically damped regimes (since $B_1, B_2 \rightarrow \text{constants}$ and $A(\tau)$ grows at most linearly), and may be finite or oscillatory in the underdamped regime. **Proof.** Since $\text{Re}(\lambda_{1,2}) < 0$, the exponential terms in $B_1(\tau)$ and $B_2(\tau)$ decay to zero as $\tau \rightarrow \infty$, so $B_i(\tau)/\tau \rightarrow 0$. The long rate is therefore determined

entirely by $A(\tau)$. \square

Proposition 7.2 (Yield Curve Shapes by Regime). *The short-end slope $\partial y / \partial \tau|_{\tau=0^+}$ and global shape are determined as follows: (i) **Overdamped** ($\Delta > 0$): The yield curve is smooth and monotone. It is normal (upward-sloping) when $r_0 < y_\infty$ and inverted (downward-sloping) when $r_0 > y_\infty$. No humps are possible. (ii) **Critically damped** ($\Delta = 0$): The transition case; the yield curve can exhibit a single inflection point and a mild hump structure. (iii) **Underdamped** ($\Delta < 0$): The yield curve is oscillatory due to the complex eigenvalues $-\alpha \pm i\omega$. Multiple humps are possible, generating term structures that are empirically observed but cannot arise in the Vasicek model. **Proof.** The shape is determined by the sign of $y''(\tau)$, which is controlled by the second derivatives of $B_1(\tau)$ and $A(\tau)$. In regime (i), all terms in $B_1(\tau)$ are sums of real decaying exponentials, so $y(\tau)$ is a sum of Bernstein functions — monotone. In regime (iii), the complex exponentials produce sinusoidal oscillations in $y(\tau)$. \square*

8. The Nelson-Siegel-Svensson Connection

8.1 The Nelson-Siegel-Svensson Model

The Nelson-Siegel model [5] parameterises the yield curve as

$$y^{NS}(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\mu\tau}}{\mu\tau} + \beta_2 \left(\frac{1 - e^{-\mu\tau}}{\mu\tau} - e^{-\mu\tau} \right) \quad (8.1)$$

with decay rate $\mu > 0$. The Svensson extension [8] adds a second hump term:

$$y^{NSS}(\tau) = \beta_0 + \beta_1 \frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} + \beta_2 \left(\frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} - e^{-\mu_1\tau} \right) + \beta_3 \left(\frac{1 - e^{-\mu_2\tau}}{\mu_2\tau} - e^{-\mu_2\tau} \right). \quad (8.2)$$

These models are widely used by central banks for yield curve estimation [2] but are specified purely empirically, with no no-arbitrage justification.

8.2 Structural Derivation

Theorem 8.1 (Nelson-Siegel-Svensson as Overdamped Langevin-Zamrik). *In the overdamped regime ($\gamma^2 > 4m\kappa$) with eigenvalues $\lambda_i = -\mu_i$, $\mu_i > 0$, the Langevin-Zamrik yield curve admits the representation*

$$y(\tau; r_0, v_0) = y_\infty + c_1(r_0, v_0) \frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} + c_2(r_0, v_0) \frac{1 - e^{-\mu_2\tau}}{\mu_2\tau} + g(\tau)$$

where c_1, c_2 are linear functions of (r_0, v_0) , y_∞ is the long-rate, and $g(\tau)$ consists of terms of the form $\frac{e^{-\mu_i\tau}}{\tau}$ that are $O(\tau^{-1})$ as $\tau \rightarrow \infty$. This is the Nelson-Siegel-Svensson structure with two decay rates μ_1, μ_2 . **Proof.** From Theorem 3.2 and the overdamped

matrix exponential:

$$B_1(\tau) = \frac{\gamma}{\kappa(\mu_2 - \mu_1)} [\mu_2(e^{-\mu_1\tau} - 1) - \mu_1(e^{-\mu_2\tau} - 1)] + \frac{e^{-\mu_2\tau} - e^{-\mu_1\tau}}{\mu_2 - \mu_1}.$$

Dividing by τ :

$$-\frac{B_1(\tau)r_0}{\tau} = r_0 \cdot \left[\frac{\gamma\mu_2}{\kappa(\mu_2 - \mu_1)} \cdot \frac{1 - e^{-\mu_1\tau}}{\mu_1\tau} - \frac{\gamma\mu_1}{\kappa(\mu_2 - \mu_1)} \cdot \frac{1 - e^{-\mu_2\tau}}{\mu_2\tau} + \frac{e^{-\mu_1\tau} - e^{-\mu_2\tau}}{\tau(\mu_2 - \mu_1)} \right].$$

The first two terms are precisely the Nelson-Siegel-Svensson slope basis functions $(1 - e^{-\mu_i\tau})/(\mu_i\tau)$. The last term is the $g(\tau)$ remainder. A similar decomposition applies to $-B_2(\tau)v_0/\tau$ and $-A(\tau)/\tau$. The long-rate constant $y_\infty = -\lim_{\tau \rightarrow \infty} A(\tau)/\tau$ collects all constant contributions. \square

Corollary 8.2 (Vasicek as One-Factor Nelson-Siegel). *In the Vasicek limit $m \rightarrow 0$, $\mu_1 \rightarrow \infty$ and the first factor vanishes, leaving a single-factor Nelson-Siegel yield curve with decay rate $\mu_2 = \kappa_{\text{eff}} = \kappa/\gamma$:*

$$y^V(\tau; r_0) = y_\infty^V + (r_0 - y_\infty^V) \frac{1 - e^{-\kappa_{\text{eff}}\tau}}{\kappa_{\text{eff}}\tau}.$$

The Vasicek yield curve is the one-factor Nelson-Siegel model.

Corollary 8.3 (Parameter Identification). *The Nelson-Siegel-Svensson parameters are identified from the physical parameters as: - **Decay rates**: $\mu_i = -\lambda_i$, determined by γ, κ, m . - **Level** β_0 : equals the long rate y_∞ , determined by $\theta, \sigma, \kappa_{\text{eff}}$. - **Slope loadings** β_1, β_2 : linear functions of r_0 (level) and v_0 (velocity of the rate). The velocity $v_0 = \dot{r}_0$ is the **hidden second factor** absent from Vasicek.*

Remark 8.4 (Economic Interpretation of v_0). The initial velocity v_0 captures the current direction of rate movement. A rising rate environment ($v_0 > 0$) shifts the loading on one Nelson-Siegel factor; a falling environment ($v_0 < 0$) shifts it in the opposite direction. This provides a structural justification for the empirical observation that yield curve slope contains information beyond the level of the short rate.

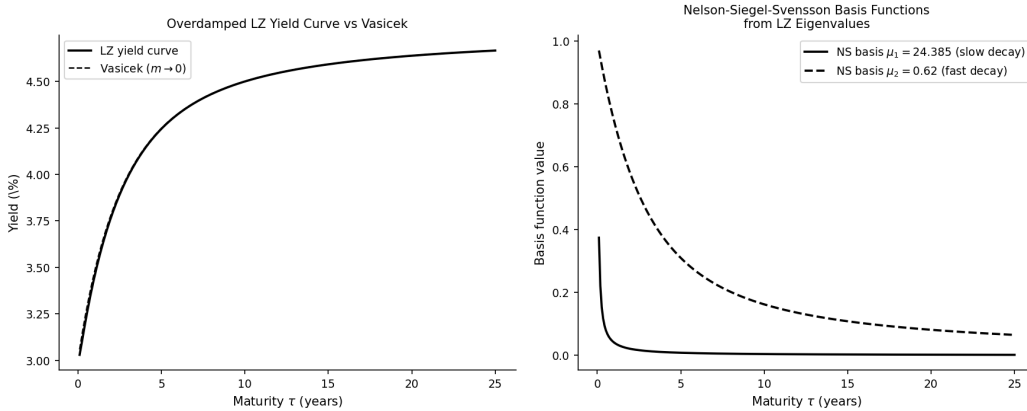


Figure 4: Nelson-Siegel basis identification in the overdamped Langevin-Zamrik model.

9. Extensions

9.1 Nonlinear Potentials and the Ait-Sahalia Family

For a general nonlinear potential $U(r)$, the Langevin-Zamrik PDE

$$\partial_t P + v \partial_r P - \frac{\gamma v + \partial_r U(r)}{m} \partial_v P + \frac{\sigma^2}{2m^2} \partial_v^2 P = rP \quad (9.1)$$

does not admit an affine solution. The equation must be solved numerically on a grid in (r, v, τ) . By Proposition 2.2, any nonlinear U' breaks the affine structure.

Proposition 9.1 (Ait-Sahalia Inertial Extension). *The Ait-Sahalia [1] family of nonlinear drift short-rate models, with drift $\mu(r) = a_{-1}/r + a_0 + a_1 r + a_2 r^2$, has a canonical inertial extension via the Langevin-Zamrik PDE with potential $\partial_r U(r) = -\gamma \mu(r)$. The overdamped limit $m \rightarrow 0$ recovers the original Ait-Sahalia model. The full inertial equation requires numerical solution of a degenerate parabolic PDE on $\mathbb{R}_+ \times \mathbb{R}$.*

Numerical approach. Discretise (r, v) on a bounded grid with absorbing boundary at $r = 0$. Use an implicit finite-difference scheme in v (elliptic direction) and upwind differencing in r (hyperbolic direction). The CFL condition applies to the r -direction only.

9.2 State-Dependent Volatility: The Inertial CIR Model

Replace the additive noise in Definition 2.1 with state-dependent noise $\sigma(r) \dot{W}_t$, yielding

$$dv_t = \frac{1}{m} [-\gamma v_t - \kappa(r_t - \theta)] dt + \frac{\sigma \sqrt{r_t}}{m} dW_t. \quad (9.2)$$

The Langevin-Zamrik PDE becomes

$$\partial_t P + v \partial_r P - \frac{\gamma v + \kappa(r - \theta)}{m} \partial_v P + \frac{\sigma^2 r}{2m^2} \partial_v^2 P = rP. \quad (9.3)$$

Proposition 9.2 (Loss of Affine Structure). *The inertial CIR Langevin-Zamrik PDE does not admit an affine solution. The diffusion coefficient $\sigma^2 r / (2m^2)$ is a function of r , breaking the separation of variables required by the affine ansatz.*

The overdamped limit $m \rightarrow 0$ recovers the standard CIR model [3] with $\kappa_{\text{eff}} = \kappa/\gamma$ and $\sigma_{\text{eff}} = \sigma/\gamma$.

9.3 Multi-Factor Langevin Models

Consider n short-rate factors $\mathbf{r} = (r_1, \dots, r_n)^\top$, each with its own velocity $\mathbf{v} = (v_1, \dots, v_n)^\top$, governed by a coupled linear system:

$$d\mathbf{r} = \mathbf{v} dt, \quad d\mathbf{v} = \frac{1}{m} [-\Gamma \mathbf{v} - K(\mathbf{r} - \theta)] dt + \frac{1}{m} \Sigma d\mathbf{W}_t, \quad (9.4)$$

where Γ, K are $n \times n$ friction and spring matrices and Σ is the noise matrix. The short rate is $r_t = \mathbf{e}^\top \mathbf{r}_t$ for some loading vector \mathbf{e} .

Proposition 9.3. *The multi-factor Langevin-Zamrik PDE in state $(\mathbf{r}, \mathbf{v}) \in \mathbb{R}^{2n}$ admits an affine solution $P = \exp\{A(\tau) + \mathbf{B}_r(\tau)^\top \mathbf{r} + \mathbf{B}_v(\tau)^\top \mathbf{v}\}$ when K is positive definite. The coefficient vectors satisfy a $2n \times 2n$ linear ODE system with IFS matrix*

$$\mathcal{M} = \begin{pmatrix} \mathbf{0} & -K/m \\ I & -\Gamma/m \end{pmatrix}.$$

The Vasicek recovery theorem extends: as $m \rightarrow 0$, $\mathbf{B}_v \rightarrow \mathbf{0}$ and the n -factor Vasicek model [6] is recovered.

9.4 Lévy Noise: The PIDE Extension

Replace the Brownian noise in Definition 2.1 with a Lévy process L_t having Lévy measure ν . The velocity SDE becomes

$$dv_t = \frac{1}{m} [-\gamma v_t - \partial_r U(r_t)] dt + \frac{\sigma}{m} dW_t + \frac{1}{m} dJ_t, \quad (9.5)$$

where $J_t = \int_0^t \int_{\mathbb{R}} \xi \tilde{N}(ds, d\xi)$ is the pure-jump component.

Proposition 9.4 (Langevin-Zamrik PIDE). *The bond price satisfies the partial integro-differential equation*

$$\begin{aligned} \partial_t P + v \partial_r P - \frac{\gamma v + \partial_r U(r)}{m} \partial_v P + \frac{\sigma^2}{2m^2} \partial_v^2 P \\ + \int_{\mathbb{R}} \left[P\left(t, T; r, v + \frac{\xi}{m}\right) - P - \frac{\xi}{m} \partial_v P \right] \nu(d\xi) = rP. \end{aligned} \quad (8.6)$$

The integral term captures the effect of velocity jumps of size ξ/m on bond prices. In the overdamped limit, the jumps pass through to the rate dynamics and the standard jump-diffusion bond pricing PIDE [3] is recovered.

10. Conclusion

We have introduced the Langevin-Zamrik PDE, a bond pricing equation derived from the full underdamped Langevin dynamics of the short rate. The main contributions are:

1. **A new PDE for bond pricing** in the two-dimensional state (r, v) , related to but distinct from the Kramers equation by adjoint structure and the substitution of friction by discounting.
2. **A closed-form affine solution** for the quadratic potential, with coefficients given by the matrix exponential of the inertia-friction-spring matrix M . Three damping regimes produce qualitatively distinct yield curve shapes.

3. **Recovery of Vasicek** in the overdamped limit $m \rightarrow 0$, proved rigorously via eigenvalue asymptotics and coefficient limits.
4. **A structural derivation of the Nelson-Siegel-Svensson model** from no-arbitrage principles: in the overdamped regime, the Langevin-Zamrik yield curve is the two-factor Nelson-Siegel-Svensson specification, with the two eigenvalues of M as the decay rates and the initial velocity v_0 as the hidden slope factor.
5. **A framework for extensions**: nonlinear potentials (Ait-Sahalia family), state-dependent volatility (inertial CIR), multi-factor systems, and Lévy noise (PIDE) are all natural generalisations of the Langevin-Zamrik PDE.
6. **A foundation for inertial forward-rate modelling**: since the LZ model belongs to the short-rate class, the forward curve is a derived quantity — affine in (r, v) with a damped-oscillatory HJM volatility structure $\sigma^{\text{HJM}}(\tau) = -(\sigma/m)B_2'(\tau)$ that is new to the literature. Lifting the inertial structure to the full forward curve in the Heath-Jarrow-Morton [10] spirit — a second-order SPDE for the forward rate in which the entire curve carries momentum — is reserved for future work.

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