Discussion Paper

Mathematical Finance

The Israeli Option: Doob-Meyer Decomposition and the Double Obstacle PDE

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1. Abstract

The Israeli option, introduced by Kifer (2000), is a financial contract in which both the holder and the writer possess the right to terminate the contract at any time, making it a zero-sum Dynkin game. We analyse the perpetual Israeli put on a geometric Brownian motion under the risk-neutral measure, and show that its value function satisfies a double obstacle variational inequality with two free boundaries $x_1^* < x_2^*$. In the continuation region (x_1^*, x_2^*) the value solves the Black–Scholes ODE, yielding the explicit form $V(x) = Ax + Bx^{-\alpha}$ with $\alpha = 2r/\sigma^2$. We derive the smooth-pasting system that determines all four unknowns (A, B, x_1^*, x_2^*), recover the perpetual American put as the limit $\delta \to \infty$, and identify the two Doob–Meyer compensators of the discounted value process as the local times of the stock price at each free boundary.

2. Introduction

The American put is the canonical single-player optimal stopping problem: the holder exercises at the first time the stock price falls below a free boundary x^*, and the discounted value process is a supermartingale whose Doob–Meyer compensator accumulates precisely at that boundary. The Israeli option, introduced by Kifer [1], extends this to a two-player setting. Both parties can act unilaterally: the holder exercises and receives the intrinsic value $G(x) = (K-x)^+$; the writer cancels and pays a penalised amount $H(x) = (K-x)^+ + \delta$ with penalty $\delta > 0$. The result is a zero-sum Dynkin game whose value function is sandwiched between two obstacles and satisfies a double obstacle variational inequality.

The Doob–Meyer decomposition provides the probabilistic backbone. For a supermartingale J_t, the decomposition $J_t = M_t - A_t$ identifies the martingale part M_t and the increasing compensator A_t. In optimal stopping, A_t accumulates exactly in the stopping region. For the Israeli option, the discounted value process $e^{-rt}V(X_t)$ decomposes with two compensators: one for each free boundary. The PDE formulation makes this structure transparent.

3. Setup and Notation

Definition 3.1 (Geometric Brownian motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space carrying a standard Brownian motion W_t. The risk-neutral stock price process satisfies

$$dX_t = rX_t\,dt + \sigma X_t\,dW_t, \qquad X_0 = x > 0,$$

where $r > 0$ is the risk-free rate and $\sigma > 0$ is the volatility. The generator of X acting on C^2 functions is

$$\mathcal{L} = rx\,\partial_x + \tfrac{1}{2}\sigma^2 x^2\,\partial_{xx}.$$

Definition 3.2 (Israeli put payoffs)**.** Fix strike $K > 0$ and penalty $\delta > 0$. Define

$$G(x) = (K - x)^+, \qquad H(x) = (K - x)^+ + \delta.$$

The holder's payoff upon exercising at τ is $G(X_\tau)$. The writer's payment upon cancelling at σ is $H(X_\sigma)$. Note $G(x) < H(x)$ for all $x \geq 0$.

Definition 3.3 (Dynkin game value)**.** The value of the Israeli put is

$$V(x) = \inf_{\sigma}\sup_{\tau}\,\mathbb{E}_x\left[e^{-r(\tau\wedge\sigma)}\Big(G(X_\tau)\mathbf{1}_{\{\tau\leq\sigma\}} + H(X_\sigma)\mathbf{1}_{\{\sigma<\tau\}}\Big)\right],$$

where the infimum is over $\mathcal{F}_t$-stopping times σ and the supremum is over $\mathcal{F}_t$-stopping times τ.

4. The Double Obstacle Variational Inequality

Theorem 4.1 (Double obstacle PDE)**.** *The value function $V \in C^2((0,\infty)\setminus\{x_1^*, x_2^*\}) \cap C^1((0,\infty))$ satisfies*

$$\max\{\min\{\mathcal{L}V - rV,\ V - G\},\ V - H\} = 0$$

subject to the constraints $G(x) \leq V(x) \leq H(x)$ for all $x > 0$.

Remark 4.2. The variational inequality partitions $(0,\infty)$ into three regions. In the **continuation region** $\mathcal{C} = \{G < V < H\}$ neither constraint binds and $\mathcal{L}V = rV$. On the **holder's contact set** $\mathcal{S}_1 = \{V = G\}$ the residual $\mathcal{L}V - rV \leq 0$. On the **writer's contact set** $\mathcal{S}_2 = \{V = H\}$ the residual $\mathcal{L}V - rV \geq 0$.

Proposition 4.3 (Three-region structure)**.** *There exist free boundaries $0 < x_1^* < x_2^*$ with $x_2^* > K$ such that*

$$\mathcal{S}_1 = (0, x_1^*], \qquad \mathcal{C} = (x_1^*, x_2^*), \qquad \mathcal{S}_2 = [x_2^*, \infty).$$

In $\mathcal{S}_1$ the holder exercises optimally; in $\mathcal{S}_2$ the writer cancels optimally; in $\mathcal{C}$ both parties wait.

Proof. For x small the put is deep in the money: the holder's immediate gain $K - x$ dominates the value of waiting, so $V = G$. For x large the put is worthless, but the writer faces an open liability; the flat penalty δ is preferable to an infinite horizon of holding costs, so $V = H = \delta$. The monotonicity of G and H and the continuity of V (proved via comparison for the variational inequality) imply the boundaries are singletons x_1^* and x_2^*. □

5. ODE Solution in the Continuation Region

Lemma 5.1 (Characteristic roots)**.** *For GBM under the risk-neutral measure the equation* $\mathcal{L}f = rf$*, i.e.*

$$\tfrac{1}{2}\sigma^2 x^2 f'' + rxf' - rf = 0,$$

is an Euler–Cauchy ODE. Substituting $f(x) = x^\beta$ *gives*

$$\tfrac{1}{2}\sigma^2\beta(\beta - 1) + r\beta - r = 0.$$

The discriminant equals $\left(r - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2 = \left(r + \frac{\sigma^2}{2}\right)^2$*, and the two roots are*

$$\beta_+ = 1, \qquad \beta_- = -\frac{2r}{\sigma^2} \equiv -\alpha < 0.$$

Proof. Expand: $\frac{1}{2}\sigma^2\beta^2 + (r - \frac{\sigma^2}{2})\beta - r = 0$. The positive root is $\beta_+ = \left[-(r - \frac{\sigma^2}{2}) + (r + \frac{\sigma^2}{2})\right]/\sigma^2 = 1$. The negative root is $\beta_- = \left[-(r - \frac{\sigma^2}{2}) - (r + \frac{\sigma^2}{2})\right]/\sigma^2 = -2r/\sigma^2$. □

Corollary 5.2. *In the continuation region* (x_1^*, x_2^*) *the value function takes the form*

$$V(x) = Ax + Bx^{-\alpha}, \qquad \alpha = \frac{2r}{\sigma^2},$$

for constants $A, B \in \mathbb{R}$ *determined by smooth pasting at both free boundaries.*

Remark 5.3. The basis function x reflects the linear payoff structure of the put; $x^{-\alpha}$ is the decreasing solution that captures the option's time value. Both terms are active in the game — unlike the American put, where the boundary condition at infinity forces $A = 0$.

6. The Smooth-Pasting System

Theorem 6.1 (Smooth-pasting conditions)**.** *The value function satisfies* $V \in C^1$ *at both free boundaries. This yields four equations:*

$$V(x_1^*) = K - x_1^* \qquad \textit{(value matching, lower)},$$

$$V'(x_1^*) = -1 \qquad \textit{(smooth pasting, lower)},$$

$$V(x_2^*) = \delta \qquad \textit{(value matching, upper, since } x_2^* > K\textit{)},$$

$$V'(x_2^*) = 0 \qquad \textit{(smooth pasting, upper)},$$

where $V(x) = Ax + Bx^{-\alpha}$ *and* $V'(x) = A - \alpha B x^{-\alpha-1}$.

Proposition 6.2 (Closed-form A and B). *The smooth-pasting conditions at the upper boundary determine*

$$A = \frac{x_2^{*-\alpha-1}}{x_1^{*-\alpha-1} - x_2^{*-\alpha-1}}, \qquad B = \frac{1}{\alpha\big(x_1^{*-\alpha-1} - x_2^{*-\alpha-1}\big)}.$$

Proof. From $V'(x_2^*) = 0$: $A = \alpha B x_2^{*-\alpha-1}$. From $V'(x_1^*) = -1$: $\alpha B(x_2^{*-\alpha-1} - x_1^{*-\alpha-1}) = -1$, giving B. Substituting back yields A. □

Remark 6.3. Substituting A and B into the two value-matching conditions produces a nonlinear 2×2 system in (x_1^*, x_2^*) that is solved numerically. The algorithm in Section 7 implements this via a standard root-finder.

7. Limiting Cases and the American Put

Theorem 7.1 (American put limit). *As $\delta \to \infty$ the writer's contact set $\mathcal{S}_2$ recedes to infinity: $x_2^* \to \infty$. The coefficient $A \to 0$ and the value function converges to the perpetual American put value*

$$V^{\mathrm{AP}}(x) = \begin{cases} K - x & x \le x_{\mathrm{AP}}^*, \\ Cx^{-\alpha} & x > x_{\mathrm{AP}}^*, \end{cases}$$

with the well-known free boundary

$$x_{\mathrm{AP}}^* = \frac{\alpha}{\alpha+1} K = \frac{2r}{2r + \sigma^2} K.$$

Proof. When $\delta \to \infty$ the writer never cancels voluntarily. The upper boundary $x_2^* \to \infty$, so $Bx_2^{*-\alpha} \to 0$ while $V(x_2^*) = \delta$ requires $Ax_2^* \to \delta$, giving $A \to 0$ in the limit. The single remaining free boundary is determined by the single smooth-pasting pair at x_1^*, recovering the American put. □

Proposition 7.2 (Zero-penalty limit). *As $\delta \to 0^+$ the two free boundaries merge: $x_2^* \searrow x_1^*$. The continuation region collapses and the writer cancels instantaneously whenever the holder would exercise, so $V(x) \to G(x)$ on the exercise region.*

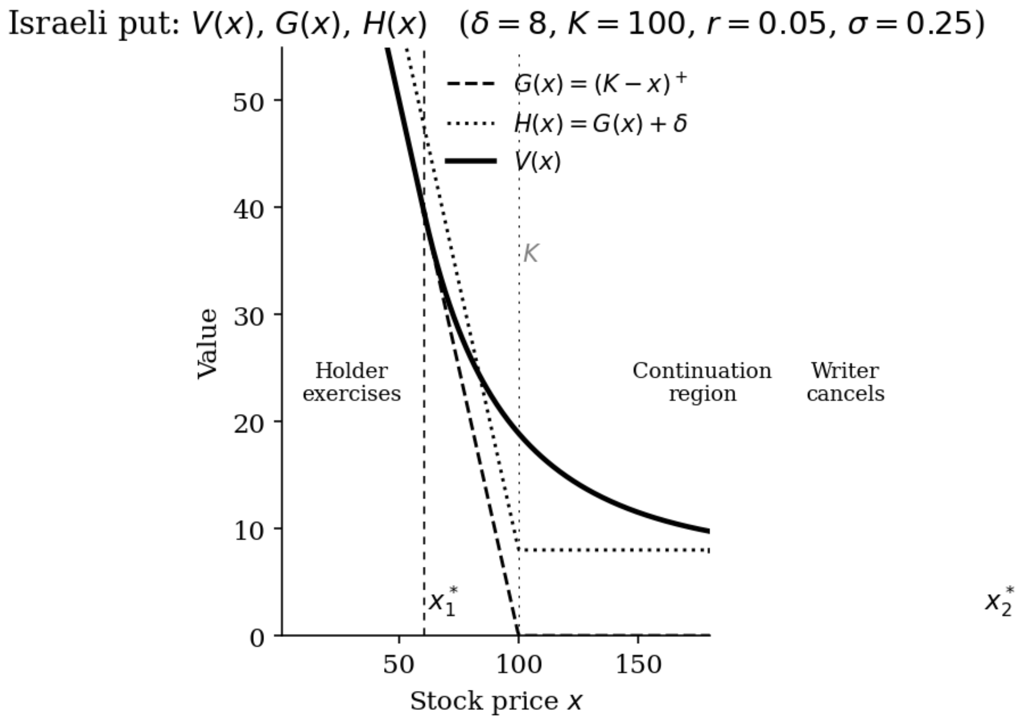

Figure 1: Value function $V(x)$ (solid) against $G(x)$ (dashed) and $H(x)$ (dotted) for $\delta = 8$, $K = 100$, $r = 0.05$, $\sigma = 0.25$. Vertical lines mark the free boundaries x_1^* and x_2^*.

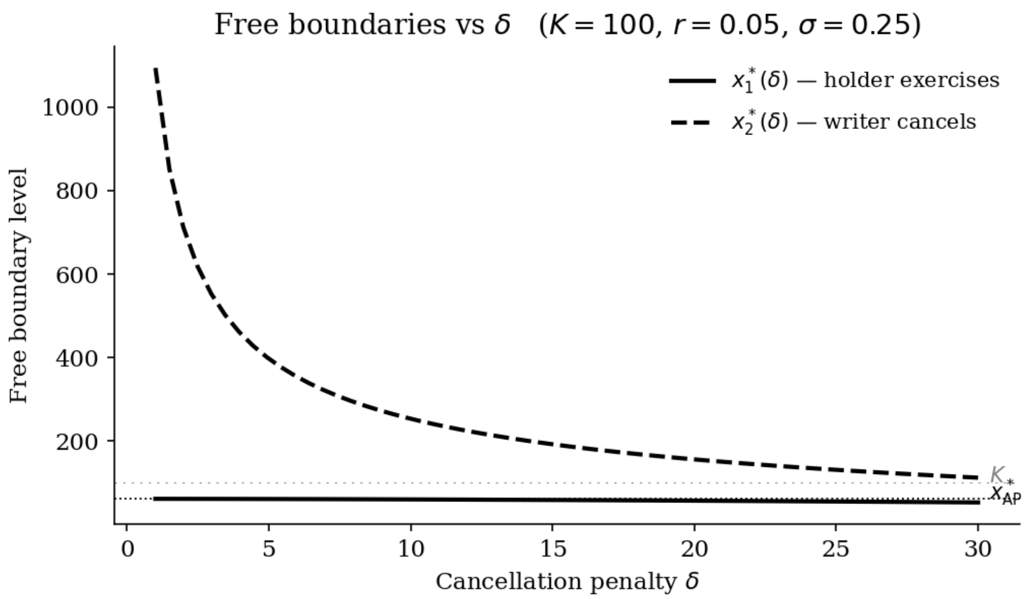

Figure 2: Free boundaries $x_1^*(\delta)$ and $x_2^*(\delta)$ as functions of the cancellation penalty δ. As $\delta \to \infty$, x_1^* converges to the perpetual American put boundary x_{AP}^* (dotted line).

8. Doob–Meyer Decomposition of the Value Process

Theorem 8.1 (Doob–Meyer structure)**.** *Let X_t evolve under the risk-neutral measure and let $\tau^* = \inf\{t : X_t \le x_1^*\}$, $\sigma^* = \inf\{t : X_t \ge x_2^*\}$ be the optimal strategies. The discounted value process $Z_t = e^{-rt}V(X_t)$ admits the decomposition*

$$Z_t = Z_0 + M_t + A_t^+ - A_t^-,$$

where $M_t = \int_0^t e^{-rs}\sigma V'(X_s)X_s\,dW_s$ is a local martingale, and A_t^+, A_t^- are continuous increasing processes satisfying

$$A_t^+ = \int_0^t e^{-rs}(rV - \mathcal{L}V)(X_s)\mathbf{1}_{\{X_s \le x_1^*\}}\,ds,$$

$$A_t^- = \int_0^t e^{-rs}(\mathcal{L}V - rV)(X_s)\mathbf{1}_{\{X_s \ge x_2^*\}}\,ds.$$

Proof. Apply Itô's formula to $e^{-rt}V(X_t)$:

$$e^{-rt}V(X_t) = V(x) + \int_0^t e^{-rs}(\mathcal{L}V - rV)(X_s)\,ds + M_t.$$

In $\mathcal{C}$, $\mathcal{L}V = rV$ so the ds integral vanishes. In $\mathcal{S}_1$, $\mathcal{L}V - rV \le 0$, contributing negatively; set $A^+ = -\int_{\mathcal{S}_1}(\mathcal{L}V - rV)\,ds \ge 0$. In $\mathcal{S}_2$, $\mathcal{L}V - rV \ge 0$; set $A^- = \int_{\mathcal{S}_2}(\mathcal{L}V - rV)\,ds \ge 0$. The decomposition follows. □

Remark 8.2. By the occupation time formula, A_t^+ and A_t^- are proportional to the local times of X at x_1^* and x_2^* respectively. Each compensator accumulates only on the contact set of the corresponding player: A^+ grows only when the holder's smooth-pasting constraint binds; A^- grows only when the writer's does. The Doob–Meyer decomposition is thus the pathwise encoding of which player is acting.

Corollary 8.3 (Optimality characterisation)**.** *The pair (τ^*, σ^*) is a saddle point of the Dynkin game if and only if $Z_{t\wedge\tau^*\wedge\sigma^*}$ is a martingale, i.e. both A^+ and A^- are flat on $[0, \tau^* \wedge \sigma^*]$. This is immediate from the definitions: A^+ grows only in $\mathcal{S}_1$ and A^- only in $\mathcal{S}_2$, and the process X started in $\mathcal{C}$ remains in $\mathcal{C}$ until it first exits.*

9. Numerical Algorithm

The free boundaries x_1^* and x_2^* satisfy the nonlinear system:

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Input: r, sigma, K, delta
Set alpha = 2*r / sigma^2

Define residuals F(x1, x2):
    denom = x1^{-alpha-1} - x2^{-alpha-1}
    B     = 1 / (alpha * denom)
    A     = alpha * B * x2^{-alpha-1}
    V(x)  = A*x + B*x^{-alpha}
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    eq1   = V(x1) - (K - x1)       [value matching at x1]
    eq2   = V(x2) - delta           [value matching at x2, x2 > K]
    return [eq1, eq2]

Initial guess: x1 = (alpha/(alpha+1))*K * 0.95
               x2 = K * 1.3

Solve F(x1, x2) = 0 via Newton / fsolve
Recover A, B from converged (x1*, x2*)
Output: x1*, x2*, A, B
        V(x) = A*x + B*x^{-alpha} on (x1*, x2*)
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10. References

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