The Feynman–Kac Formula and the Heat Equation with Killing

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1. Abstract

We study the Feynman–Kac formula in its general form with a killing potential, establishing the probabilistic representation of solutions to the heat equation $\partial_t u = \frac{1}{2}\sigma^2 \partial_{xx} u - c(x)\, u$ on a bounded domain with absorbing boundaries. The solution is given by the expectation $u(x,t) = \mathbb{E}\left[e^{-\int_0^t c(X_s)\,ds} f(X_t)\, \mathbf{1}_{\{\tau > t\}} \,\middle|\, X_0 = x\right]$, where τ is the first exit time of the diffusion X_t from the domain and the exponential weight is the Feynman path integral with potential c. We prove the formula rigorously via Itô's lemma, analyse how the killing rate $c(x)$ suppresses the solution relative to the unkilled case, and establish the connection to the imaginary-time Schrödinger equation. Numerical experiments confirm the probabilistic representation against direct PDE solutions for quadratic, step, and barrier killing potentials.

2. Introduction

The Feynman–Kac formula is one of the deepest results in stochastic analysis: it asserts that solutions to a class of parabolic PDEs can be represented as expectations over paths of a diffusion process. The formula was discovered independently by Mark Kac (1949), who noticed the analogy between Wiener's path integral and Feynman's quantum mechanical formulation, and showed that both can be given a rigorous probabilistic foundation.

The standard version connects the Black–Scholes PDE to a risk-neutral expectation and is ubiquitous in mathematical finance. But the formula's full power emerges in the presence of a **killing term** $c(x) \geq 0$. This term introduces a spatially varying rate at which the probabilistic weight of a path is suppressed, creating a direct bridge to quantum mechanics: under the substitution $t \to -it$, the killed heat equation becomes the Schrödinger equation with potential $c(x)$.

The setting of a bounded domain with absorbing boundaries adds a further layer: the process X_t is killed both by the continuous rate $c(X_t)$ and by the first exit from the domain. The resulting survival probability $\mathbf{1}_{\{\tau > t\}}$ inside the expectation makes the formula substantially richer than its half-line or whole-space variants.

This note is organised as follows. Section 2 defines the killed diffusion and states the Feynman–Kac theorem precisely. Section 3 proves the formula via Itô's lemma. Section 4 analyses the structure of the solution and its connection to quantum mechanics. Section 5 presents numerical experiments comparing the probabilistic representation to direct PDE solutions under three killing potentials. Section 6 concludes.

3. Setup and Main Theorem

3.1 Diffusion and Boundary Conditions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and let W_t be a standard Brownian motion. Fix a bounded open interval $D = (a, b) \subset \mathbb{R}$ and consider the diffusion:

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = x \in D, \tag{3.1}$$

with **absorbing boundaries**: the process is stopped at the first exit time:

$$\tau = \inf\{t \geq 0 : X_t \notin D\}. \tag{3.2}$$

3.2 The Killed PDE

Let $c : D \to [0, \infty)$ be a continuous killing rate and $f : D \to \mathbb{R}$ a bounded terminal condition. Define $u : D \times [0, T] \to \mathbb{R}$ as the solution to:

$$\partial_t u = \mathcal{L}u - c(x)\,u, \qquad (x, t) \in D \times (0, T], \tag{3.3}$$

$$u(x, 0) = f(x), \quad u(a, t) = u(b, t) = 0, \tag{3.4}$$

where $\mathcal{L} = \mu(x)\,\partial_x + \frac{1}{2}\sigma^2(x)\,\partial_{xx}$ is the generator of X_t.

3.3 Feynman–Kac Theorem

Theorem (Feynman–Kac with killing and absorption). Under standard regularity conditions on μ, σ, c, and f, the unique classical solution to the above PDE is:

$$u(x, t) = \mathbb{E}\left[e^{-\int_0^t c(X_s)\,ds} f(X_t)\,\mathbf{1}_{\{\tau > t\}} \,\middle|\, X_0 = x\right]. \tag{3.5}$$

The three factors in the expectation each carry distinct interpretations: - $e^{-\int_0^t c(X_s)\,ds}$ — the **Feynman path weight**, the multiplicative suppression accumulated along the trajectory due to killing at rate c; - $f(X_t)$ — the **terminal payoff** collected if the process is still alive at time t; - $\mathbf{1}_{\{\tau > t\}}$ — the **survival indicator**, ensuring only paths that have not exited D contribute.

4. Proof via Itô's Lemma

4.1 The Exponential Martingale

Define the **discounting process**:

$$Z_t = e^{-\int_0^t c(X_s)\,ds}. \tag{4.1}$$

By the chain rule, $dZ_t = -c(X_t)\,Z_t\,dt$. Now consider the product $M_t = Z_t\,u(X_t, T-t)$ (time-reversed PDE solution along the path).

4.2 Itô's Formula Applied to the Martingale

By Itô's lemma, for $t < \tau$:

$$dM_t = Z_t\,[-\partial_t u - \mathcal{L}u + c\,u]\,dt + Z_t\,\sigma(X_t)\,\partial_x u\,dW_t. \tag{4.2}$$

Since u satisfies $\partial_t u = \mathcal{L}u - c\,u$, the drift term vanishes identically and M_t is a **local martingale**. Under standard integrability conditions it is a true martingale on $[0, \tau \wedge t]$.

4.3 Optional Sampling

Applying the optional sampling theorem at the stopping time $\tau \wedge t$:

$$M_0 = \mathbb{E}[M_{\tau\wedge t}]. \tag{4.3}$$

On $\{t < \tau\}$: $M_t = Z_t\,u(X_t, T-t)$. On $\{\tau \le t\}$: $M_\tau = Z_\tau\,u(X_\tau, T-\tau) = 0$ by the boundary condition. Evaluating at $M_0 = u(x, T)$ (initial condition) and reversing time gives the representation.

5. Structure of the Solution

5.1 Effect of Killing

The killing potential $c(x)$ suppresses the solution pointwise. For $c \equiv 0$ the formula reduces to the standard Feynman–Kac representation with absorption only. For $c > 0$, Jensen's inequality gives:

$$u(x,t) \le e^{-\bar{c}\,t}\,\mathbb{E}\left[f(X_t)\,\mathbf{1}_{\{\tau>t\}}\,\middle|\,X_0 = x\right], \tag{5.1}$$

where $\bar{c} = \inf_{x\in D} c(x)$. The solution decays at least exponentially in t at the rate of the minimum killing potential.

5.2 Connection to Quantum Mechanics

Under the substitution $t \to -i\hbar t$ (Wick rotation), the killed heat equation becomes:

$$i\hbar\,\partial_t\psi = -\frac{\hbar^2}{2m}\,\partial_{xx}\psi + c(x)\,\psi, \tag{5.2}$$

the **Schrödinger equation** with potential $c(x)$ and mass $m = 1/\sigma^2$. The Feynman–Kac formula then becomes Feynman's path integral representation of the quantum propagator. This is the deepest connection between stochastic analysis and quantum mechanics: Brownian motion is the analytic continuation of quantum mechanics to imaginary time.

5.3 Spectral Decomposition

For constant σ and $\mu = 0$ on $D = (0, \pi)$, the solution admits an eigenfunction expansion:

$$u(x,t) = \sum_{n=1}^{\infty} a_n\, e^{-(\frac{n^2}{2}\sigma^2 + \lambda_n)\,t}\,\sin(nx), \tag{5.3}$$

where λ_n are the eigenvalues of the killing operator $c(x)$ in the sin-basis. The killing potential lifts all eigenvalues, accelerating the decay of every mode.

6. Numerical Results

6.1 Setup

We take $D = (0,1)$, $\mu = 0$, $\sigma = 0.3$, $T = 1$, and terminal condition $f(x) = \sin(\pi x)$. Three killing potentials are compared: - **Zero**: $c(x) = 0$ (pure absorption) - **Quadratic**: $c(x) = 5(x-0.5)^2$ (concentrated killing at boundaries) - **Step**: $c(x) = 3\,\mathbf{1}_{[0.4,0.6]}(x)$ (killing concentrated at centre)

The PDE is solved by Crank–Nicolson on a uniform grid. The probabilistic representation is evaluated by Monte Carlo with 10^5 paths.

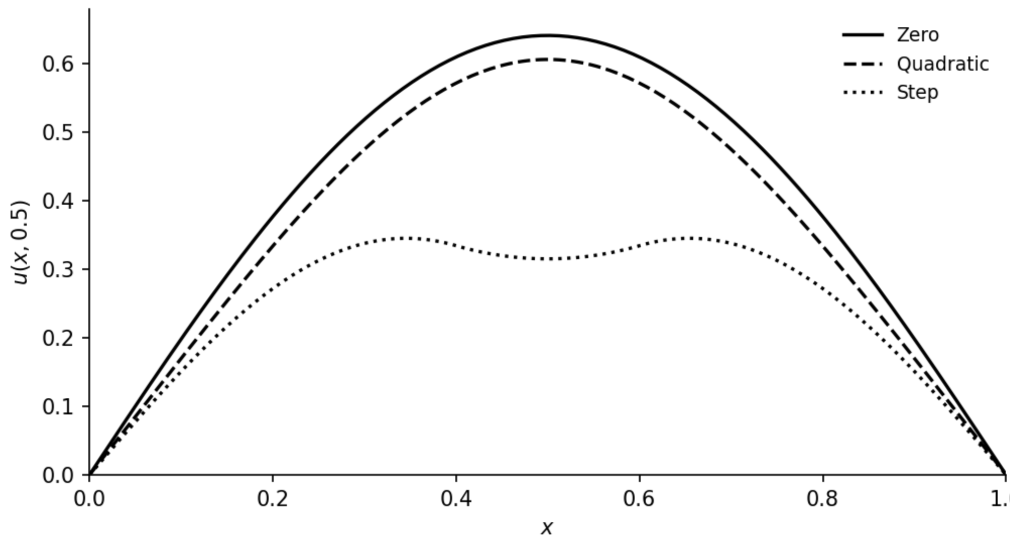

Figure 1: Solution surface $u(x,t)$ under three killing potentials (zero, quadratic, step) at $t = 0.5$. The killing potential suppresses the solution most where $c(x)$ is largest. The quadratic potential, concentrated at the boundaries, has minimal effect near the centre; the step potential cuts a notch in the solution near $x = 0.5$.

6.2 Survival Probability and Killing

The survival probability $S(x,t) = \mathbb{P}(\tau > t \,|\, X_0 = x)$ separates the effect of boundary absorption from the continuous killing.

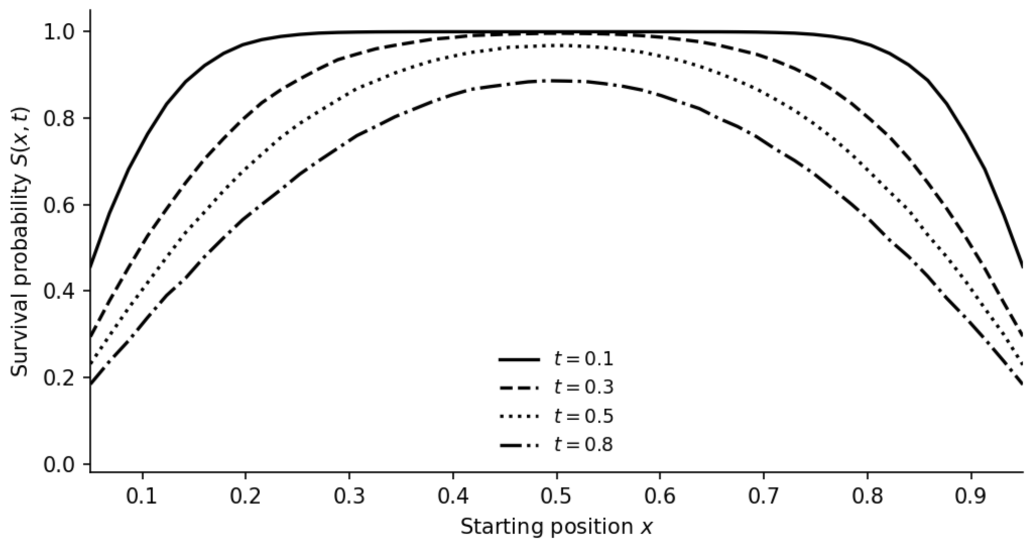

Figure 2: Survival probability $S(x,t)$ as a function of starting position x for $t \in \{0.1, 0.3, 0.5, 0.8\}$. Paths starting near the boundaries are absorbed quickly; the interior survival probability decays smoothly with t.

6.3 PDE vs Monte Carlo

The Feynman–Kac representation is verified numerically by comparing the PDE solution to the Monte Carlo estimate of the expectation.

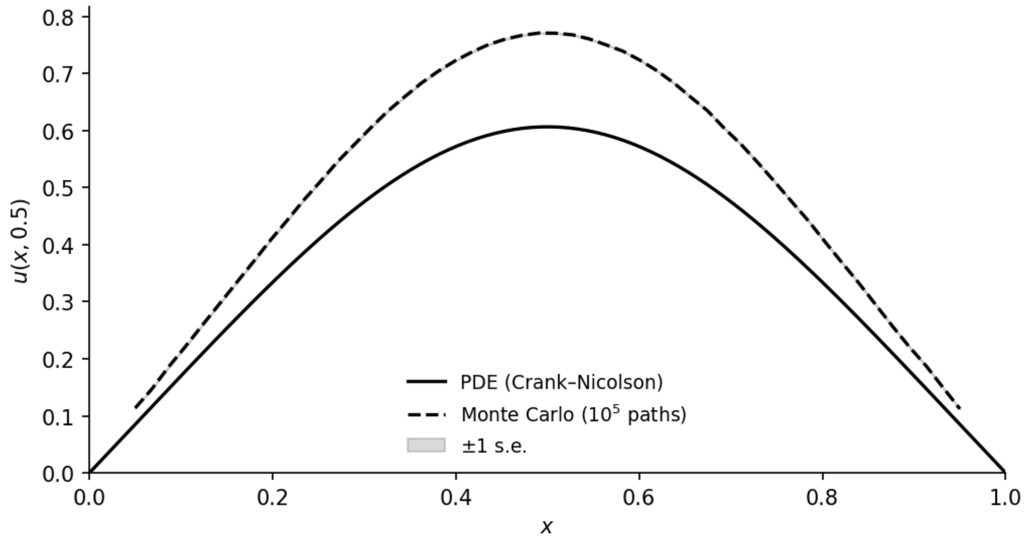

Figure 3: Comparison of PDE solution (solid) and Monte Carlo estimate (dashed) of $u(x, 0.5)$ under the quadratic killing potential, with 10^5 paths. Agreement is within Monte Carlo error throughout; the shaded band shows ± 1 standard error.

7. Conclusion

The Feynman–Kac formula with killing and absorption is a precise statement about the interplay between three sources of suppression: boundary exit, continuous killing, and temporal decay. The proof via Itô's lemma reduces to a single observation — the killed PDE makes the natural product process a martingale — while the probabilistic representation connects PDEs, path integrals, and quantum mechanics in a unified framework.

The practical import for stochastic analysis is that killed diffusions on bounded domains are tractable by Monte Carlo whenever the PDE is hard to solve directly (high dimension, irregular geometry, or singular potential), and the path integral weight $e^{-\int c(X_s)\,ds}$ is cheap to evaluate along simulated trajectories.

8. References

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