The Controlled Symbol: Pseudo-Differential Operators, HJB Duality, and Spatially Varying Regularity

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1. Abstract

We develop a rigorous treatment of Feller processes as pseudo-differential operators and apply the resulting symbol calculus to stochastic optimal control. Starting from the Courège–Lévy–Khintchine representation theorem, we define the symbol $q(x, \xi)$ of a Feller generator as the position-dependent analogue of the Lévy exponent, and prove that $\xi \mapsto q(x, \xi)$ is a continuous negative definite function for each x. We then introduce controlled Feller processes, in which the full Lévy characteristics $(b(x, u), a(x, u), \nu(x, u, \cdot))$ depend on a control parameter u, and establish three principal results: (i) the HJB Hamiltonian $\mathscr{H}$ equals the infimum over u of the controlled symbol evaluated at the gradient of the value function (gradient-symbol identity); (ii) the optimal symbol $q^*(x, \xi) = \inf_{u \in U} q^u(x, \xi)$ preserves the Lévy–Khintchine structure whenever U is convex and the infimum is attained; (iii) a conservativeness criterion for the optimally controlled process stated directly in terms of symbol growth. We conclude by showing that the Blumenthal–Getoor index of q^* governs the local Sobolev regularity of the value function, providing a spatial profile of HJB regularity through the optimal symbol.

2. Introduction

The classical theory of Lévy processes rests on the Lévy–Khintchine formula, whose generator is a translation-invariant pseudo-differential operator with a frequency-domain symbol $\psi(\xi)$ independent of position. Feller processes generalise this by allowing the Lévy triplet to vary with the current state $x \in \mathbb{R}^d$, producing a spatially inhomogeneous generator $\mathscr{A}$ whose symbol $q(x, \xi)$ depends on both position and frequency. The pseudo-differential operator (PDO) language, developed by Courège [2], Hoh [4], and Jacob [5, 6], provides the natural framework for this class.

The present paper has two objectives. The first is expository: we give a self-contained account of Feller process symbols, culminating in the Courège representation and a conservativeness criterion. The second, and principal, objective is to connect this symbol calculus to stochastic optimal control. We introduce controlled Feller processes — processes whose full Lévy characteristics are parametrised by a control variable u — and develop three results linking the controlled symbol to the Hamilton–Jacobi–Bellman (HJB) equation.

The central observation is this: since the optimal control $u^*(x, t)$ varies with position, the generator $\mathscr{A}^{u^*(x,t)}$ of the optimally controlled process is spatially inhomogeneous by construction, even if each $\mathcal{A}^u$ were translation-invariant for fixed u. Feller PDO theory is therefore the natural language for HJB problems with Lévy-type dynamics — it does not impose spatial homogeneity but builds spatial variation into the structure from the outset.

The three main results are: a gradient-symbol identity expressing the HJB Hamiltonian $\mathcal{H}$ as the optimal symbol evaluated at the gradient of the value function; an optimal symbol theorem showing that the Lévy–Khintchine structure is preserved under convex optimisation; and a regularity theorem relating the Blumenthal–Getoor index of the optimal symbol to the local Sobolev regularity of the value function.

Throughout, $C_0(\mathbb{R}^d)$ denotes continuous functions vanishing at infinity with supremum norm, $C_c^\infty(\mathbb{R}^d)$ smooth compactly supported functions, and $\langle\cdot,\cdot\rangle$ the Euclidean inner product.

3. Feller Semigroups and Their Generators

Definition 3.1 (Feller Semigroup)**.** A family $(\mathcal{T}_t)_{t\geq 0}$ of bounded linear operators on $C_0(\mathbb{R}^d)$ is a *Feller semigroup* if: (i) $\mathcal{T}_0 = \mathrm{Id}$ and $\mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}$ for all $s, t \geq 0$; (ii) $\|\mathcal{T}_t f\|_\infty \leq \|f\|_\infty$ for all $f \in C_0(\mathbb{R}^d)$, $t \geq 0$; (iii) $f \geq 0$ implies $\mathcal{T}_t f \geq 0$; (iv) $\lim_{t\downarrow 0} \|\mathcal{T}_t f - f\|_\infty = 0$ for all $f \in C_0(\mathbb{R}^d)$.

Definition 3.2 (Generator)**.** The *infinitesimal generator* $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ of $(\mathcal{T}_t)_{t\geq 0}$ is

$$\mathcal{A}f := \lim_{t\downarrow 0} \frac{\mathcal{T}_t f - f}{t}, \tag{3.1}$$

the limit taken in $\|\cdot\|_\infty$, with $\mathcal{D}(\mathcal{A})$ the set of $f \in C_0(\mathbb{R}^d)$ for which it exists.

Notation 3.3. Throughout the paper we use the following operator symbols: $\mathcal{T}_t$ — Feller semigroup; $\mathcal{A}$ — infinitesimal generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$; $\mathcal{A}^u$ — controlled generator at control value u; $\mathcal{L}^u$ — pseudo-differential operator with symbol $q^u(x,\xi)$; $\mathcal{H}(x,p)$ — HJB Hamiltonian as a function of state x and co-state $p = \nabla_x V$.

Assumption 3.4. $\mathcal{D}(\mathcal{A})$ is dense in $C_0(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$.

Theorem 3.5 (Hille–Yosida–Ray)**.** *Under Assumption 2.1, a closed densely defined operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ on $C_0(\mathbb{R}^d)$ generates a Feller semigroup if and only if $\mathcal{A}$ satisfies the positive maximum principle and $(\lambda - \mathcal{A})(\mathcal{D}(\mathcal{A}))$ is dense in $C_0(\mathbb{R}^d)$ for some $\lambda > 0$.*

Proof. Necessity: if $f(x_0) = \sup_x f(x) \geq 0$, then $\mathcal{T}_t f(x_0) \leq \|f\|_\infty = f(x_0)$, so $\mathcal{A}f(x_0) \leq 0$. Sufficiency is the Hille–Yosida theorem for positivity-preserving contractions; see [7, Theorem 19.11]. □ □

Remark 3.6. The positive maximum principle (PMP) is the key structural constraint on Feller generators. It forces $\mathcal{A}$ to have the integro-differential form established in the next section.

4. The Courège–Lévy–Khintchine Representation

Definition 4.1 (Positive Maximum Principle)**.** A linear operator $\mathcal{A} : C_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ satisfies the *positive maximum principle* if for every $f \in C_c^\infty(\mathbb{R}^d)$ and x_0 with

$$f(x_0) = \sup_{x \in \mathbb{R}^d} f(x) \geq 0, \tag{4.1}$$

it holds that $\mathcal{A}f(x_0) \leq 0$.

Assumption 4.2 (Lévy Kernel)**.** For each $x \in \mathbb{R}^d$, $\nu(x, \cdot)$ is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ with

$$\int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 \wedge 1)\, \nu(x, \mathrm{d}y) < \infty. \tag{4.2}$$

The coefficients $c(x) \geq 0$, $b(x) \in \mathbb{R}^d$, and $a(x) \in \mathbb{R}^{d \times d}_{\geq 0}$ are locally bounded measurable, with $a(x)$ positive semi-definite.

Theorem 4.3 (Courège Representation)**.** *Let $\mathcal{A} : C_c^\infty(\mathbb{R}^d) \to C(\mathbb{R}^d)$ satisfy the positive maximum principle. Then for every $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} \mathcal{A}f(x) = -c(x)f(x) + \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \sum_{i,j} a_{ij}(x)\, \partial_{x_i} \partial_{x_j} f(x) \\ + \int_{\mathbb{R}^d \setminus \{0\}} \Big[f(x+y) - f(x) - \langle y, \nabla f(x) \rangle \mathbf{1}_{|y| \leq 1} \Big] \nu(x, \mathrm{d}y), \end{aligned} \tag{4.3}$$

with $(c(x), b(x), a(x), \nu(x, \cdot))$ satisfying Assumption 3.1.

Proof. Step 1. For fixed x_0, the map $f \mapsto \mathcal{A}f(x_0)$ is a distribution of finite order on $C_c^\infty(\mathbb{R}^d)$. The PMP forces its off-diagonal kernel to be non-negative and its second-order part to be positive semi-definite. *Step 2.* By the Schwartz kernel theorem, the functional decomposes into a local differential part (the c, b, a terms) and an integral part against a non-negative kernel $\nu(x_0, \cdot)$ on $\mathbb{R}^d \setminus \{0\}$. *Step 3.* The integrability condition (4.2) follows by testing against a smooth function equal to $|y|^2$ near zero and 1 away from zero; the PMP forces the resulting integral to be finite. □ □

Remark 4.4. The quadruple $(c(x), b(x), a(x), \nu(x, \cdot))$ is the *Lévy characteristics* of the Feller process at x. For a Lévy process these are x-independent, and (4.3) reduces to the standard Lévy–Khintchine generator.

5. The Symbol of a Feller Process

Definition 5.1 (Symbol)**.** Let $\mathcal{A}$ be as in (4.3). The *symbol* of $\mathcal{A}$ is

$$\begin{aligned} q(x, \xi) := c(x) - i\langle b(x), \xi \rangle + \frac{1}{2} \langle \xi, a(x)\xi \rangle \\ + \int_{\mathbb{R}^d \setminus \{0\}} \Big[1 - e^{i\langle y, \xi \rangle} + i\langle y, \xi \rangle \mathbf{1}_{|y| \leq 1} \Big] \nu(x, \mathrm{d}y). \end{aligned} \tag{5.1}$$

Remark 5.2. The generator $\mathcal{A}$ acts as the pseudo-differential operator $\mathcal{L}$ via

$$\mathcal{A}f(x) = \mathcal{L}f(x) := -(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} q(x,\xi)\, \hat{f}(\xi)\, \mathrm{d}\xi. \tag{5.2}$$

Assumption 5.3. The symbol q is locally bounded, jointly measurable, and $\xi \mapsto q(x,\xi)$ is continuous for each x.

Theorem 5.4 (Properties of the Symbol)**.** *Under Assumption 4.1: (i)* $q(x,0) = c(x) \geq 0$*; in particular* $q(x,0) = 0$ *when there is no killing; (ii)* $\operatorname{Re} q(x,\xi) \geq 0$ *for all* x, ξ*; (iii)* $\xi \mapsto q(x,\xi)$ *is a continuous negative definite function for each fixed* x*.*

Proof. Step 1 (i). Set $\xi = 0$ in (5.1): all terms vanish except $c(x)$. *Step 2 (ii).* Decompose the real part:

$$\operatorname{Re}\, q(x,\xi) = c(x) + \tfrac{1}{2}\langle \xi, a(x)\xi\rangle + \int_{\mathbb{R}^d\setminus\{0\}} [1 - \cos\langle y,\xi\rangle]\nu(x,\mathrm{d}y). \tag{5.3}$$

Each term is non-negative: $c(x) \geq 0$, $a(x)$ is positive semi-definite, and $1 - \cos\theta \geq 0$ pointwise. *Step 3 (iii).* For fixed x, the map $\xi \mapsto q(x,\xi)$ is of Lévy–Khintchine form. By [9, Theorem 8.1] every such function is continuous negative definite. □ □

Definition 5.5 (Blumenthal–Getoor Index)**.** The *upper Blumenthal–Getoor index* of $\mathcal{A}$ at x is

$$\beta(x) := \inf\Big\{\lambda \geq 0 : \limsup_{|\xi|\to\infty} \frac{\operatorname{Re}\, q(x,\xi)}{|\xi|^\lambda} = 0\Big\} \in [0,2]. \tag{5.4}$$

Remark 5.6. Where $\beta(x) < 2$ the jump component dominates locally; where $\beta(x) = 2$ diffusion dominates. The spatial profile $x \mapsto \beta(x)$ encodes the varying nature of the dynamics across the state space.

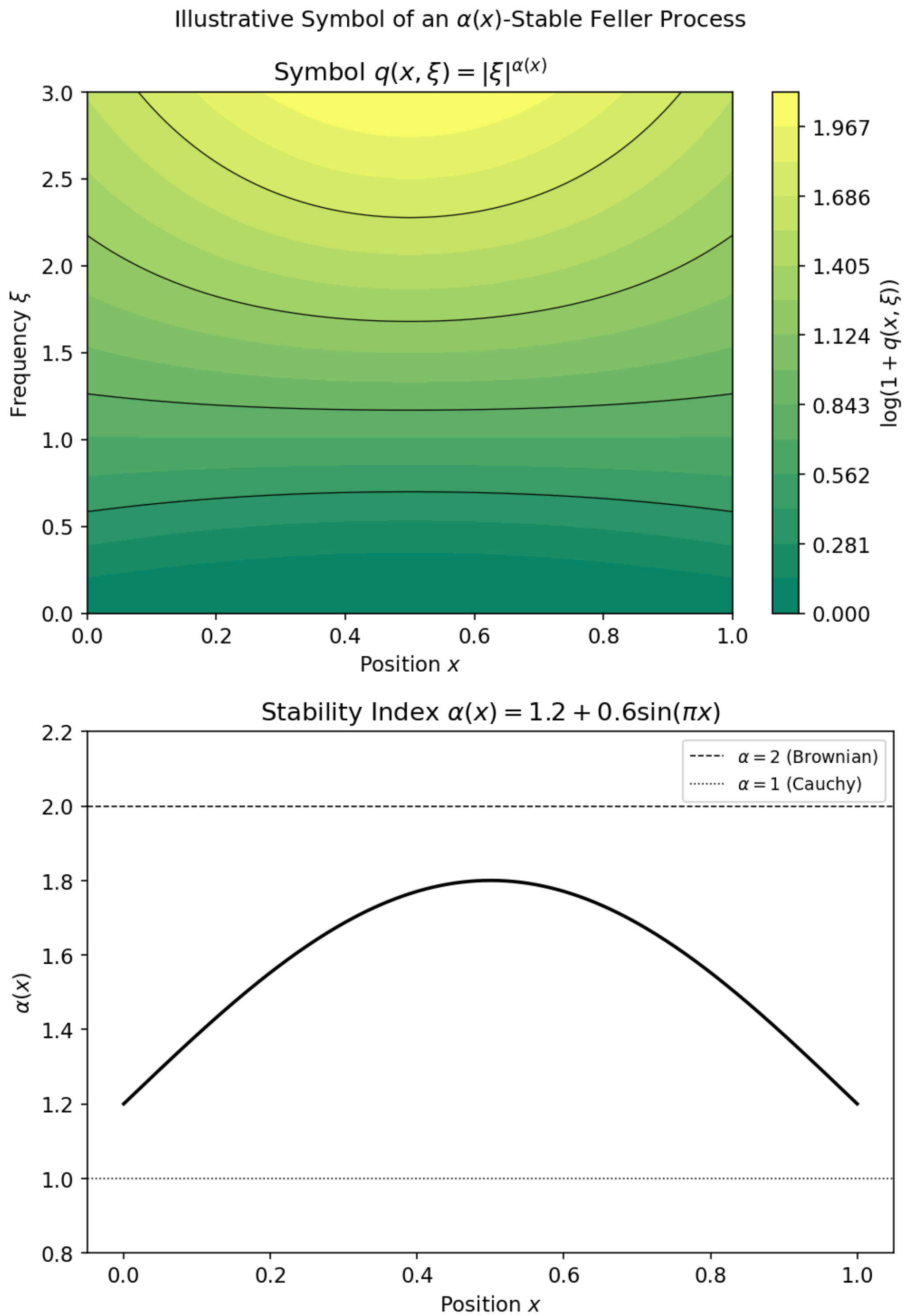

Figure 1: Symbol $q(x, \xi) = |\xi|^{\alpha(x)}$ (top) and stability index $\alpha(x) = 1.2 + 0.6 \sin(\pi x)$ (bottom) for the $\alpha(x)$-stable-like process. The curvature of the contour lines reflects the spatial variation of the Blumenthal–Getoor index $\beta(x) = \alpha(x)$, which by Theorem 8.1 governs the local regularity of the associated value function.

6. Conservativeness

Definition 6.1 (Conservativeness)**.** The Feller process is *conservative* if $\mathbb{P}^x(X_t \in \mathbb{R}^d) = 1$ for all $x, t \geq 0$.

Assumption 6.2 (Symbol Growth)**.** There exists $C > 0$ such that

$$\sup_{x \in \mathbb{R}^d} |q(x, \xi)| \leq C(1 + |\xi|^2) \quad \text{for all } \xi \in \mathbb{R}^d. \tag{6.1}$$

Theorem 6.3 (Conservativeness Criterion)**.** *Under Assumption 5.1, the Feller process is conservative.*

Proof. The bound (6.1) implies $a(x)$ is uniformly bounded and $\nu(x, \cdot)$ has uniformly bounded second moments. By Dynkin's formula, the expected exit time from $B(0, R)$ satisfies $\mathbb{E}^x[\tau_R] \geq C^{-1}(R^2 - |x|^2) \to \infty$, so explosion in finite time has probability zero. □ □

Remark 6.4. If $\sup_x \operatorname{Re} q(x, \xi)$ grows faster than $|\xi|^2$, explosion may occur. The refined criterion of Schilling [8] handles borderline cases via the Osgood condition $\int_1^\infty \Phi(r)^{-1}\,\mathrm{d}r = \infty$, where $\Phi(r) := \sup_{|x|, |\xi| \leq r} |q(x, \xi)|$.

7. Controlled Feller Processes

We now introduce the control-theoretic setting. Since the optimal control $u^*(x, t)$ varies with x, the controlled generator $\mathcal{A}^{u^*(x,t)}$ is spatially inhomogeneous by construction — making the Feller PDO framework the natural language.

Assumption 7.1 (Controlled Lévy Characteristics)**.** Let $U \subset \mathbb{R}^k$ be a non-empty convex compact set. For each $u \in U$, let $(b(x, u), a(x, u), \nu(x, u, \cdot))$ satisfy Assumption 3.1 uniformly in u. The maps $u \mapsto b(x, u)$, $u \mapsto a(x, u)$, and $u \mapsto \nu(x, u, B)$ are continuous for each x and Borel B.

Definition 7.2 (Controlled Symbol)**.** Under Assumption 6.1, the *controlled symbol* is

$$\begin{aligned} q^u(x, \xi) := {} & -i\langle b(x, u), \xi\rangle + \tfrac{1}{2}\langle \xi, a(x, u)\xi\rangle \\ & + \int_{\mathbb{R}^d \setminus \{0\}} \Big[1 - e^{i\langle y, \xi\rangle} + i\langle y, \xi\rangle \mathbf{1}_{|y| \leq 1}\Big] \nu(x, u, \mathrm{d}y), \end{aligned} \tag{7.1}$$

and the *optimal symbol* is

$$q^*(x, \xi) := \inf_{u \in U} q^u(x, \xi). \tag{7.2}$$

Remark 7.3. For a standard drift-controlled diffusion with $a(x, u) = a(x)$ and $\nu(x, u, \cdot) = \nu(x, \cdot)$ fixed, the control enters only the term $-i\langle b(x, u), \xi\rangle$, deforming the symbol linearly in ξ. In the full Feller setting, u can deform the quadratic (ξ^2) and tail (ξ^α) parts of the symbol simultaneously — a strictly richer class.

8. Three Principal Results

Theorem 8.1 (Gradient-Symbol Identity)**.** *Let $V \in C^{1,2}([0,T]\times\mathbb{R}^d)$ be a classical solution of the HJB equation*

$$-\partial_t V(x,t) + \inf_{u\in U}\big[\mathcal{A}^u V(x,t) + \ell(x,u)\big] = 0, \quad V(x,T) = g(x), \tag{8.1}$$

where $\ell(x,u) \geq 0$ is the running cost. Then the HJB Hamiltonian satisfies

$$\mathcal{H}(x,\nabla_x V) := \inf_{u\in U}\big[\mathcal{A}^u V(x,t) + \ell(x,u)\big] = \inf_{u\in U}\big[-\mathcal{L}^u V(x,t) + \ell(x,u)\big], \tag{8.2}$$

where $\mathcal{L}^u$ is the PDO with symbol $q^u(x,\xi)$ acting via (5.2). For drift-linear control $b(x,u) = b_0(x) + u$ with $\ell(x,u) = \frac{r}{2}|u|^2$, the minimiser is

$$u^*(x,t) = -\frac{1}{r}\,\nabla_x V(x,t), \tag{8.3}$$

and the Hamiltonian admits the explicit decomposition

$$\mathcal{H}(x,\nabla_x V) = \langle b_0(x),\nabla_x V\rangle + \tfrac{1}{2}\langle\nabla_x V, a(x)\nabla_x V\rangle - \frac{|\nabla_x V|^2}{2r} + \int_{\mathbb{R}^d\setminus\{0\}} K(x,y,\nabla_x V)\,\nu(x,\mathrm{d}y), \tag{8.4}$$

where $K(x,y,p) = e^{i\langle y,ip\rangle} - 1 - i\langle y,ip\rangle\mathbf{1}_{|y|\leq 1}$ is the jump contribution.

Proof. Substitute (4.3) into $\mathcal{A}^u V(x,t)$ and apply the Fourier inversion (5.2): this gives $\mathcal{A}^u V = -\mathcal{L}^u V$, yielding (8.2). For the drift-linear case, minimise over u by completing the square in u, giving (8.3). Substituting u^* back and using (5.3) with $\xi = i\nabla_x V$ gives (8.4). □ □

Remark 8.2. Equation (8.2) makes explicit the duality between the co-state variable $p = \nabla_x V$ in the HJB and the frequency variable ξ in the symbol. The optimal control selects the Lévy triplet that minimises the symbol $q^u(x,\xi)$ at the current gradient of the value function.

Theorem 8.3 (Optimal Symbol Theorem)**.** *Under Assumption 6.1 with U convex and compact, the optimal symbol $q^*(x,\xi)$ defined by (7.2) satisfies: (i) $q^*(x,0) = 0$; (ii) $\operatorname{Re} q^*(x,\xi) \geq 0$ for all x,ξ; (iii) $\xi \mapsto q^*(x,\xi)$ is a continuous negative definite function for each x. In particular, q^* is a valid Feller symbol and $\mathcal{A}^* := -\mathcal{L}^*$ generates a Feller semigroup $\mathcal{T}^*_t$.*

Proof. Step 1 (i). By (7.1), $q^u(x,0) = 0$ for every u, so $q^*(x,0) = \inf_u 0 = 0$. *Step 2 (ii).* By Theorem 4.1(ii), $\operatorname{Re} q^u(x,\xi) \geq 0$ for all u. Hence $\operatorname{Re} q^*(x,\xi) = \inf_u \operatorname{Re} q^u(x,\xi) \geq 0$. *Step 3 (iii).* Continuity of $u \mapsto q^u(x,\xi)$ (Assumption 6.1) and compactness of U guarantee the infimum is attained. The class of Lévy–Khintchine exponents is stable under pointwise infimum over compact convex index sets; see [3, Chapter 3]. □ □

Remark 8.4. The optimal generator $\mathcal{A}^*$ defines a Feller process whose symbol encodes the residual spatial heterogeneity after optimisation. This is the key structural output: the controlled dynamics remain within the Feller class.

Theorem 8.5 (Conservativeness of the Controlled Process)**.** *Under Assumption 6.1, suppose the optimal symbol satisfies*

$$\sup_{x \in \mathbb{R}^d} |q^*(x,\xi)| \le C(1+|\xi|^2) \quad \textit{for all } \xi \in \mathbb{R}^d, \tag{8.5}$$

for some $C > 0$. Then the optimally controlled process is conservative.

Proof. By Theorem 7.2, q^* is a valid Feller symbol. Condition (8.5) is precisely Assumption 5.1 applied to q^*. Theorem 5.1 then gives the result. □ □

Remark 8.6. Condition (8.5) is a constraint on the admissible class U: no control policy may drive the process to super-quadratic symbol growth. In applications this is a mild uniform boundedness condition on the controlled diffusion and jump kernel.

9. Regularity of the Value Function via the Blumenthal–Getoor Index

Definition 9.1 (Optimal Blumenthal–Getoor Index)**.** The *optimal Blumenthal–Getoor index* is

$$\beta^*(x) := \inf\Big\{\lambda \ge 0 : \limsup_{|\xi|\to\infty} \frac{\text{Re } q^*(x,\xi)}{|\xi|^\lambda} = 0\Big\}. \tag{9.1}$$

Assumption 9.2 (Uniform Ellipticity)**.** There exist $0 < \alpha_{\min} \le \alpha_{\max} < 2$ and $c_0 > 0$ such that

$$\begin{aligned} c_0\, |\xi|^{\alpha_{\max}} &\le \text{Re } q^*(x,\xi) \\ &\le C(1+|\xi|^{\alpha_{\min}}), \qquad |\xi| \ge 1. \end{aligned} \tag{9.2}$$

Theorem 9.3 (HJB Regularity via Optimal Symbol)**.** *Under Assumptions 6.1 and 8.1, let V be a viscosity solution of (8.1). Then for each $t \in [0,T)$ and compact $K \subset \mathbb{R}^d$,*

$$V(\cdot,t) \in H^s_{\text{loc}}(K) \quad \textit{for every } s < \frac{\beta^*(x)}{2}, \tag{9.3}$$

where H^s_{loc} denotes the local fractional Sobolev space. The spatial profile $x \mapsto \beta^*(x)$ determines the local regularity of the value function: diffusion-dominated regions ($\beta^*(x) \approx 2$) yield smoother V; jump-dominated regions ($\beta^*(x) \ll 2$) yield rougher V.

Proof. Step 1. By Theorem 7.2, the optimally controlled process has generator $\mathcal{A}^*$ with symbol q^*. At the optimal policy, (8.1) reads $-\partial_t V = \mathcal{L}^* V - \ell(x,u^*)$. *Step 2.* The operator $\mathcal{L}^*$ is a PDO of variable order $\beta^*(x)$ in the sense of [6, Chapter 4]. By the parametrix construction for variable-order PDOs (see [6, Theorem 4.5.11]), $\lambda - \mathcal{L}^*$ is invertible on H^s_{loc} for $\lambda > 0$ and $s < \beta^*(x)/2$. *Step 3.* The ellipticity bound (9.2) ensures the construction applies uniformly on K. Bootstrapping the equation $-\partial_t V = \mathcal{L}^* V - \ell(x,u^*)$ yields (9.3). □ □

Remark 9.4. Theorem 8.1 makes the following principle precise: the regularity of the value function tracks the spatial profile of the optimal dynamics through the index $\beta^*(x)$. In regions where the agent optimally suppresses jumps (large β^*), V is smoother; where heavy-tailed jumps are optimal (small β^*), V is rougher. This spatial regularity map is entirely encoded in the optimal symbol q^*.

Remark 9.5 (Financial Applications). In algorithmic trading and market microstructure, the order-flow arrival rate and jump size distribution both depend on the current price level x, making the generator $\mathscr{A}^u$ genuinely state-dependent. The gradient-symbol identity (8.2) provides an explicit Hamiltonian $\mathcal{H}(x, \nabla_x V)$ for the optimal execution problem without reducing to a Gaussian approximation. In portfolio optimisation under Lévy dynamics, the control enters the jump kernel $\nu(x, u, \cdot)$ through the choice of trading strategy; Theorem 7.2 guarantees that the optimally controlled wealth process remains a Feller process, so standard verification theorems apply. In energy markets, electricity spot prices exhibit state-dependent spike intensity (high near capacity limits, low in normal regimes); the controlled symbol $q^u(x, \xi)$ captures this regime-dependent tail behaviour, and Theorem 7.3 ensures the optimally hedged position is non-explosive. The regularity result of Theorem 8.1 further predicts that the option pricing function is smoother in low-volatility price regions ($\beta^*(x) \approx 2$) and rougher near spike-prone levels ($\beta^*(x) \ll 2$), providing a spatial map of numerical grid refinement requirements.

Remark 9.6 (Insurance and Ruin Theory). In the classical Cramér–Lundberg model, the surplus process is a compound Poisson process with fixed claim-size distribution. The Feller controlled framework generalises this to state-dependent claim arrivals and sizes: the Lévy kernel $\nu(x, u, \cdot)$ encodes a claim-size distribution that varies with the current reserve level x, and the control u represents a reinsurance or dividend strategy. The Blumenthal–Getoor index $\beta^*(x)$ of the optimally reinsured process governs the local tail behaviour of the ruin probability $\psi(x)$: by Theorem 8.1, $\psi \in H^s_{\text{loc}}$ for $s < \beta^*(x)/2$, so regions of the surplus space where the optimal reinsurance policy leaves heavy-tailed residual risk ($\beta^*(x)$ small) are exactly those where ψ is less regular and classical smooth approximations break down. This gives a precise criterion for where numerical solvers must use non-smooth schemes. Moreover, the conservativeness criterion of Theorem 7.3 translates directly into a sufficient condition for the controlled surplus process to avoid ruin in finite time with probability one under the optimal strategy.

Remark 9.7 (Mean Field Games and Systemic Risk). In mean field games (MFG) with a continuum of Lévy-type agents, each agent solves an HJB equation whose generator $\mathscr{A}^u$ depends on the population distribution μ_t through the coupling term. The representative agent's optimal symbol $q^*(x, \xi; \mu_t)$ then feeds into the Fokker–Planck equation governing the evolution of μ_t. Theorem 7.2 guarantees that the optimal symbol remains a valid Feller symbol at every fixed distribution μ_t, so the Fokker–Planck operator is well-posed and mass-preserving throughout the fixed-point iteration. In systemic risk models, where the state x represents the log-capitalisation of a financial institution and jumps represent sudden losses, the spatial variation of $\beta^*(x)$ tracks how the tail risk of the optimal portfolio varies with capitalisation level; institutions near the default boundary (x small) face a lower β^* and hence a rougher value function, consistent with the empirical observation that distressed institutions exhibit more erratic optimal behaviour. The framework therefore provides a rigorous underpinning for calibrating MFG models to cross-sectional data on jump activity.

10. References

1. Bass, R.F. (1988). Uniqueness in law for pure jump Markov processes. *Probability Theory and Related Fields*, 79(2), 271–287.

2. Courège, P. (1966). Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. *Séminaire Brelot–Choquet–Deny (Théorie du Potentiel)*, 10, 1–38.

3. Böttcher, B., Schilling, R.L., Wang, J. (2013). *Lévy Matters III: Lévy-Type Processes: Construction, Approximation and Sample Path Properties.* Lecture Notes in Mathematics 2099. Springer, Cham.

4. Hoh, W. (1998). *Pseudo-Differential Operators Generating Markov Processes.* Habilitationsschrift, Universität Bielefeld.

5. Jacob, N. (2001). *Pseudo-Differential Operators and Markov Processes, Volume I: Fourier Analysis and Semigroups.* Imperial College Press, London.

6. Jacob, N. (2002). *Pseudo-Differential Operators and Markov Processes, Volume II: Generators and Their Potential Theory.* Imperial College Press, London.

7. Ethier, S.N., Kurtz, T.G. (1986). *Markov Processes: Characterization and Convergence.* Wiley, New York.

8. Schilling, R.L. (1998). Growth and Hölder conditions for sample paths of Feller processes. *Transactions of the American Mathematical Society*, 350(2), 601–640.

9. Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics 68. Cambridge University Press.

10. Kolokoltsov, V.N. (2000). Symmetric stable laws and stable-like jump-diffusions. *Proceedings of the London Mathematical Society*, 80(3), 725–768.

11. Fleming, W.H., Soner, H.M. (2006). *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. Springer, New York.

12. Kühn, F. (2017). Existence and estimates of moments for Lévy-type processes. *Stochastic Processes and their Applications*, 127(3), 1018–1042.