The Three-Player War of Attrition as a Dynkin Game

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1. Abstract

We study the three-player war of attrition as a Dynkin game in continuous time, driven by a geometric Brownian motion representing market demand. Each of three symmetric firms controls a stopping time; the last firm to exit captures the entire market. Because the game is not zero-sum, pure strategy Nash equilibria generically fail to exist and equilibrium requires mixed stopping strategies. The model has three regions: an exit region below a common threshold x^*, a mixed strategy region in which firms randomise at a state-dependent hazard rate, and a certainty continuation region above the firm-specific break-even level $\bar{x}_n = c/\pi_n$. We show that in the mixed strategy region the value function satisfies $\mathcal{L}V_n - rV_n = 0$ — an ODE collapse that is the mathematical signature of indifference — with the game interaction encoded entirely in the hazard rate and the upper matching condition. The hazard rate $\lambda_n(x)$ is non-negative throughout and vanishes at $\bar{x}_n$, confirming a smooth transition to the certainty region. The recursive structure — each n-player problem uses the $(n-1)$-player value as a boundary condition — yields a tractable system solved by backward induction.

2. Introduction

The war of attrition is a paradigmatic model of strategic delay: each player incurs a flow cost while waiting for rivals to exit, and the last survivor collects a prize. The two-player case is classical in evolutionary game theory [1] and has been extensively studied in industrial organisation [2, 3]. Its continuous-time formulation as a two-player Dynkin game is well understood, and existence of Nash equilibria in mixed strategies follows from general results [4].

The three-player extension is substantially harder. With three players the game is no longer zero-sum, and the standard minimax theory does not apply. Pure strategy Nash equilibria fail in general: if any firm committed to a deterministic exit threshold, the others would free-ride by waiting one instant longer. Equilibrium therefore requires each firm to randomise its exit time, leading to a mixed strategy characterised by a state-dependent hazard rate.

The key structural insight is recursive. When the first firm exits, the game reduces to a two-player war of attrition whose value is known. The three-player problem is then solved by treating the two-player value as a boundary condition at the upper edge of the mixed strategy region.

Previous analyses of multi-player Dynkin games include [5, 6, 7]. The contribution of this note is to work out the three-region structure explicitly — exit region, mixed strategy region, and certainty continuation — and to show that the value function in the mixed region satisfies a reduced ODE whose game content is concentrated in the boundary conditions and the hazard rate.

3. Model

Let $X = (X_t)_{t\geq 0}$ be a geometric Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t, \quad X_0 = x > 0, \tag{3.1}$$

with $\mu < r$ and $\sigma > 0$, where $r > 0$ is the common discount rate. Three symmetric firms operate in the market. When $n \in \{1, 2, 3\}$ firms remain active, each active firm earns instantaneous net flow profit

$$\pi_n X_t - c, \tag{3.2}$$

where $\pi_1 > \pi_2 > \pi_3 > 0$ and $c > 0$. The process X_t represents market demand; π_n captures the per-firm share of that demand under n-firm competition; c is a fixed operating cost.

Definition 3.1 (Break-even demand)**.** For each n, define the break-even demand level $\bar{x}_n = c/\pi_n$. The flow profit $\pi_n x - c$ is negative for $x < \bar{x}_n$ and positive for $x > \bar{x}_n$. Since $\pi_1 > \pi_2 > \pi_3$, we have $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$.

Assumption 3.2 (Profitability ordering)**.** $\pi_1 > \pi_2 > \pi_3 > 0$ and $c > 0$. The monopolist is more profitable per unit of demand than the duopolist, which is more profitable than the triopolist.

Each firm i chooses a stopping time τ_i with respect to the natural filtration $(\mathcal{F}_t)$. The game ends at $\tau = \min_i \tau_i$. The firm that stops first exits and receives zero continuation value. The remaining firms play a reduced game with one fewer active player.

Definition 3.3 (Value function)**.** The expected payoff to an active firm when n firms remain, demand is at x, and all firms play symmetrically is denoted $V_n(x)$.

4. Three-Region Structure

The equilibrium has a universal three-region structure indexed by the current state x:

$$\text{Exit region: } x < x^*, \quad \text{Mixed region: } x^* \le x \le \bar{x}_n, \quad \text{Certainty region: } x > \bar{x}_n. \tag{4.1}$$

The threshold x^* is common to all n and equals the monopolist's smooth-pasting exit threshold x_1^*. The upper boundary $\bar{x}_n$ is firm-configuration-specific.

Proposition 4.1 (Common exit threshold)**.** *In the symmetric equilibrium, all n-player games share the same exit threshold $x^* = x_1^*$. Below x^*, the monopolist's continuation value is zero, so the prospect of rivals exiting has no value. No active firm has an incentive to stay.*

Proof. Suppose $n = 2$. The value of staying for an instant dt is $\lambda_2(x)(V_1(x) - V_2(x))dt + (\pi_2 x - c)dt$. If $V_1(x) = 0$ — that is, if $x \le x_1^*$ — then the exit prize vanishes and the flow payoff is $\pi_2 x - c < 0$ (since $x < \bar{x}_2$). The firm strictly prefers to exit. An identical argument applies for $n = 3$. □

The generator of X is $\mathcal{L} = \mu x \partial_x + \frac{1}{2}\sigma^2 x^2 \partial_x^2$. The characteristic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ has roots $\beta_+ > 1 > 0 > \beta_-$.

5. The Monopoly Value

The monopoly value $V_1(x)$ solves the standard single-agent exit problem. In the certainty region $x > \bar{x}_1$ the firm never exits. In the mixed region $x^* < x < \bar{x}_1$ there are no rivals and no randomisation; the firm simply optimises its exit time.

Theorem 5.1 (Monopoly value)**.** *There exists a unique threshold $x^* = x_1^*$ and a unique value function $V_1 \in C^2((x^*, \infty))$ satisfying*

$$\mathcal{L}V_1(x) - rV_1(x) + \pi_1 x - c = 0, \quad x > x^*, \tag{5.1}$$

with boundary conditions $V_1(x^*) = 0$, $\partial_x V_1(x^*) = 0$ (smooth pasting), and $V_1(x)/x \to \pi_1/(r - \mu)$ as $x \to \infty$. The solution is

$$V_1(x) = B_1 x^{\beta_-} + \frac{\pi_1}{r - \mu}x - \frac{c}{r}, \quad x > x^*, \tag{5.2}$$

where $x^* = \frac{c}{r} \cdot \frac{r-\mu}{\pi_1} \cdot \frac{\beta_-}{\beta_- - 1}$ and $B_1 = -\frac{\pi_1}{(r-\mu)\beta_-}(x^*)^{1-\beta_-}$.

6. The Two-Player Subgame

When two firms remain, the war of attrition operates in the mixed region $(x^*, \bar{x}_2)$. Above $\bar{x}_2$, the duopolist is flow-profitable and stays with certainty.

Proposition 6.1 (Two-player indifference)**.** *In the symmetric mixed-strategy equilibrium, each firm exits at state x at hazard rate $\lambda_2(x)$ chosen so that the other firm is exactly indifferent:*

$$\lambda_2(x)(V_1(x) - V_2(x)) = c - \pi_2 x. \tag{6.1}$$

The left side is the flow benefit from a rival exiting (rate times value jump); the right side is the flow loss from staying.

Theorem 6.2 (Duopoly value — three regions)**.** *The duopoly value function is:*

$$V_2(x) = \begin{cases} 0 & x \leq x^*, \\ A_2[x^{\beta_+} - (x^*)^{\beta_+ - \beta_-} x^{\beta_-}] & x^* < x \leq \bar{x}_2, \\ D_2 x^{\beta_-} + \dfrac{\pi_2}{r-\mu} x - \dfrac{c}{r} & x > \bar{x}_2, \end{cases} \tag{6.2}$$

where $A_2 > 0$ and D_2 are determined by continuity and smooth matching at $\bar{x}_2$.

Proof. In the exit region $x \leq x^*$, the result is immediate from Proposition 3.1. In the certainty region $x > \bar{x}_2$, the firm makes no exit decision (flow profit is positive, no randomisation). The ODE is $\mathcal{L}V_2 - rV_2 + \pi_2 x - c = 0$, with the x^{β_+} term suppressed by the growth condition. In the mixed region, substituting the indifference condition of Proposition 5.1 into the value equation $\mathcal{L}V_2 - rV_2 + \lambda_2(V_1 - V_2) + \pi_2 x - c = 0$ yields $\mathcal{L}V_2 - rV_2 = 0$. The general solution is $A_2 x^{\beta_+} + B_2 x^{\beta_-}$. The lower boundary condition $V_2(x^*) = 0$ gives $B_2 = -A_2(x^*)^{\beta_+ - \beta_-}$. The constants A_2 and D_2 are then uniquely determined by continuity and smooth matching at $\bar{x}_2$. □

Remark 6.3 (ODE collapse and rent dissipation). The reduction $\mathcal{L}V_n - rV_n = 0$ in the mixed region is not an accident — it is the mathematical signature of indifference in a mixed strategy equilibrium. The hazard rate is chosen precisely so that the lambda term cancels the flow loss term. Importantly, the resulting ODE does NOT imply $V_2 = 0$: the lower boundary condition $V_2(x^*) = 0$ pins one constant, but the upper matching condition at $\bar{x}_2$ (continuity and smooth derivatives with the certainty solution) pins the remaining constant $A_2 > 0$. The game interaction is thus encoded in the hazard rate $\lambda_2(x)$ and in the upper boundary conditions, not in the ODE itself.

Proposition 6.4 (Hazard rate non-negativity)**.** *The hazard rate* $\lambda_2(x) = (c - \pi_2 x)/(V_1(x) - V_2(x))$ *is non-negative on* $(x^*, \bar{x}_2)$ *and satisfies* $\lambda_2(\bar{x}_2) = 0$*.*

Proof. For $x \in (x^*, \bar{x}_2)$: the numerator $c - \pi_2 x > 0$ since $x < \bar{x}_2 = c/\pi_2$. For the denominator, $V_1(x) > V_2(x)$ follows from the fact that the monopolist earns strictly higher flow profit $\pi_1 > \pi_2$ and faces no exit competition. At $x = \bar{x}_2$: $c - \pi_2 \bar{x}_2 = 0$, so $\lambda_2 = 0$. The firm transitions smoothly from randomising to staying with certainty. □

7. The Three-Player Game

The analysis for $n = 3$ is identical in structure, using V_2 as the prize value.

Proposition 7.1 (Three-player indifference)**.** *Each firm exits at hazard rate* $\lambda_3(x)$ *such that:*

$$2\lambda_3(x)\big(V_2(x) - V_3(x)\big) = c - \pi_3 x. \tag{7.1}$$

The factor 2 reflects the fact that either of the two rivals may exit first.

Theorem 7.2 (Triopoly value — three regions)**.** *The triopoly value function is:*

$$V_3(x) = \begin{cases} 0 & x \le x^*, \\ A_3\left[x^{\beta_+} - (x^*)^{\beta_+ - \beta_-} x^{\beta_-}\right] & x^* < x \le \bar{x}_3, \\ D_3 x^{\beta_-} + \dfrac{\pi_3}{r-\mu}x - \dfrac{c}{r} & x > \bar{x}_3, \end{cases} \tag{7.2}$$

where $A_3 > 0$ and D_3 are determined by continuity and smooth matching at $\bar{x}_3 = c/\pi_3$. The constants satisfy $A_3 < A_2$, reflecting the lower competitive rents in the three-player stage.

Corollary 7.3 (Ordering of break-even thresholds)**.** *The break-even thresholds satisfy $\bar{x}_1 < \bar{x}_2 < \bar{x}_3$. The triopoly firm's mixed strategy region $(x^*, \bar{x}_3)$ contains the duopoly mixed region $(x^*, \bar{x}_2)$. A firm operating under three-firm competition endures losses over a wider range of demand before the flow profit turns positive.*

Remark 7.4 (Hazard rate non-negativity for $n = 3$). For $x \in (x^*, \bar{x}_3)$, the numerator $c - \pi_3 x > 0$ and $V_2(x) > V_3(x)$, so $\lambda_3(x) \ge 0$. At $\bar{x}_3$, $\lambda_3 = 0$: the triopoly firm transitions smoothly to certainty continuation, where all three firms are profitable and exit is suboptimal.

8. Dynkin Game Formulation

The three-player war of attrition is naturally embedded in the theory of multi-player Dynkin games [5, 6].

Definition 8.1 (Multi-player Dynkin game)**.** A three-player Dynkin game is a tuple $(\tau_1, \tau_2, \tau_3, R)$ where each τ_i is a stopping time and $R(\tau_1, \tau_2, \tau_3)$ is the payoff vector. The game ends at $\tau = \min(\tau_1, \tau_2, \tau_3)$.

Definition 8.2 (Nash equilibrium in stopping times)**.** A triple $(\tau_1^*, \tau_2^*, \tau_3^*)$ is a Nash equilibrium if, for each i,

$$\mathbb{E}[e^{-r\tau^*} V_i(\tau^*)] \ge \mathbb{E}[e^{-r\tau_i} V_i(\tau_i, \tau_{-i}^*)] \tag{8.1}$$

for all stopping times τ_i.

Theorem 8.3 (Existence of Nash equilibrium)**.** *Under Assumption 2.1, the three-player war of attrition admits a symmetric Nash equilibrium in mixed stopping strategies. The equilibrium is characterised by the common exit threshold x^*, the break-even thresholds $\bar{x}_2$, $\bar{x}_3$, and the state-dependent hazard rates $\lambda_2(x)$, $\lambda_3(x)$ derived in Sections 5 and 6.*

Remark 8.4. Uniqueness holds within the class of symmetric Markov strategies. Asymmetric equilibria may exist but are not studied here.

9. Recursive Solution Algorithm

The following algorithm computes the equilibrium value functions and hazard rates by backward induction on the number of active players.

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Algorithm: Backward Induction for the Three-Player War of Attrition

Input:
  Parameters: mu, sigma, r, pi_1, pi_2, pi_3, c
  Derived: beta_plus, beta_minus (roots of characteristic equation)
           x_bar_n = c / pi_n  (break-even thresholds)

Stage 1 - Monopoly (n = 1):
  Compute x* = (c/r) * ((r-mu)/pi_1) * (beta_minus / (beta_minus - 1))
  Compute B_1 = -(pi_1 / ((r-mu)*beta_minus)) * x*^(1 - beta_minus)
  V_1(x) = B_1 * x^beta_minus + pi_1/(r-mu)*x - c/r  for x > x*

Stage 2 - Duopoly (n = 2):
  Mixed region [x*, x_bar_2]:
    V_2(x) = A_2 * [x^beta_plus - (x*)^(beta_plus - beta_minus) * x^beta_minus]
  Certainty region [x_bar_2, inf):
    V_2(x) = D_2 * x^beta_minus + pi_2/(r-mu)*x - c/r
  Solve 2x2 system for (A_2, D_2) from:
    (i)  continuity at x_bar_2: mixed value = certainty value
    (ii) smooth matching at x_bar_2: mixed slope = certainty slope
  Hazard rate: lambda_2(x) = (c - pi_2*x) / (V_1(x) - V_2(x))
  Verify: lambda_2 >= 0 on (x*, x_bar_2),  lambda_2(x_bar_2) = 0

Stage 3 - Triopoly (n = 3):
  Mixed region [x*, x_bar_3]:
    V_3(x) = A_3 * [x^beta_plus - (x*)^(beta_plus - beta_minus) * x^beta_minus]
  Certainty region [x_bar_3, inf):
    V_3(x) = D_3 * x^beta_minus + pi_3/(r-mu)*x - c/r
  Solve 2x2 system for (A_3, D_3) from continuity + smooth matching at x_bar_3
  Hazard rate: lambda_3(x) = (c - pi_3*x) / (2*(V_2(x) - V_3(x)))
  Verify: lambda_3 >= 0 on (x*, x_bar_3),  lambda_3(x_bar_3) = 0

Output:
  V_1, V_2, V_3 - value functions on [x*, inf)
  lambda_2, lambda_3 - equilibrium hazard rates
  x*, x_bar_2, x_bar_3 - regime boundaries
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10. Figures

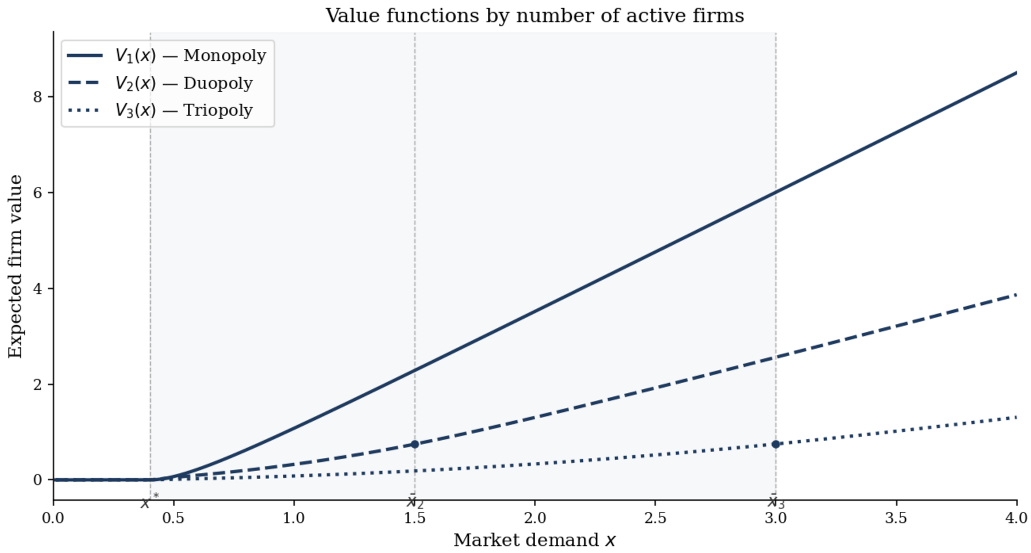

Figure 1: Value functions $V_1(x)$, $V_2(x)$, $V_3(x)$ for the calibrated model. All three vanish below x^*. In the mixed strategy region each V_n rises from zero following the $\mathcal{L}V - rV = 0$ solution, then transitions at $\bar{x}_n$ to the full ODE solution in the certainty region. The monopoly value V_1 dominates throughout; $V_2 > V_3$ for all $x > x^*$.

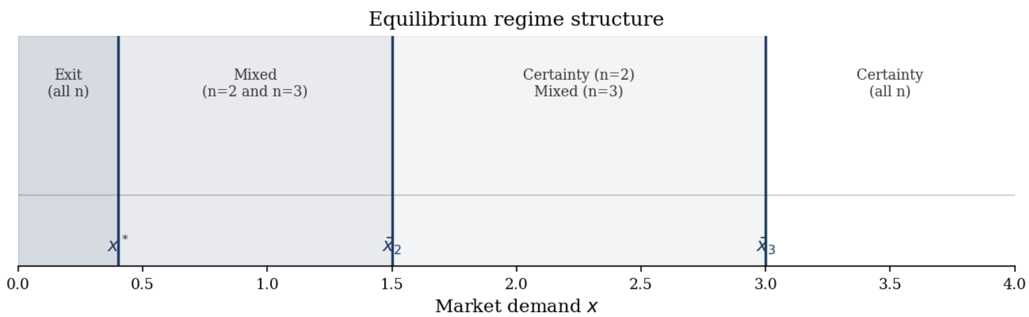

Figure 2: The three-region structure of the equilibrium. Below x^* all firms exit. Between x^* and $\bar{x}_2$, both duopolists and triopolists randomise. Between $\bar{x}_2$ and $\bar{x}_3$, a duopolist stays with certainty while a triopolist still randomises. Above $\bar{x}_3$ all active firms stay with certainty.

11. Discussion

The three-region structure derived here clarifies two points that the earlier two-region treatment obscured.

First, the ODE collapse $\mathcal{L}V_n - rV_n = 0$ in the mixed region is correctly interpreted as an indifference condition, not as rent dissipation. The value $V_n(x)$ is strictly positive for $x > x^*$ in the mixed region: the positive coefficient $A_n > 0$ is pinned by the upper matching condition, which carries the information about the value of surviving to the certainty region. The game-theoretic content — the option value of outlasting rivals — enters through A_n and through the hazard rate, not through the ODE.

Second, the relevant threshold ordering is $x^* < \bar{x}_1 < \bar{x}_2 < \bar{x}_3$. The common exit threshold x^* is determined by the monopolist's smooth-pasting condition and does not shift with the number of players. What changes across player counts is the upper boundary of the mixed region: triopolists must endure a wider range of loss-making demand before the certainty region begins, because their per-firm profit coefficient π_3 is lower.

Third, the hazard rates λ_2 and λ_3 are non-negative throughout their respective mixed regions and vanish exactly at the break-even thresholds, providing a smooth transition to certainty continuation. This confirms the internal consistency of the equilibrium.

The recursive structure extends naturally to N players. At each stage n, the value V_n is determined by a 2×2 linear system at the upper matching boundary $\bar{x}_n$, with V_{n-1} as the certainty continuation. The chain of ODEs and matching conditions is solved by backward induction from $n = 1$.

12. References

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