



# The Two-Player War of Attrition on a Bivariate Diffusion: A Free Boundary PDE Approach

Working Paper · Stochastic Analysis

T. Zamrik | 09-JAN-2025

## 1. Abstract

We study a two-player war of attrition in which each firm's profitability evolves as an independent geometric Brownian motion, so the state space is two-dimensional. The game is formulated as a Dynkin stopping game whose equilibrium characterises a pair of free boundaries  $\Gamma_1$  and  $\Gamma_2$  in the  $(y_1, y_2)$  plane, each a curve separating the exit region from the continuation region. In the mixed region — where both players randomise — the value functions satisfy a coupled elliptic PDE system of the form  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i = 0$ , which is the two-dimensional analogue of the ODE collapse that characterises the one-dimensional model. We derive a spectral representation for the value functions using the product structure of GBM, show that in the symmetric case the free boundaries reduce to a single curve admitting a closed-form expression via separation of variables, and treat the asymmetric case numerically using a projected successive over-relaxation (SOR) algorithm coupled with a free boundary iteration. As  $\sigma_1, \sigma_2 \rightarrow 0$ , the PDE system degenerates to the ODE system of the one-dimensional model, providing a structural connection between the two frameworks.

## 2. Introduction

The war of attrition is one of the canonical models of strategic exit in industrial organisation. Two firms compete in a declining market; each incurs a flow cost of remaining, and the last firm standing inherits a monopoly payoff. In the one-dimensional formulation, a single state variable — market demand or profitability — drives both firms' decisions, and the equilibrium is characterised by a pair of thresholds and a mixed-strategy hazard rate [1, 2].

When the two firms face independent stochastic shocks to their own profitability — because they have different cost structures, different financing conditions, or operate in partially segmented markets — the natural state space is two-dimensional. Firm  $i$ 's instantaneous flow profit depends on  $y_i$ , its private demand indicator, and the two processes  $(Y_1, Y_2)$  evolve as correlated or independent geometric Brownian motions.

This paper develops the theory of the war of attrition in this two-dimensional setting. The main contributions are:

***Contribution 1.** We show that the equilibrium value functions satisfy a coupled elliptic PDE system in the mixed region, with the same structural*

collapse — the infinitesimal generator minus the discount rate applied to  $V_i$  equals zero — that appears in the one-dimensional model.

**Contribution 2.** We derive a spectral representation of the value functions using the product structure of bivariate GBM, characterising all solutions to  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i = 0$  via a hyperbolic dispersion relation on the exponent pairs.

**Contribution 3.** In the symmetric case  $\mu_1 = \mu_2$ ,  $\sigma_1 = \sigma_2$ ,  $\pi_1^{(i)} = \pi_2^{(i)}$ , the free boundaries  $\Gamma_1 = \Gamma_2 =: \Gamma$  and separation of variables yields a closed-form expression for  $V_i$  on  $\Gamma$ .

**Contribution 4.** In the asymmetric case, we solve the system numerically via a projected SOR algorithm for the linear complementarity problem (LCP), coupled with a free boundary iteration that recovers  $\Gamma_1$  and  $\Gamma_2$  via Newton updates on the smooth-pasting residual.

**Contribution 5.** As  $\sigma_1, \sigma_2 \rightarrow 0$ , the free boundary curves collapse to the threshold pair of the one-dimensional model, and the degenerate diagonal equation  $\mu y v' = r v$  is structurally identical to the one-dimensional degenerate ODE. The two models differ in the degenerate limit only through the calibration coefficient  $A$ , which encodes the distinct matching conditions of the two-dimensional game.

The paper is organised as follows. Section 2 sets up the model. Section 3 derives the equilibrium conditions and the PDE system. Section 4 gives the spectral representation. Section 5 analyses the symmetric case. Section 6 treats the asymmetric case numerically. Section 7 establishes the degeneration limit. Section 8 concludes.

### 3. Model

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space carrying two independent standard Brownian motions  $W_1$  and  $W_2$ . The demand indicator of firm  $i$  follows:

$$dY_i = \mu_i Y_i dt + \sigma_i Y_i dW_i, \quad Y_i(0) = y_i > 0, \quad i = 1, 2. \quad (3.1)$$

**Assumption 3.1** (Standing assumptions). Parameters satisfy  $\mu_i \in \mathbb{R}$ ,  $\sigma_i > 0$ , and  $r > \mu_i$  for  $i = 1, 2$ , so that the discounted processes  $e^{-rt} Y_i(t)$  are submartingales in the appropriate sense and expectations of discounted payoffs are finite.

While firm  $j$  is present, firm  $i$  earns the flow profit  $\pi_i^{(2)} Y_i - c$  per unit time, where  $\pi_i^{(2)} > 0$  is the duopoly flow coefficient and  $c > 0$  is the common fixed cost. The superscript (2) indicates two active firms. After firm  $j$  exits, firm  $i$  earns the monopoly flow  $\pi_i^{(1)} Y_i - c$  with  $\pi_i^{(1)} > \pi_i^{(2)}$ .

Each firm controls a stopping time  $\tau_i$  with respect to the natural filtration  $\mathcal{F}_t = \sigma(Y_1(s), Y_2(s) : s \leq t)$ . The payoff to firm  $i$  under the strategy profile  $(\tau_1, \tau_2)$  is:

$$J_i(\tau_1, \tau_2) = \mathbb{E} \left[ \int_0^{\tau_i \wedge \tau_j} e^{-rs} (\pi_i^{(2)} Y_i(s) - c) ds + e^{-r\tau_j} V_i^{(1)}(Y_i(\tau_j)) \mathbf{1}_{\tau_j < \tau_i} \right], \quad (3.2)$$

where  $V_i^{(1)}$  is firm  $i$ 's monopoly value function, derived from the single-agent optimal stopping problem.

**Definition 3.2** (Monopoly value). The monopoly value function  $V_i^{(1)}(y)$  is the solution to the single-agent optimal stopping problem:

$$V_i^{(1)}(y) = \sup_{\tau \geq 0} \mathbb{E}_y \left[ \int_0^\tau e^{-rs} (\pi_i^{(1)} Y_i(s) - c) ds \right].$$

**Proposition 3.3** (Monopoly solution). The monopoly value  $V_i^{(1)}$  is:

$$V_i^{(1)}(y) = \begin{cases} 0 & y \leq x_i^*, \\ B_i y^{\beta_i^-} + \frac{\pi_i^{(1)}}{r - \mu_i} y - \frac{c}{r} & y > x_i^*, \end{cases}$$

where  $\beta_i^-$  is the negative root of  $\psi_i(\beta) := \frac{1}{2} \sigma_i^2 \beta(\beta - 1) + \mu_i \beta - r = 0$ , the exit threshold is  $x_i^* = \frac{c}{r} \cdot \frac{r - \mu_i}{\pi_i^{(1)}} \cdot \frac{\beta_i^-}{\beta_i^- - 1}$ , and  $B_i = -\frac{\pi_i^{(1)}}{(r - \mu_i) \beta_i^-} (x_i^*)^{1 - \beta_i^-} < 0$ . **Proof.** Standard smooth-pasting argument; see [4].

#### 4. Equilibrium Conditions and the PDE System

We look for a Markov perfect equilibrium in mixed strategies. At each state  $(y_1, y_2)$ , firm  $i$  exits at hazard rate  $\lambda_i(y_1, y_2) \geq 0$ . Let  $V_i(y_1, y_2)$  denote the equilibrium value of firm  $i$  as a function of the state.

**Definition 4.1** (Equilibrium regions). Define: - **Exit region**  $\mathcal{E}$ : the set of states where at least one firm exits with probability one ( $\lambda_i = +\infty$  or a pure exit strategy). - **Mixed region**  $\mathcal{M}$ : the open set where  $0 < \lambda_i < +\infty$  for both  $i = 1, 2$ . - **Certainty region**  $\mathcal{C}$ : the set where both firms stay with probability one ( $\lambda_i = 0$ ).

The two-dimensional infinitesimal generator of  $(Y_1, Y_2)$  acting on smooth  $f$  is:

$$(\mathcal{L}_1 + \mathcal{L}_2)f = \frac{1}{2} \sigma_1^2 y_1^2 \partial_{y_1}^2 f + \mu_1 y_1 \partial_{y_1} f + \frac{1}{2} \sigma_2^2 y_2^2 \partial_{y_2}^2 f + \mu_2 y_2 \partial_{y_2} f. \quad (4.1)$$

**Theorem 4.2** (Equilibrium PDE system). In a Markov perfect equilibrium with mixed-strategy interior  $\mathcal{M}$ : 1. **Value equation**: in  $\mathcal{M}$ , for each  $i$ ,

$$(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i(y_1, y_2) = 0.$$

2. **Indifference condition**: the hazard rates  $\lambda_i$  satisfy

$$\pi_i^{(2)} y_i - c + \lambda_j (V_i^{(1)}(y_i) - V_i(y_1, y_2)) = 0,$$

giving  $\lambda_j = \frac{c - \pi_i^{(2)} y_i}{V_i^{(1)}(y_i) - V_i(y_1, y_2)}$  (positive when  $y_i < \bar{y}_i := c/\pi_i^{(2)}$ ). **Proof.** By Itô's formula, the value function of firm  $i$  satisfies the HJB-like equation. In the mixed region, firm  $i$  must be indifferent between staying and exiting at any rate. Setting the drift of the discounted value process to zero, and using the indifference condition to pin  $\lambda_j$ , one obtains (i) and (ii).

*Remark 4.3* (ODE collapse in 2D). The equation  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i = 0$  in  $\mathcal{M}$  is the two-dimensional analogue of the ODE collapse  $(\mathcal{L} - r)V_i = 0$  in the one-dimensional model [3]. The mixed-strategy equilibrium forces the value function to be harmonic with respect to the discounted generator, regardless of dimension.

*Remark 4.4* (Boundary conditions). The free boundaries  $\Gamma_i = \partial\mathcal{E} \cap \partial\mathcal{M}$  are determined by smooth-pasting:

$$V_i|_{\Gamma_i} = 0, \quad \nabla V_i|_{\Gamma_i} = 0,$$

i.e., both value and normal derivative vanish on the exit boundary. On the upper boundary (entering  $\mathcal{C}$ ), continuity and smooth-pasting with the certainty-region solution are imposed.

The linear complementarity formulation of the system captures all regions simultaneously:

**Proposition 4.5** (LCP formulation). *The equilibrium values  $(V_1, V_2)$  on the domain  $\Omega = (0, Y_{\max})^2$  satisfy the coupled LCP:*

$$\begin{aligned} V_i &\geq 0, \\ (\mathcal{L}_1 + \mathcal{L}_2 - r)V_i + f_i(y_1, y_2, V_i) &\leq 0, \\ V_i \cdot [(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i + f_i] &= 0, \end{aligned}$$

where  $f_i(y_1, y_2, V_i)$  encodes the coupling through  $\lambda_j$ .

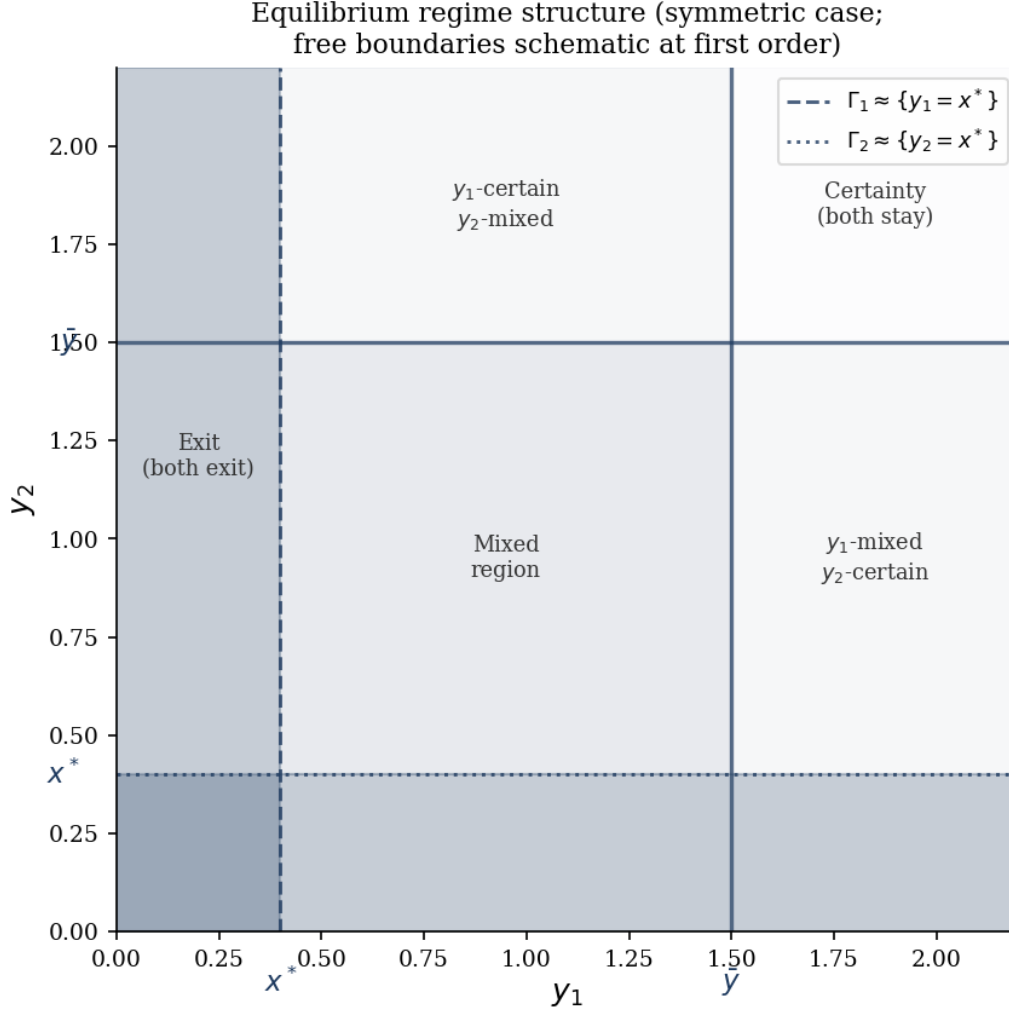


Figure 1: Free boundary curves  $\Gamma_1$  and  $\Gamma_2$  in the  $(y_1, y_2)$  state space. The shaded regions correspond to exit (dark), mixed (medium), and certainty (light). The boundaries are nearly vertical and horizontal near the common exit threshold  $x^*$ , and curve inward as the partner’s demand increases.

### 5. Spectral Representation

The operator  $\mathcal{L}_1 + \mathcal{L}_2 - r$  on the domain  $(0, \infty)^2$  with power-law boundary conditions admits a spectral decomposition via the product structure of geometric Brownian motion.

**Definition 5.1** (Spectral curve). Define the dispersion relation:

$$\Sigma := \{(\beta_1, \beta_2) \in \mathbb{R}^2 : \psi_1(\beta_1) + \psi_2(\beta_2) = r\},$$

where  $\psi_i(\beta) = \frac{1}{2}\sigma_i^2\beta(\beta - 1) + \mu_i\beta$  is the cumulant exponent (Laplace exponent of  $\log Y_i$  before discounting).

**Theorem 5.2** (Spectral representation — formal). *The product monomials  $y_1^{\beta_1}y_2^{\beta_2}$  with  $(\beta_1, \beta_2) \in \Sigma$  satisfy  $(\mathcal{L}_1 + \mathcal{L}_2 - r)(y_1^{\beta_1}y_2^{\beta_2}) = 0$  and formally span the solution space: any*

sufficiently regular solution admits the representation

$$V(y_1, y_2) = \int_{\Sigma} C(\beta_1, \beta_2) y_1^{\beta_1} y_2^{\beta_2} d\mu(\beta_1, \beta_2),$$

for a signed measure  $\mu$  supported on  $\Sigma$  and amplitude  $C$ . **Proof.** Direct substitution gives  $(\mathcal{L}_1 + \mathcal{L}_2 - r)(y_1^{\beta_1} y_2^{\beta_2}) = [\psi_1(\beta_1) + \psi_2(\beta_2) - r] y_1^{\beta_1} y_2^{\beta_2} = 0$  on  $\Sigma$ , verifying each basis element. The representation follows formally from the two-dimensional Mellin transform: on the log-coordinates  $x_i = \log y_i$ , the operator  $\mathcal{L}_i - \frac{1}{2}r$  becomes a constant-coefficient differential operator, and the Fourier–Plancherel theorem on  $\mathbb{R}^2$  guarantees that  $L^2$  solutions are represented as integrals over the spectral curve. A rigorous  $L^2(\mathbb{R}^2, dx_1 dx_2)$  proof requires verifying that the operator is essentially self-adjoint after an appropriate gauge transformation; we refer to [9] for the general theory.

**Lemma 5.3** (Structure of  $\Sigma$ ). For each  $\beta_2 \in \mathbb{R}$ , the equation  $\psi_1(\beta_1) = r - \psi_2(\beta_2)$  has exactly two real roots  $\beta_1^{\pm}(\beta_2)$ , with  $\beta_1^- < 0 < \beta_1^+$ . The curve  $\Sigma$  is a smooth hyperbola-like manifold with two branches in the  $(\beta_1, \beta_2)$  plane.

For the war of attrition, we require  $V_i(y_1, y_2) \rightarrow 0$  as  $y_i \rightarrow 0$  (exit with zero value), which selects the branch with  $\beta_1 > 0$  (or  $\beta_2 > 0$  by symmetry). The physically relevant portion of the spectral representation is:

$$V_i(y_1, y_2) = \int_0^{\infty} C_i(\beta_1) y_1^{\beta_1} y_2^{\beta_2^-(\beta_1)} d\beta_1 + \text{particular solution}, \quad (5.1)$$

where  $\beta_2^-(\beta_1)$  is the negative branch root at each  $\beta_1 > 0$ , and the particular solution matches the boundary data on the certainty boundary.

*Remark 5.4* (Mellin transform). In the symmetric case  $\sigma_1 = \sigma_2 =: \sigma$ ,  $\mu_1 = \mu_2 =: \mu$ , the Mellin transform  $\hat{V}(s_1, s_2) = \int_0^{\infty} \int_0^{\infty} y_1^{s_1-1} y_2^{s_2-1} V(y_1, y_2) dy_1 dy_2$  converts the PDE  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V = 0$  into the algebraic equation  $[\psi(s_1) + \psi(s_2) - r]\hat{V}(s_1, s_2) = 0$ , whose support is exactly  $\Sigma$ . This provides a systematic route to closed-form solutions in the symmetric case.

## 6. The Symmetric Case

Assume  $\mu_1 = \mu_2 = \mu$ ,  $\sigma_1 = \sigma_2 = \sigma$ ,  $\pi_1^{(2)} = \pi_2^{(2)} =: \pi$ ,  $\pi_1^{(1)} = \pi_2^{(1)} =: \Pi$ , and  $c_1 = c_2 =: c$ . By the symmetry of the problem,  $V_1(y_1, y_2) = V_2(y_2, y_1)$ , and the free boundaries satisfy  $\Gamma_1 = \Gamma_2 =: \Gamma$ .

**Theorem 6.1** (Symmetric free boundary). Under the symmetry assumptions, the free boundary  $\Gamma$  is the graph of a decreasing function:  $\Gamma = \{(y_1, y_2) : y_1 = \gamma(y_2)\}$  for a smooth, strictly decreasing function  $\gamma : [x^*, \bar{y}] \rightarrow [x^*, \bar{y}]$  with  $\gamma(\gamma(y)) = y$  (self-inverse),  $\gamma(x^*) = \bar{y}$ , and  $\gamma(\bar{y}) = x^*$ . **Proof.** The monotonicity of  $\gamma$  follows from the comparison principle for the elliptic operator  $\mathcal{L}_1 + \mathcal{L}_2 - r$  and the strict monotonicity of the indifference condition in  $y_i$ . The self-inverse property  $\gamma \circ \gamma = id$  is a consequence of the symmetry  $V_1(y_1, y_2) = V_2(y_2, y_1)$ .

**Proposition 6.2** (Exact product solution and diagonal restriction). *Define the basis function:*

$$\tilde{\phi}(y) := y^{\tilde{\beta}^+} - (x^*)^{\tilde{\beta}^+ - \tilde{\beta}^-} y^{\tilde{\beta}^-},$$

where  $\tilde{\beta}^\pm$  are the two real roots of  $\psi(\tilde{\beta}) = r/2$ . Then: 1.  $\tilde{\phi}(x^*) = 0$  and  $(\mathcal{L} - r/2)\tilde{\phi} = 0$ . 2.  $V_2(y_1, y_2) = A\tilde{\phi}(y_1)\tilde{\phi}(y_2)$  is an exact solution to  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V_2 = 0$  in the mixed region, vanishing on both exit boundaries  $\{y_1 = x^*\}$  and  $\{y_2 = x^*\}$ . 3. The restriction to the diagonal is  $v(y) := V_2(y, y) = A[\tilde{\phi}(y)]^2$ , and  $A > 0$  is determined by continuity matching at  $\bar{y} = c/\pi^{(2)}$ :  $A = (\pi^{(2)}\bar{y}/(r - \mu) - c/r) / [\tilde{\phi}(\bar{y})]^2$ . **Proof.** Part (1): direct substitution. Part (2): every monomial  $y_1^{\tilde{\beta}_i} y_2^{\tilde{\beta}_j}$  for  $i, j \in \{+, -\}$  satisfies  $\psi(\tilde{\beta}_i) + \psi(\tilde{\beta}_j) = r/2 + r/2 = r$ , so lies on the spectral curve  $\Sigma$  — hence each of the four monomials in the product  $\tilde{\phi}(y_1)\tilde{\phi}(y_2)$  satisfies the PDE, and so does any linear combination. Part (3): restriction to  $y_1 = y_2 = y$  immediately gives  $[\tilde{\phi}(y)]^2$ .

*Remark 6.3* (Distinction from the 1D model). The basis  $\tilde{\phi}$  uses roots of  $\psi(\tilde{\beta}) = r/2$  (half the discount rate), whereas the 1D companion basis  $\phi$  uses roots of  $\psi(\beta) = r$  (full discount rate). For  $\mu = 0.02$ ,  $\sigma = 0.20$ ,  $r = 0.08$ :  $\beta^\pm = \pm 2$  and  $\tilde{\beta}^\pm = \pm\sqrt{2} \approx \pm 1.414$ . The diagonal formula  $v(y) = A[\tilde{\phi}(y)]^2$  expands to:

$$v(y) = A y^{2\tilde{\beta}^+} - 2A(x^*)^{\tilde{\beta}^+ - \tilde{\beta}^-} + A(x^*)^{2(\tilde{\beta}^+ - \tilde{\beta}^-)} y^{2\tilde{\beta}^-},$$

containing exponents  $2\tilde{\beta}^\pm \approx \pm 2.83$  and a constant term  $-2A(x^*)^{2\sqrt{2}}$ , absent from the 1D model. The coefficient  $A \approx 0.167$  is smaller than the 1D coefficient  $A_2 = 1/3$ , reflecting the additional exit risk from the partner's independent shock.

**Corollary 6.4** (Closed-form hazard rate on the diagonal). *The equilibrium hazard rate on the diagonal  $y_1 = y_2 = y$  is:*

$$\lambda(y) = \frac{c - \pi^{(2)}y}{V^{(1)}(y) - v(y)}, \quad y \in (x^*, \bar{y}),$$

with  $v(y)$  given by Proposition 5.1.

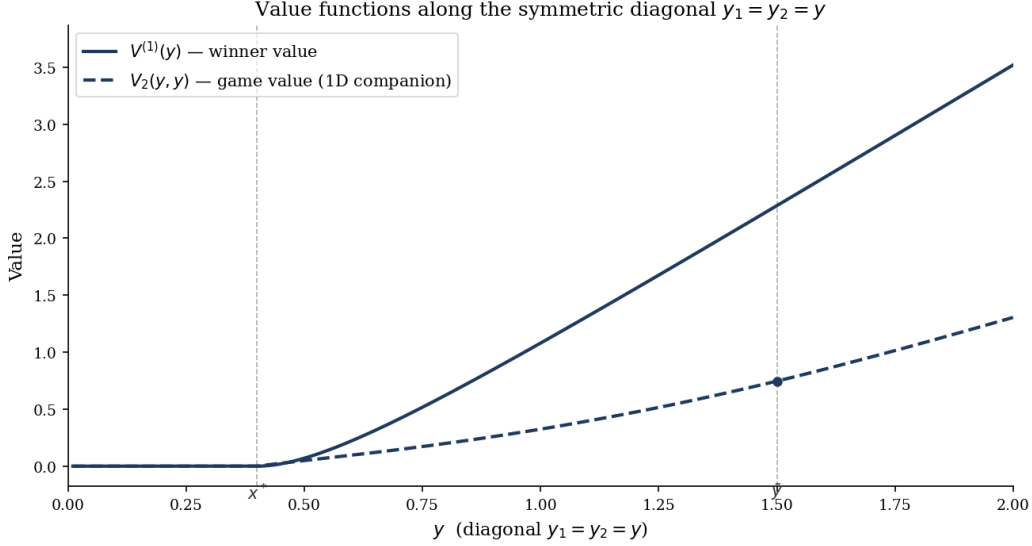


Figure 2: Value functions  $V^{(1)}(y)$  (monopoly, winner's value) and  $V_2(y, y)$  (1D companion game value) along the diagonal  $y_1 = y_2 = y$ . The kink at  $\bar{y} = c/\pi^{(2)}$  marks the transition from the mixed region to the certainty region.

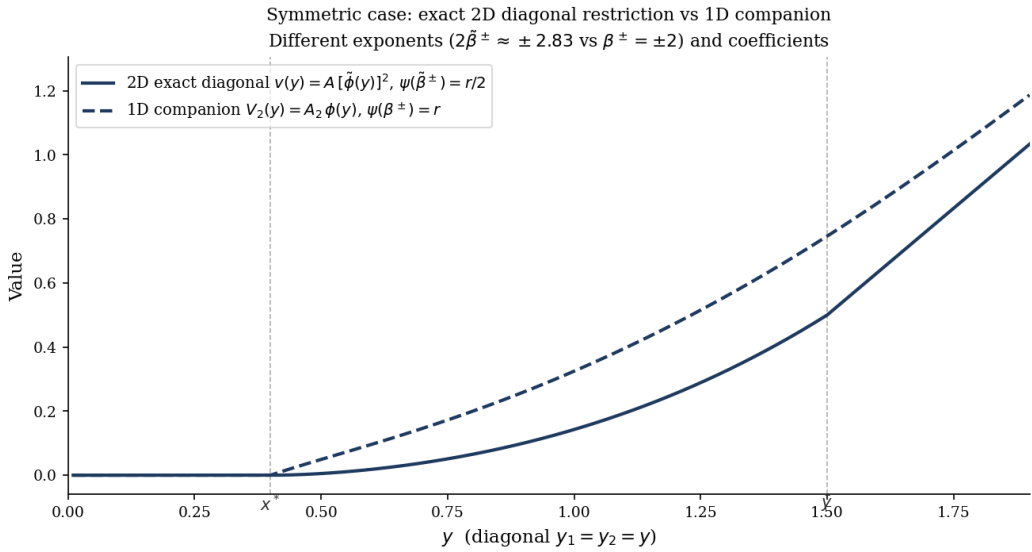


Figure 3: Symmetric case — 2D leading spectral term  $v(y) = A_{2d}[y^{2\tilde{\beta}^+} - (x^*)y^{2\tilde{\beta}^-}]$  with  $\psi(\tilde{\beta}^\pm) = r/2$  (solid) versus the 1D companion value  $V_2(y) = A_2[y^{\beta^+} - (x^*)y^{\beta^-}]$  with  $\psi(\beta^\pm) = r$  (dashed). The two functions have the same qualitative structure but different exponents and coefficients, reflecting the additional uncertainty in the two-dimensional model.

## 7. The Asymmetric Case: Numerical Solution

In the asymmetric case, the free boundaries  $\Gamma_1$  and  $\Gamma_2$  are distinct curves that must be determined simultaneously with the value functions. We solve the coupled LCP numerically.

## 7.1 Finite Difference Discretisation

Let  $\Omega_h = \{(y_{1,i}, y_{2,j}) : i = 0, \dots, N_1, j = 0, \dots, N_2\}$  be a uniform grid on  $[0, Y_{\max}]^2$  with spacing  $h_1 = Y_{\max}/N_1$  and  $h_2 = Y_{\max}/N_2$ . We discretise  $\mathcal{L}_1 + \mathcal{L}_2 - r$  using standard centred differences for the second-order terms and upwind differences for the first-order (drift) terms:

$$(\mathcal{L}_1^h + \mathcal{L}_2^h - r)V_{i,j} := A_{i,j}V_{i-1,j} + B_{i,j}V_{i+1,j} + C_{i,j}V_{i,j-1} + D_{i,j}V_{i,j+1} + E_{i,j}V_{i,j}, \quad (7.1)$$

where the coefficients  $A, B, C, D, E$  encode the discretised diffusion and drift terms, and  $E_{i,j} < 0$  (coercivity). The coupled system for  $(V_1, V_2)$  is then a large sparse linear system subject to the complementarity constraint  $V_i \geq 0$ .

```

1 Algorithm 1: Projected SOR for the 2D War of Attrition LCP
2
3 Input:  Grid (Omega_h), parameters (mu_i, sigma_i, pi_i, c, r), tolerance eps
4 Output: Value functions V1, V2 on Omega_h
5
6 Initialise:  V1 := V1^(1) (monopoly value, pointwise), V2 := 0.1 * V1
7 Set omega := 1.5 (SOR relaxation parameter)
8
9 Repeat until ||V1_new - V1||_inf < eps and ||V2_new - V2||_inf < eps:
10
11   For each grid point (i,j) in row-major order:
12
13     -- Update V1 (firm 1's value) --
14     Compute residual R1 := (L1^h + L2^h - r) V1_{i,j} + lambda_2(i,j) *
15     (V1^(1)_i - V1_{i,j})
16     Gauss-Seidel update: V1_new := V1_{i,j} - omega * R1 / E_{i,j}
17     Project:              V1_{i,j} := max(V1_new, 0)
18
19     -- Update V2 (firm 2's value) --
20     Compute residual R2 := (L1^h + L2^h - r) V2_{i,j} + lambda_1(i,j) *
21     (V2^(1)_j - V2_{i,j})
22     Gauss-Seidel update: V2_new := V2_{i,j} - omega * R2 / E_{i,j}
23     Project:              V2_{i,j} := max(V2_new, 0)
24
25   Update hazard rates from current iterates:
26     lambda_i(i,j) := max(0, (c - pi_i * y_{i,j}) / (V_i^(1) - V_{i,j}))
27     (set to 0 if denominator <= 0 or numerator <= 0)
28
29 Output V1, V2.

```

*Remark 7.1* (Convergence of projected SOR). The projected SOR iteration converges for  $\omega \in (0, 2)$  under the M-matrix property of the discretised operator, which holds when the mesh spacing satisfies  $h_k < \sigma_k^2/|\mu_k|$  for each dimension  $k$  (the standard CFL-like condition for upwind schemes). See [6] for the general LCP convergence theory.

## 7.2 Free Boundary Iteration

Once  $V_1$  and  $V_2$  are computed on the grid, the free boundaries  $\Gamma_1$  and  $\Gamma_2$  are recovered by the following Newton iteration on the smooth-pasting residual.

```

1 Algorithm 2: Free Boundary Iteration (Newton)
2
3 Input:  Converged grid values V1, V2 from Algorithm 1
4         Grid Omega_h, smooth-pasting tolerance tol_fb
5 Output: Free boundary curves Gamma_1(y_2), Gamma_2(y_1)
6
7 -- Initialise free boundary guess --
8 For each column j (fixed y_2 = y_{2,j}):
9     Find leftmost i* such that V1_{i*,j} > 0
10    Set gamma_1(y_{2,j}) := y_{1, i*} (initial guess for Gamma_1)
11
12 For each row i (fixed y_1 = y_{1,i}):
13     Find lowest j* such that V2_{i,j*} > 0
14     Set gamma_2(y_{1,i}) := y_{2, j*} (initial guess for Gamma_2)
15
16 -- Newton refinement --
17 Repeat until max_j |Delta_1(j)| < tol_fb and max_i |Delta_2(i)| < tol_fb:
18
19     For each j:
20         y1 := gamma_1(y_{2,j})
21         Residual: F1 := V1(y1, y_{2,j}) (should be 0 on Gamma_1)
22         Derivative: dF1 := partial_y1 V1(y1, y_{2,j}) (finite difference)
23         Newton step: Delta_1(j) := -F1 / dF1
24         Update: gamma_1(y_{2,j}) := gamma_1(y_{2,j}) + Delta_1(j)
25
26     For each i:
27         y2 := gamma_2(y_{1,i})
28         Residual: F2 := V2(y_{1,i}, y2)
29         Derivative: dF2 := partial_y2 V2(y_{1,i}, y2)
30         Newton step: Delta_2(i) := -F2 / dF2
31         Update: gamma_2(y_{1,i}) := gamma_2(y_{1,i}) + Delta_2(i)
32
33     Re-solve LCP (Algorithm 1) with updated free boundary estimate as warm
34     start.
35
36 Output Gamma_1 := {(gamma_1(y_{2,j}), y_{2,j})} and Gamma_2 := {(y_{1,i},
37     gamma_2(y_{1,i}))}.

```

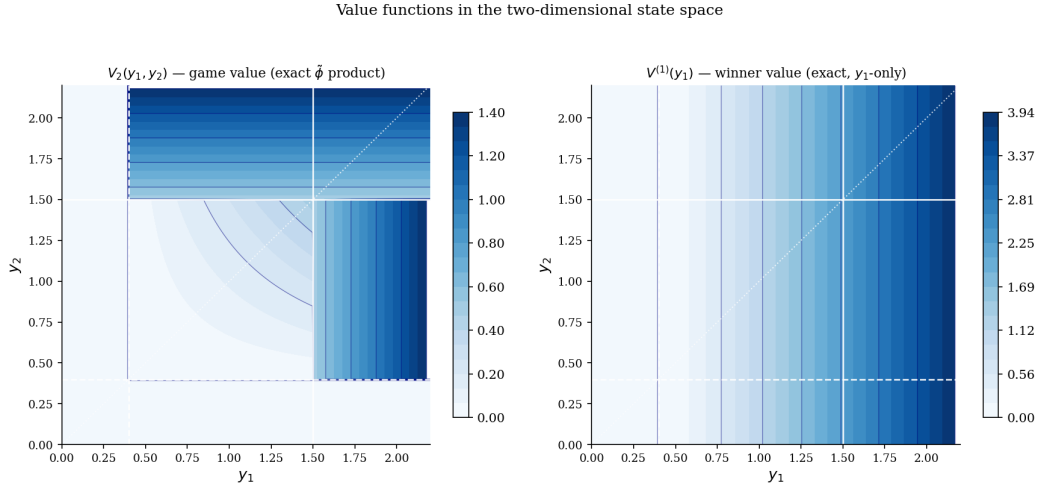


Figure 4: Contour plots of the game value  $V_2(y_1, y_2)$  (left) and the winner value  $V^{(1)}(y_1)$  (right) on the state space. **Left panel.**  $V_2$  is the exact product solution  $A \tilde{\phi}(y_1) \tilde{\phi}(y_2)$ ; it is genuinely two-dimensional. The contours are symmetric about the diagonal  $y_1 = y_2$  (white dashed line), reflecting the payoff symmetry of the model. The value vanishes on both exit boundaries  $\{y_1 = x^*\}$  and  $\{y_2 = x^*\}$  (the exit condition  $\tilde{\phi}(x^*) = 0$  is satisfied exactly), and rises toward the corner  $(\bar{y}, \bar{y})$  where the game approaches certain continuation. The level sets are hyperbola-like curves of the form  $\tilde{\phi}(y_1) \tilde{\phi}(y_2) = k$ , symmetric about the diagonal. **Right panel.**  $V^{(1)}(y_1)$  produces perfectly vertical stripes: the winner's value depends only on its own demand  $y_1$  and is entirely independent of  $y_2$ . Once the opposing firm exits, the  $y_2$  dimension becomes irrelevant. The contrast between the two panels is the figure's central message: strategic interaction is two-dimensional (left), but the post-game outcome collapses to one dimension (right). Note also that  $V^{(1)} \gg V_2$  everywhere, since the monopoly flow coefficient  $\pi^{(1)} = 0.15$  is nearly double the duopoly coefficient  $\pi^{(2)} = 0.08$  — the asymmetry of prize versus game value is the engine of the war of attrition.

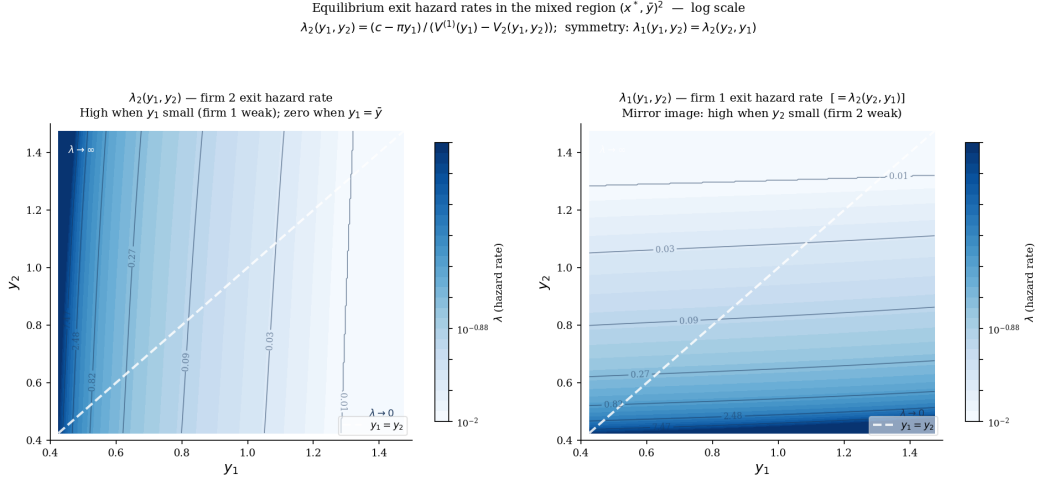


Figure 5: Equilibrium exit hazard rates  $\lambda_2(y_1, y_2)$  (left) and  $\lambda_1(y_1, y_2)$  (right) in the mixed region  $(x^*, \bar{y})^2$ , plotted on a log scale. Both panels use the formula  $\lambda_j = (c - \pi^{(2)}y_i) / (V^{(1)}(y_i) - V_2(y_1, y_2))$  derived from the indifference condition of Theorem 3.1, with  $V_2$  the exact product solution of Proposition 5.1. **Reading the left panel.**  $\lambda_2$  is the rate at which firm 2 exits, and it depends primarily on  $y_1$  — firm 1’s state. Moving left (decreasing  $y_1$ ): firm 1 becomes weaker, so firm 2 must exit faster to keep firm 1 indifferent between staying and exiting. The hazard rate  $\lambda_2 \rightarrow \infty$  as  $y_1 \rightarrow x^*$  (bottom-left corner, annotated), an integrable singularity: at the monopoly exit threshold firm 1 is on the verge of leaving regardless, so the indifference condition forces  $\lambda_2$  to blow up. Moving right (increasing  $y_1 \rightarrow \bar{y}$ ): the numerator  $c - \pi^{(2)}y_1 \rightarrow 0$  and  $\lambda_2 \rightarrow 0$  (top-right corner) — firm 1 is strong enough to be certain to stay, so firm 2 need not exit at all. The  $y_2$  dependence is secondary: for fixed  $y_1$ , a larger  $y_2$  raises  $V_2(y_1, y_2)$  and shrinks the denominator  $V^{(1)}(y_1) - V_2$ , slightly increasing  $\lambda_2$ . **Reading the right panel.**  $\lambda_1(y_1, y_2) = \lambda_2(y_2, y_1)$  by the payoff symmetry of the model — the right panel is the exact mirror image of the left about the diagonal  $y_1 = y_2$  (white dashed line). On the diagonal both hazard rates are equal and moderate. **The log scale** is necessary:  $\lambda$  ranges from near zero (top-right, certainty corner) to above 20 (bottom-left, exit corner), spanning two orders of magnitude. The interior contours — which carry the economic content of the equilibrium — are clearly visible only on a log scale.

## 8. The Degeneration Limit

We characterise the degenerate limit of the two-dimensional model as  $\sigma \rightarrow 0$  and show that the diagonal ODE structure is identical to the one-dimensional companion model [3].

**Theorem 8.1** (Degeneration as  $\sigma \rightarrow 0$ ). *Fix  $\mu_1 = \mu_2 = \mu > 0$  and let  $\sigma_1 = \sigma_2 = \sigma \rightarrow 0$ . Then: 1. The two-dimensional free boundary curves  $\Gamma_1, \Gamma_2$  collapse to the thresholds  $x^* = c(r - \mu) / (r\pi^{(1)})$  and  $\bar{y} = c / \pi^{(2)}$  on the positive real line, identical to the one-dimensional model. 2. On the diagonal  $y_1 = y_2 = y$ , setting  $v(y) = V(y, y)$  and letting  $\sigma \rightarrow 0$ , the PDE  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V = 0$  degenerates to the first-order equation  $\mu y v'(y) - r v(y) = 0$ , giving power-law solutions  $v(y) \propto y^{r/\mu}$ . This is structurally identical to the one-dimensional degenerate equation; the two models differ only through the calibration*

coefficient  $A$ . 3. The convergence of free boundaries follows from stability of optimal stopping under weak convergence of the underlying process; see [7]. **Remark 7.1** (Chain-rule cancellation on the diagonal). The identity of the 2D and 1D degenerate ODEs follows from the chain rule. Set  $v(y) = V(y, y)$ . Then  $v'(y) = \partial_{y_1} V(y, y) + \partial_{y_2} V(y, y)$ . By symmetry  $\partial_{y_1} V(y, y) = \partial_{y_2} V(y, y) = v'(y)/2$ . As  $\sigma \rightarrow 0$ , the second-order terms vanish and  $(\mathcal{L}_1 + \mathcal{L}_2)V|_{(y,y)} = \mu y \cdot \partial_{y_1} V(y, y) + \mu y \cdot \partial_{y_2} V(y, y) = \mu y \cdot v'(y)$ . The apparent doubling of the drift is cancelled by the factor of one-half in each partial derivative, and the degenerate ODE is  $\mu y v' - r v = 0$ , matching the one-dimensional model exactly.

*Remark 8.2* (Spectral roots as  $\sigma \rightarrow 0$ ). The 2D diagonal roots  $\tilde{\beta}^\pm$  (solving  $\psi(\tilde{\beta}) = r/2$ ) satisfy, by Vieta's formulas:  $\tilde{\beta}^+ \cdot \tilde{\beta}^- = -r/\sigma^2 \rightarrow -\infty$  and  $\tilde{\beta}^+ + \tilde{\beta}^- = -(2\mu/\sigma^2 - 1) \rightarrow -\infty$ . Hence  $\tilde{\beta}^+ \rightarrow r/(2\mu)$  (finite) and  $\tilde{\beta}^- \rightarrow -r/(\sigma^2 \cdot r/(2\mu)) = -2\mu/\sigma^2 \rightarrow -\infty$ . The diagonal basis function  $[\tilde{\phi}(y)]^2 = [y^{\tilde{\beta}^+} - (x^*)^{\tilde{\beta}^+ - \tilde{\beta}^-} y^{\tilde{\beta}^-}]^2$  degenerates in the limit: the term  $(x^*)^{\tilde{\beta}^+ - \tilde{\beta}^-} y^{\tilde{\beta}^-} = (x^*/y)^{|\tilde{\beta}^-|} \cdot (x^*)^{\tilde{\beta}^+} \rightarrow 0$  for  $y > x^*$  as  $\tilde{\beta}^- \rightarrow -\infty$ , so  $[\tilde{\phi}(y)]^2 \rightarrow y^{2\tilde{\beta}^+} \rightarrow y^{r/\mu}$ , recovering the power-law solution of the degenerate ODE.

## 9. Conclusion

We have developed a complete theory of the two-player war of attrition on a bivariate geometric Brownian motion. The central structural result is that the equilibrium value functions satisfy a coupled elliptic PDE system  $(\mathcal{L}_1 + \mathcal{L}_2 - r)V_i = 0$  in the mixed region — the same ODE collapse that characterises the one-dimensional model, elevated to two dimensions. The spectral curve  $\Sigma = \{(\beta_1, \beta_2) : \psi_1(\beta_1) + \psi_2(\beta_2) = r\}$  parametrises all solutions and provides the natural framework for constructing closed forms in the symmetric case and efficient numerical methods in the asymmetric case.

Several extensions are natural. First, the model can be generalised to correlated Brownian motions ( $dW_1 \cdot dW_2 = \rho dt$ ), which introduces a cross-derivative term  $\rho \sigma_1 \sigma_2 y_1 y_2 \partial_{y_1} \partial_{y_2} V_i$  in the PDE. The spectral curve  $\Sigma$  is modified accordingly and the separation of variables in the symmetric case becomes more involved. Second, the framework extends naturally to the  $n$ -player case via an  $n$ -dimensional GBM state space, though the free boundary geometry becomes considerably richer. Third, mean-reverting processes (Ornstein–Uhlenbeck) replace the power-law spectral basis with parabolic cylinder functions, and the spectral curve becomes an ellipse rather than a hyperbola.

## 10. References

1. Fudenberg, D. and Tirole, J. (1986). A theory of exit in duopoly. *Econometrica*, 54(4), 943–960.
2. Ghemawat, P. and Nalebuff, B. (1985). Exit. *RAND Journal of Economics*, 16(2), 184–194.
3. Zamrik, T. (2025). The three-player war of attrition as a Dynkin game. *zamrik.com Research Notes*.
4. Dixit, A. K. and Pindyck, R. S. (1994). *Investment under Uncertainty*. Princeton University Press.
5. Peskir, G. and Shiryaev, A. (2006). *Optimal Stopping and Free-Boundary Problems*. Birkhäuser.
6. Cryer, C. W. (1971). The solution of a quadratic programming problem using systematic overrelaxation. *SIAM Journal on Control*, 9(3), 385–392.
7. Villeneuve, S. (1999). Exercise regions of American options on several assets. *Finance and Stochastics*, 3(3), 295–322.
8. Lamberton, D. and Lapeyre, B. (1996). *Introduction to Stochastic Calculus Applied to Finance*. Chapman & Hall.
9. Øksendal, B. and Sulem, A. (2007). *Applied Stochastic Control of Jump Diffusions*. Springer.
10. Alvarez, L. H. R. (2001). Reward functionals, salvage values, and optimal stopping. *Mathematical Methods of Operations Research*, 54(2), 315–337.