The Doob h-Transform: Harmonic Functions, Conditioned Brownian Motion, and the Martin Boundary

Measure Change, Bessel Processes, and Minimal Harmonic Functions

Working Paper · Stochastic Analysis

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1. Abstract

The Doob h-transform is a fundamental technique for conditioning Markov processes on rare events. Given a strictly positive harmonic function h for the generator $\mathscr{A}$ of a Markov process X, the h-transform reweights the original measure via the local martingale $M_t = h(X_t)/h(X_0)$, producing a new Markov process whose generator is $\mathscr{A}^h f = h^{-1}\mathscr{A}(hf)$. We develop the theory systematically: harmonic functions and Dynkin's formula, the measure-change construction, conditioning standard Brownian motion to stay positive (yielding the three-dimensional Bessel process $\mathrm{BES}(3)$), conditioning to hit a fixed point (yielding the Brownian bridge), and Doob's general theorem connecting h-transforms to conditional distributions. We conclude with Martin boundary theory, which classifies all positive harmonic functions via minimal harmonic functions and provides the canonical integral representation against the Martin kernel $K(x, \xi)$.

2. Introduction

The problem of conditioning a stochastic process on an event of probability zero — staying positive forever, hitting a specific point, remaining in a prescribed region — lies at the heart of modern probability theory. Classical conditional probability requires a non-null conditioning event; extending the notion to null events demands deeper structural insight.

J. L. Doob's resolution, developed in the 1950s and formalised in the 1960s, is both elegant and profound. Given a Markov process X and a strictly positive harmonic function h for its generator $\mathscr{A}$, one constructs a new probability measure under which X behaves as if conditioned on the rare event encoded by h. The construction is explicit: weight the original measure by the martingale $M_t = h(X_t)/h(X_0)$. The resulting process — the h-transform of X — inherits the Markov property and has a modified generator $\mathscr{A}^h f = h^{-1}\mathscr{A}(hf)$, which amounts to adding a logarithmic-gradient drift $a\,\nabla \log h$ to the original dynamics.

The power of the h-transform lies in its universality. Conditioning standard Brownian motion to stay positive corresponds to $h(x) = x$ and recovers the three-dimensional Bessel process. Conditioning to hit a fixed point y at time T corresponds to $h(t,x) = p(T-t,x,y)$, the transition density, and recovers the Brownian bridge. Doob's theorem unifies these as special cases: every h-transform of a killed Markov process is a conditional process.

The deepest manifestation is Martin boundary theory. The collection of all positive harmonic functions for $\mathscr{A}$ admits a canonical decomposition into minimal harmonic functions — those that cannot be written as sums of other positive harmonic functions. Every positive harmonic function is a unique integral over minimal ones with respect to a measure on the Martin boundary $\partial_m E$. This provides a complete classification of all possible conditionings.

This paper develops the complete theory. Section 2 establishes setup and notation. Sections 3 and 4 cover harmonic functions and the h-transform construction. Sections 5 and 6 treat the Bessel process. Section 7 treats the Brownian bridge. Section 8 states and proves Doob's theorem. Section 9 develops Martin boundary theory. Section 10 discusses the fine topology and quasi-left-continuity. Section 11 presents numerical illustrations.

3. Setup and Notation

Assumption 3.1 (State space and process)**.** Let $E \subset \mathbb{R}^d$ be an open connected domain. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}_x)$ be a filtered probability space satisfying the usual conditions, where $\mathbb{P}_x$ denotes the law of a time-homogeneous strong Markov diffusion $X = (X_t)_{t\geq 0}$ started at $x \in E$ with continuous sample paths.

Assumption 3.2 (Generator)**.** The infinitesimal generator $\mathscr{A}$ acts on $f \in C^2(E)$ by

$$\mathscr{A}f(x) = b(x)\cdot\nabla f(x) + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\,\partial^2_{x_i x_j} f(x), \tag{3.1}$$

where $b : E \to \mathbb{R}^d$ is the drift and $a(x) = \sigma(x)\sigma(x)^\top$ is the diffusion matrix, assumed uniformly elliptic: $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda > 0$ and all $\xi \in \mathbb{R}^d$.

Notation 3.3. Throughout: $\mathscr{A}$ denotes the generator (3.1); $\mathscr{T}_t f(x) = \mathbb{E}_x[f(X_t)]$ denotes the Markov semigroup; $\tau_D = \inf\{t > 0 : X_t \notin D\}$ denotes the first exit time from $D \subset E$; $G_D(x,y)$ denotes the Green function of $\mathscr{A}$ on D; $\mathbb{E}^h_x$ denotes expectation under the h-transformed measure $\mathbb{P}^h_x$.

4. Harmonic Functions and Dynkin's Formula

Definition 4.1 (Harmonic function)**.** A function $h \in C^2(E)$ is *harmonic* for $\mathscr{A}$ on E if

$$\mathscr{A}h(x) = 0 \qquad \text{for all } x \in E. \tag{4.1}$$

It is *superharmonic* if $\mathscr{A}h \leq 0$ and *subharmonic* if $\mathscr{A}h \geq 0$.

Definition 4.2 (Admissible harmonic function)**.** We say h is *admissible* if h is harmonic on E, strictly positive ($h(x) > 0$ for all $x \in E$), and belongs to $C^2(E)$.

Theorem 4.3 (Dynkin's formula)**.** *Let $f \in \mathcal{D}(\mathcal{A})$ and let τ be a stopping time with $\mathbb{E}_x[\tau] < \infty$. Then*

$$\mathbb{E}_x[f(X_\tau)] = f(x) + \mathbb{E}_x\Big[\int_0^\tau \mathcal{A}f(X_s)\,ds\Big]. \tag{4.2}$$

Proof. *By Itô's formula applied to $f(X_t)$,*

$$f(X_t) = f(x) + \int_0^t \mathcal{A}f(X_s)\,ds + M_t^f, \tag{4.3}$$

where $M_t^f = \int_0^t \nabla f(X_s) \cdot \sigma(X_s)\,dW_s$ is a local martingale. Stopping at τ, applying optional stopping under $\mathbb{E}_x[\tau] < \infty$, and noting that $\mathbb{E}_x[M_\tau^f] = 0$ yields (4.2). □

Remark 4.4. When $\mathcal{A}h = 0$, Dynkin's formula gives $\mathbb{E}_x[h(X_\tau)] = h(x)$ for any bounded stopping time τ. This is the stochastic mean value property, the precise analogue of the classical mean value property for harmonic functions.

Corollary 4.5 (Harmonic functions generate martingales)**.** *If h is admissible then $(h(X_t))_{t\geq 0}$ is a local martingale under $\mathbb{P}_x$.*

Proof. Itô's formula gives $h(X_t) = h(x) + \int_0^t \mathcal{A}h(X_s)\,ds + M_t^h$. Since $\mathcal{A}h = 0$, the Lebesgue integral vanishes and $h(X_t) = h(x) + M_t^h$ is a local martingale. □ □

Example 4.6. For standard Brownian motion B_t on $(0, \infty)$ with generator $\mathcal{A} = \frac{1}{2}\partial_{xx}$, the function $h(x) = x$ satisfies $\mathcal{A}h = \frac{1}{2}h'' = 0$. Thus $(B_t)_{t\geq 0}$ is itself a local martingale — consistent with the optional stopping theorem.

Example 4.7. For two-dimensional Brownian motion with generator $\mathcal{A} = \frac{1}{2}\Delta$, the function $h(x) = \log|x|$ satisfies $\Delta(\log|x|) = 0$ for $x \neq 0$. The corresponding h-transform conditions the two-dimensional Brownian motion to avoid the origin.

5. The Doob h-Transform

Definition 5.1 (h-transform measure)**.** Let h be admissible. For each $x \in E$ and $T > 0$, define the probability measure $\mathbb{P}_x^h$ on $\mathcal{F}_T$ by

$$\frac{d\mathbb{P}_x^h}{d\mathbb{P}_x}\Big|_{\mathcal{F}_T} = \frac{h(X_T)}{h(x)} =: M_T. \tag{5.1}$$

Remark 5.2. The density M_T in (5.1) is well-defined since $h > 0$ and, when $h(X_t)$ is a true (not merely local) martingale, $\mathbb{E}_x[M_T] = h(x)/h(x) = 1$. The process $(M_t)_{t\geq 0}$ defined by $M_t = h(X_t)/h(x)$ is the Radon-Nikodym martingale generating the family $(\mathbb{P}_x^h|_{\mathcal{F}_t})_{t\geq 0}$ consistently.

Theorem 5.3 (Generator of the h-transform)**.** *Under $\mathbb{P}_x^h$, the process X is a Markov diffusion with generator*

$$\mathcal{A}^h f(x) = \frac{1}{h(x)}\,\mathcal{A}(hf)(x). \tag{5.2}$$

Equivalently, the drift of X under $\mathbb{P}_x^h$ is

$$b^h(x) = b(x) + a(x)\,\nabla \log h(x). \tag{5.3}$$

Proof. *Step 1. Under $\mathbb{P}_x$, Itô's formula gives:*

$$f(X_t) = f(x) + \int_0^t \mathscr{A}f(X_s)\,ds + \int_0^t \nabla f(X_s)\cdot\sigma(X_s)\,dW_s. \tag{5.4}$$

Step 2. By Girsanov's theorem applied to the density (5.1), under $\mathbb{P}_x^h$ the process

$$\tilde{W}_t = W_t - \int_0^t \sigma(X_s)^\top \nabla \log h(X_s)\,ds \tag{5.5}$$

is a standard Brownian motion. Substituting (5.5) into (5.4):

$$f(X_t) = f(x) + \int_0^t [\mathscr{A}f(X_s) + \nabla f(X_s)\cdot a(X_s)\nabla \log h(X_s)]ds + \int_0^t \nabla f(X_s)\cdot\sigma(X_s)\,d\tilde{W}_s. \tag{5.6}$$

Step 3. Expanding $h^{-1}\mathscr{A}(hf)$ by the product rule:

$$h^{-1}\mathscr{A}(hf) = \mathscr{A}f + h^{-1}[b\cdot f\nabla h + a : f\nabla^2 h + a\nabla h\cdot\nabla f] = \mathscr{A}f + \nabla f\cdot a\nabla\log h, \tag{5.7}$$

which matches the drift in (5.6), confirming (5.2) and (5.3). □

Remark 5.4. Formula (5.3) has a transparent interpretation: the h-transform adds a logarithmic-gradient drift $a\nabla \log h$ to the original dynamics. This biases the process toward regions of large h, with magnitude proportional to the diffusion coefficient. It is precisely the score function of the conditioning event.

Proposition 5.5 (Preservation of Markov property)**.** *The process X under $\mathbb{P}_x^h$ is a strong Markov process with semigroup*

$$\mathscr{T}_t^h f(x) = \frac{1}{h(x)}\mathscr{T}_t(hf)(x). \tag{5.8}$$

Proof. *The Markov property of X under $\mathbb{P}_x$, combined with the tower property applied to the martingale $M_t = h(X_t)/h(x)$, gives for $s < t$:*

$$\mathbb{E}_x^h[f(X_t)\mid\mathscr{F}_s] = \frac{1}{h(X_s)}\mathbb{E}_x[h(X_t)f(X_t)\mid\mathscr{F}_s] = \mathscr{T}_{t-s}^h f(X_s), \tag{5.9}$$

which is the Markov property for $\mathbb{P}_x^h$. □

6. Conditioning Brownian Motion to Stay Positive

Let B_t be standard Brownian motion on $\mathbb{R}$ with generator $\mathscr{A} = \frac{1}{2}\partial_{xx}$. We work on $E = (0,\infty)$ with absorbing boundary at 0.

Assumption 6.1. Let $B_0 = x > 0$ and let $\tau_0 = \inf\{t > 0 : B_t = 0\}$ be the first hitting time of zero. We study the killed process $B^{(0,\infty)}$, which is B stopped at τ_0.

Definition 6.2. The *Brownian motion conditioned to stay positive* is the h-transform of $B^{(0,\infty)}$ with the harmonic function $h(x) = x$.

The function $h(x) = x$ is harmonic for $\mathscr{A} = \frac{1}{2}\partial_{xx}$ since $\mathscr{A}h = \frac{1}{2}h'' = 0$.

Theorem 6.3 (BM conditioned to stay positive is BES(3). *). Under the h-transform with $h(x) = x$, the process satisfies the SDE*

$$dX_t = dW_t + \frac{1}{X_t}\,dt, \qquad X_0 = x > 0. \tag{6.1}$$

This is the three-dimensional Bessel process BES(3)*, and the process never reaches zero.* ***Proof.*** *Applying (5.3) with $b = 0$, $a = 1$, and $h(x) = x$:*

$$b^h(x) = 0 + 1 \cdot \partial_x \log x = \frac{1}{x}. \tag{6.2}$$

The SDE (6.1) follows. Non-attainability of zero: the scale function of (6.1) is $s(x) = -1/x$ (since $\mathscr{A}^h s = \frac{1}{2}s'' + (1/x)s' = 0$ on $(0,\infty)$). As $x \to 0^+$, $s(x) \to -\infty$, so zero is an entrance boundary inaccessible from $x > 0$ in finite time. □

Remark 6.4. The drift $1/X_t$ in (6.1) is a Coulomb repulsion: as $X_t \to 0$, the restoring force diverges, preventing the process from reaching zero. The h-transform encodes the conditioning event $\{\tau_0 = \infty\}$ pathwise through this modified drift.

7. The Three-Dimensional Bessel Process

Definition 7.1 (Bessel process of dimension δ)**.** For $\delta > 0$, the δ-dimensional Bessel process BES(δ) is the unique strong solution to

$$dX_t = \frac{\delta - 1}{2X_t}\,dt + dW_t, \qquad X_0 = x > 0. \tag{7.1}$$

For $\delta = 3$, (7.1) reduces to (6.1).

Proposition 7.2 (BES(3). *as Euclidean norm). Let $\mathbf{B}_t = (B_t^1, B_t^2, B_t^3)$ be standard Brownian motion in $\mathbb{R}^3$. Then $\|\mathbf{B}_t\|$ is a* BES(3) *process started at $\|\mathbf{B}_0\|$.* ***Proof.*** *By Itô's formula in $\mathbb{R}^3$,*

$$d\|\mathbf{B}_t\| = \sum_{i=1}^{3} \frac{B_t^i}{\|\mathbf{B}_t\|}\,dB_t^i + \frac{1}{2}\left(\frac{3}{\|\mathbf{B}_t\|} - \frac{\|\mathbf{B}_t\|^2}{\|\mathbf{B}_t\|^3}\right)dt = d\hat{W}_t + \frac{1}{\|\mathbf{B}_t\|}\,dt, \tag{7.2}$$

where $\hat{W}_t = \int_0^t (\mathbf{B}_s/\|\mathbf{B}_s\|) \cdot d\mathbf{B}_s$ is a standard Brownian motion by Lévy's characterisation. The SDE (7.2) is exactly (6.1). □

Remark 7.3. Proposition 6.1 gives geometric content to Theorem 5.1: conditioning one-dimensional BM on $(0,\infty)$ to stay positive is equivalent to observing the Euclidean norm of a three-dimensional Brownian motion.

Definition 7.4 (Scale function and speed measure)**.** The *scale function* of $\mathrm{BES}(3)$ is

$$s(x) = -\frac{1}{x}, \qquad x > 0, \tag{7.3}$$

and the *speed measure* is $m(dx) = 2x^2\,dx$.

Theorem 7.5 (Boundary classification)**.** *For* $\mathrm{BES}(3)$ *on* $(0,\infty)$*: - The boundary 0 is an entrance boundary (inaccessible from* $x > 0$ *in finite time). - The boundary* $+\infty$ *is a natural boundary. - The process is transient:* $\mathbb{P}_x(X_t \to \infty) = 1$*. **Proof.** Using Feller's boundary classification with scale function* (7.3) *and speed measure m: the criterion for 0 to be entrance requires* $\int_0^\epsilon s'(x)\,m(0,x)\,dx < \infty$ *where* $m(0,x) = \int_0^x m(dy) = \frac{2}{3}x^3$*. Then* $\int_0^\epsilon x^{-2}\cdot\frac{2}{3}x^3\,dx = \frac{2}{3}\int_0^\epsilon x\,dx < \infty$*, confirming 0 is entrance. Transience follows since* $s(0^+) = -\infty$ *while* $s(+\infty) = 0$ *is finite, so the process drifts to* $+\infty$ *by the recurrence / transience criterion.* □

Remark 7.6. The transience of $\mathrm{BES}(3)$ mirrors the transience of three-dimensional Brownian motion (Pólya's theorem), which is unsurprising given Proposition 6.1. The h-transform framework makes the connection structural rather than coincidental.

8. Conditioning Brownian Motion to Hit a Point: The Brownian Bridge

Let B_t be standard Brownian motion started at $x \in \mathbb{R}$. We condition it to satisfy $B_T = y$ for fixed $T > 0$ and $y \in \mathbb{R}$.

Definition 8.1 (Transition density)**.** The transition density of standard Brownian motion is

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-x)^2}{2t}\right), \qquad t > 0,\ x,y \in \mathbb{R}. \tag{8.1}$$

Definition 8.2 (Harmonic function for the bridge)**.** Define the time-inhomogeneous function

$$h(t,x) = p(T-t,x,y) = \frac{1}{\sqrt{2\pi(T-t)}}\exp\left(-\frac{(y-x)^2}{2(T-t)}\right), \qquad t \in [0,T). \tag{8.2}$$

This satisfies $(\partial_t + \mathscr{A})h = 0$, the time-inhomogeneous harmonicity condition.

Theorem 8.3 (Brownian bridge as h-transform)**.** *Under the h-transform with h given by* (8.2)*, the process satisfies*

$$dX_t = \frac{y - X_t}{T-t}\,dt + dW_t, \qquad X_0 = x, \quad t \in [0,T), \tag{8.3}$$

and $X_T = y$ *almost surely. This is the Brownian bridge from x to y over* $[0,T]$*. **Proof.** The logarithmic spatial gradient of* (8.2) *is*

$$\partial_x \log h(t,x) = \frac{y-x}{T-t}. \tag{8.4}$$

Applying the time-inhomogeneous version of (5.3) *with* $b = 0$ *and* $a = 1$ *gives* $b^h(t,x) = (y-x)/(T-t)$*, which yields* (8.3)*. Almost-sure convergence* $X_T = y$ *follows from the* *explicit representation* $X_t = x(1-t/T) + y(t/T) + (T-t)\int_0^t (T-s)^{-1}\,dW_s$*, whose variance vanishes as* $t \to T$. □

Remark 8.4. The drift $(y - X_t)/(T-t)$ in (8.3) pulls the process toward y with increasing strength as $t \to T$. This is the pathwise mechanism by which conditioning on the zero-probability event $\{B_T = y\}$ is implemented.

Proposition 8.5 (Explicit conditional distribution)**.** *Under* $\mathbb{P}_x^h$*, the conditional distribution of* X_t *given* $X_s = z$ *for* $0 \le s < t < T$ *is*

$$X_t \mid X_s = z \;\sim\; \mathcal{N}\left(z + \frac{(y-z)(t-s)}{T-s},\; \frac{(T-t)(t-s)}{T-s}\right). \tag{8.5}$$

Proof. *The Brownian bridge inherits Gaussian marginals. The mean follows from* (8.3) *by linearity; the variance* $(T-t)(t-s)/(T-s)$ *follows from the joint Gaussian structure of* (B_t, B_T) *via the standard conditional Gaussian formula, using* $\mathrm{Cov}(B_t, B_T) = \min(t,T) = t$ *and* $\mathrm{Var}(B_T) = T$. □

Remark 8.6. The conditional variance in (8.5) satisfies $(T-t)(t-s)/(T-s) \to 0$ as $t \to T$, confirming that the process collapses to y deterministically at the terminal time.

9. Doob's Theorem: Conditioning on Sets of Measure Zero

We now state and prove the general theorem unifying the examples of Sections 5 and 7.

Definition 9.1 (Killed process)**.** For a domain $D \subset E$ with exit time τ_D, the *killed process* X^D coincides with X for $t < \tau_D$ and enters a cemetery state ∂ for $t \ge \tau_D$.

Assumption 9.2. Let $h : E \to (0,\infty)$ be an admissible harmonic function for the generator $\mathcal{A}^D$ of the killed process X^D, and let $\mathbb{P}_x^{D,h}$ denote the corresponding h-transform measure.

Theorem 9.3 (Doob's h-transform theorem)**.** *Let* $A \in \mathcal{F}$ *be a measurable event and suppose* $h(x) = \mathbb{P}_x(A) > 0$ *for all* $x \in E$*. Then: (i) h is harmonic for* $\mathcal{A}$*. (ii) The h-transform* $\mathbb{P}_x^h$ *equals the conditional measure* $\mathbb{P}_x(\cdot \mid A)$. ***Proof.*** *Step 1 (harmonicity). By the Markov property,*

$$h(X_t) = \mathbb{P}_{X_t}(A) = \mathbb{E}_x[\mathbf{1}_A \mid \mathcal{F}_t], \tag{9.1}$$

so $(h(X_t))_{t\ge 0}$ *is a* $\mathbb{P}_x$*-martingale. By Corollary 3.1,* $\mathcal{A}h = 0$*, establishing (i). Step 2 (conditional measure). For any* $\mathcal{F}_T$*-measurable bounded* f,

$$\mathbb{E}_x^h[f(X_T)] = \mathbb{E}_x\left[\frac{h(X_T)}{h(x)} f(X_T)\right] = \frac{1}{h(x)}\,\mathbb{E}_x[\mathbb{P}_{X_T}(A) f(X_T)]. \tag{9.2}$$

By the Markov property and tower property, $\mathbb{E}_x[\mathbb{P}_{X_T}(A) f(X_T)] = \mathbb{E}_x[\mathbf{1}_A f(X_T)]$*, so* (9.2) *equals*

$$\frac{\mathbb{E}_x[\mathbf{1}_A f(X_T)]}{\mathbb{P}_x(A)} = \mathbb{E}_x[f(X_T) \mid A], \tag{9.3}$$

establishing (ii). □

Remark 9.4. Theorem 8.1 shows that the h-transform with $h = \mathbb{P}_x(A)$ is a genuine conditional distribution. The converse — every positive harmonic function arises as $\mathbb{P}_x(A)$ for some A — is the content of Martin boundary theory.

10. Martin Boundary Theory

Martin boundary theory answers the question: what are all positive harmonic functions for $\mathscr{A}$? The answer is encoded in the Martin boundary, a compactification of E that captures the limiting behaviour of the process as it exits.

Definition 10.1 (Green function)**.** Fix a reference point $x_0 \in E$. The *Green function* of $\mathscr{A}$ on E is

$$G(x,y) = \int_0^\infty p_E(t,x,y)\,dt, \qquad x,y \in E, \tag{10.1}$$

where $p_E(t,x,y)$ is the transition density of X killed on exiting E. We assume $G(x,y) < \infty$ for $x \neq y$ (transient case).

Definition 10.2 (Martin kernel)**.** The *Martin kernel* with reference point x_0 is

$$K(x,y) = \frac{G(x,y)}{G(x_0,y)}, \qquad x \in E, \quad y \in E \setminus \{x_0\}. \tag{10.2}$$

Definition 10.3 (Martin compactification and boundary)**.** The *Martin compactification* $\bar{E}^M$ of E is the smallest compactification of E such that the map $y \mapsto K(\cdot,y)$ extends continuously from E to $\bar{E}^M$ in the topology of locally uniform convergence. The *Martin boundary* is $\partial_M E = \bar{E}^M \setminus E$.

Remark 10.4. For each $\xi \in \partial_M E$, the function $K(\cdot,\xi) = \lim_{y\to\xi} K(\cdot,y)$ is a positive harmonic function. Distinct points ξ on the Martin boundary correspond to distinct limiting behaviours of the killed process.

Definition 10.5 (Minimal harmonic function)**.** A strictly positive harmonic function h is *minimal* if whenever $h = h_1 + h_2$ with h_1, h_2 positive and harmonic, then $h_i = c_i h$ for constants $c_i \in [0,1]$ with $c_1 + c_2 = 1$. The *minimal Martin boundary* $\partial_m E \subseteq \partial_M E$ is the set of boundary points ξ for which $K(\cdot,\xi)$ is minimal.

Theorem 10.6 (Martin representation theorem)**.** *Let h be a strictly positive harmonic function for $\mathscr{A}$ on E. Then there exists a unique finite measure μ on the minimal Martin boundary $\partial_m E$ such that*

$$h(x) = \int_{\partial_m E} K(x,\xi)\,\mu(d\xi), \qquad x \in E. \tag{10.3}$$

Proof. *Step 1. Let $\mathscr{H}^+ = \{h > 0 : \mathscr{A}h = 0,\ h(x_0) = 1\}$. By elliptic regularity and the maximum principle, $\mathscr{H}^+$ is compact and convex in the topology of locally uniform convergence. Step 2. The extreme points of $\mathscr{H}^+$ are exactly the minimal harmonic functions normalised at x_0. By Definition 9.4, these are parametrised by $\partial_m E$ via $\xi \mapsto K(\cdot,\xi)/K(x_0,\xi) = K(\cdot,\xi)$ (since $K(x_0,\xi) = G(x_0,\xi)/G(x_0,\xi) = 1$). Step 3.*

The Choquet representation theorem applied to the compact convex set $\mathcal{H}^+$ guarantees that every $h \in \mathcal{H}^+$ is an integral over extreme points with respect to a unique measure, yielding (10.3). □

Remark 10.7. Theorem 9.1 is the harmonic-function analogue of the Krein-Milman theorem. The measure μ in (10.3) encodes the conditioning event: $\mu = \delta_\xi$ (a point mass) corresponds to conditioning X to exit E through the boundary point $\xi \in \partial_m E$, in the sense of Theorem 8.1.

Corollary 10.8 (Canonical examples)**.** *For standard Brownian motion: (i) On $E = (0, \infty)$, the minimal Martin boundary is $\partial_m E = \{+\infty\}$. The unique minimal harmonic function is $h(x) = x$, corresponding to conditioning BM to drift to $+\infty$. (ii) On the unit disk $D \subset \mathbb{R}^2$, the minimal Martin boundary is $\partial_m D = \partial D = S^1$ and $K(x, \xi) = P(x, \xi)$, the Poisson kernel.*

11. Quasi-Left-Continuity and the Fine Topology

Definition 11.1 (Excessive function)**.** A function $u : E \to [0, \infty]$ is *α-excessive* for $\alpha \geq 0$ if $e^{-\alpha t}\mathcal{T}_t u \leq u$ for all $t > 0$ and $e^{-\alpha t}\mathcal{T}_t u \nearrow u$ as $t \to 0$. The case $\alpha = 0$ gives *excessive* functions; superharmonic functions are excessive.

Definition 11.2 (Fine topology)**.** The *fine topology* on E is the coarsest topology making every excessive function continuous. A set $A \subset E$ is *thin* at x if x is not a fine limit point of $A \setminus \{x\}$.

Definition 11.3 (Regular and irregular boundary points)**.** A boundary point $\xi \in \partial E$ is *regular* for D if $\mathbb{P}_\xi(\tau_D = 0) = 1$. It is *irregular* if $\mathbb{P}_\xi(\tau_D = 0) < 1$.

Theorem 11.4 (Blumenthal 0-1 law under h-transform)**.** *Let $\mathcal{F}_{0^+} = \bigcap_{t>0} \mathcal{F}_t$. Under $\mathbb{P}_x^h$, every event $A \in \mathcal{F}_{0^+}$ satisfies $\mathbb{P}_x^h(A) \in \{0, 1\}$.* ***Proof.*** *Since $\mathbb{P}_x^h \ll \mathbb{P}_x$ on each $\mathcal{F}_T$, trivial events under $\mathbb{P}_x$ remain trivial under $\mathbb{P}_x^h$. The Blumenthal 0-1 law holds for $\mathbb{P}_x$ (strong Markov property at $t = 0$), so $\mathcal{F}_{0^+}$ is $\mathbb{P}_x$-trivial and hence $\mathbb{P}_x^h$-trivial.* □

Proposition 11.5 (Quasi-left-continuity)**.** *Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of stopping times with $\tau_n \nearrow \tau < \infty$. Under $\mathbb{P}_x^h$,*

$$X_{\tau_n} \to X_\tau \quad \mathbb{P}_x^h\text{-a.s.} \tag{11.1}$$

Proof. *Quasi-left-continuity follows from the path continuity of X (Assumption 2.1): since $t \mapsto X_t$ is continuous $\mathbb{P}_x$-a.s., it is also continuous $\mathbb{P}_x^h$-a.s. (as $\mathbb{P}_x^h \ll \mathbb{P}_x$), and* (11.1) *holds trivially for continuous paths.* □

Remark 11.6. The regular boundary points of ∂E in the fine topology are precisely those at which the conditioned process can be extended continuously. Irregular points correspond to cusps or re-entrant corners where the Brownian motion may or may not hit the boundary instantaneously, depending on the geometry.

Proposition 11.7 (Fine topology and thinness)**.** *Under $\mathbb{P}_x^h$, the process X^h avoids every set that is thin at x in the fine topology.* ***Proof.*** *Thinness of A at x means $\mathbb{P}_x(\tau_A = 0) = 0$ where $\tau_A = \inf\{t > 0 : X_t \in A\}$ (by the Wiener regularity criterion). Since $\mathbb{P}_x^h \ll \mathbb{P}_x$, this implies $\mathbb{P}_x^h(\tau_A = 0) = 0$.* □

12. Numerical Illustrations

We illustrate the four main objects of the theory. All simulations use the Euler-Maruyama scheme with step size $\Delta t = 10^{-3}$ on the interval $[0, 1]$.

```
Input: x0, T, dt, N_paths, target y (for bridge)
Output: sample paths for BM, BES(3), Bridge, and Martin kernel contours

Algorithm BM_absorbed:
    X[0] = x0
    For k = 0 to T/dt - 1:
        dW = sqrt(dt) * randn()
        X[k+1] = X[k] + dW
        if X[k+1] <= 0: X[k+1:] = 0; break

Algorithm BES3:
    X[0] = x0
    For k = 0 to T/dt - 1:
        dW = sqrt(dt) * randn()
        X[k+1] = max(X[k] + dt / X[k] + dW, eps)

Algorithm Bridge(x0, y, T):
    X[0] = x0
    For k = 0 to T/dt - 2:
        tau = T - k * dt
        dW = sqrt(dt) * randn()
        X[k+1] = X[k] + (y - X[k]) / tau * dt + dW
    X[-1] = y

Algorithm Martin_contours(xi_list):
    For each xi in xi_list:
        Compute K(x, xi) = P(x, xi) on grid over unit disk D
        Plot contour lines of K(., xi)
```

12.1 Brownian Motion vs BES(3)

Figure 1 contrasts three absorbed Brownian motion paths (dashed) with three BES(3) paths (solid). The absorbed paths eventually hit zero; the BES(3) paths are repelled by the Coulomb drift and drift to infinity.

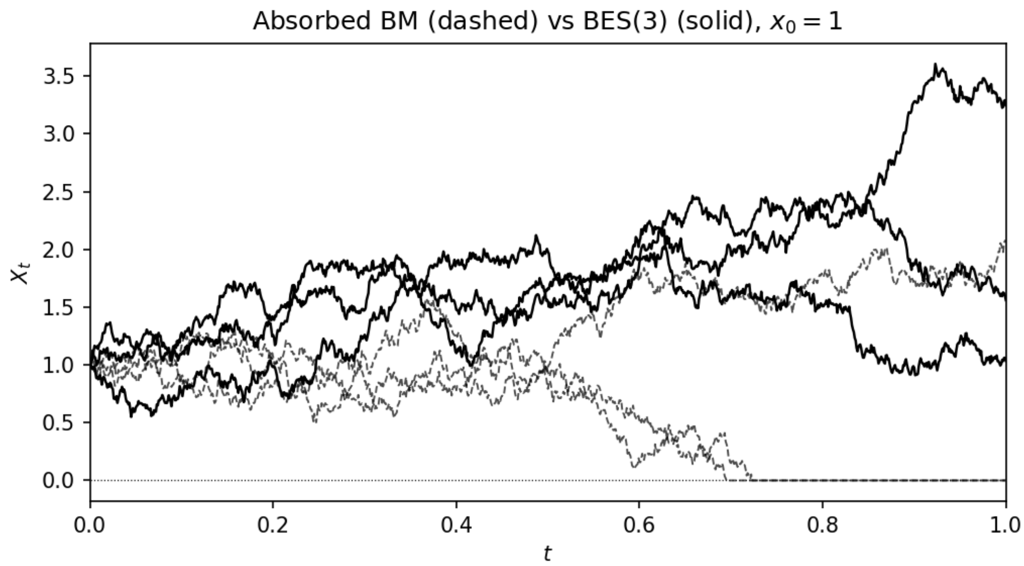

Figure 1: Three absorbed Brownian motion paths (dashed) started at $x_0 = 1$ on $(0, \infty)$ versus three BES(3) paths (solid). The BES(3) drift $1/X_t$ prevents absorption at zero, producing transient upward motion.

12.2 h-Transform Martingale Weight

Figure 2 plots the Radon-Nikodym martingale $M_t = X_t/x_0$ along three Brownian motion paths, illustrating how the measure change selectively upweights trajectories that stay large.

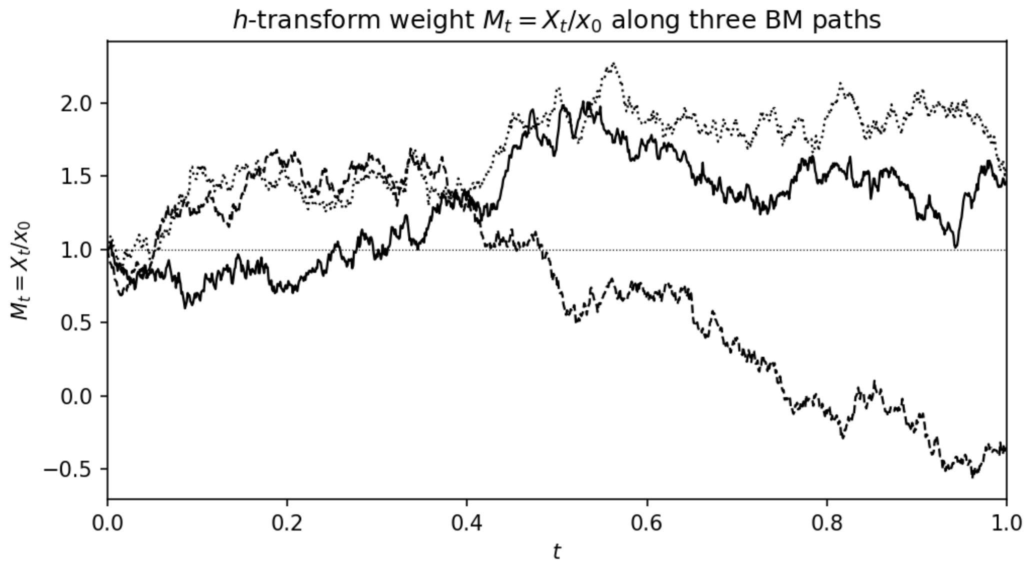

Figure 2: The h-transform weight $M_t = X_t/x_0$ along three Brownian motion paths started at $x_0 = 1$. Paths approaching zero accumulate small weight; paths growing large accumulate large weight, encoding the selective conditioning implicit in the measure change.

12.3 Brownian Bridge Family

Figure 3 shows Brownian bridge paths conditioning on four distinct terminal values, illustrating how the drift $(y - X_t)/(T - t)$ pulls each path to its prescribed endpoint.

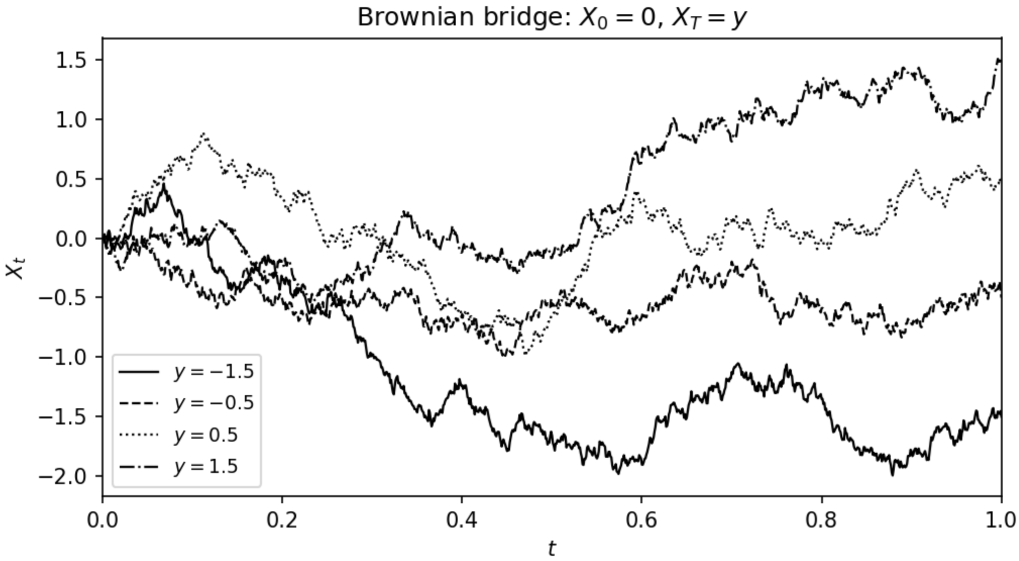

Figure 3: Brownian bridge paths from $x = 0$ at $t = 0$ to four terminal values $y \in \{-1.5, -0.5, 0.5, 1.5\}$ at $T = 1$. Each path satisfies $dX_t = (y - X_t)/(1 - t)\,dt + dW_t$. The pull toward y intensifies as $t \to 1$.

12.4 Martin Boundary and Poisson Kernel

Figure 4 illustrates the Martin boundary of the unit disk via contour lines of the Poisson kernel $K(x, \xi) = P(x, \xi) = (1 - |x|^2)/|x - \xi|^2$ for three boundary points. Each family of contours is a set of Apollonius circles — classical loci of the form $\{x : |x - \xi|^2 = c(1 - |x|^2)\}$ for varying $c > 0$ — which are genuine circular arcs inside the disk. The three families, one per boundary point, intersect to produce the petal-and-lens pattern visible in the figure. The concentrating behaviour near each ξ_i reflects the probabilistic meaning: $K(x, \xi)$ is, up to normalisation, the density of the exit distribution of Brownian motion started at x with respect to the exit distribution started at the reference point x_0. A Brownian motion conditioned (via the h-transform with $h = K(\cdot, \xi)$) to exit through ξ spends most of its time in the high-contour region near ξ, visible as the tight clustering of curves close to each boundary point.

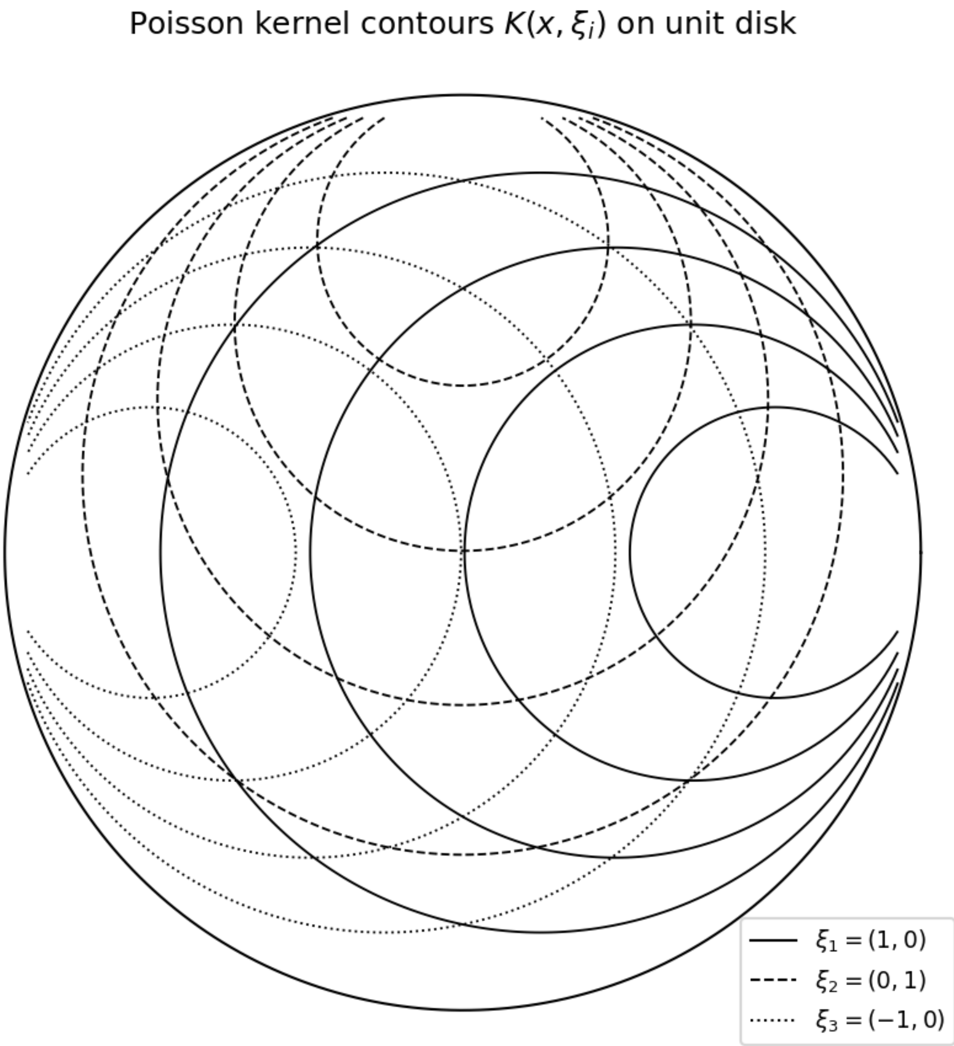

Figure 4: Contour lines of three minimal harmonic functions $K(x, \xi_i) = P(x, \xi_i)$ (Poisson kernel) on the unit disk D, for boundary points $\xi_1 = (1, 0)$ (solid), $\xi_2 = (0, 1)$ (dashed), $\xi_3 = (-1, 0)$ (dotted). Each family of contours is a nested set of Apollonius circles concentrating near its boundary point; the intersecting pattern reflects the three independent conditioning directions for exit-conditioned Brownian motion.

13. References

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