Bessel Processes, Fractional Dynamics, and Long Memory in Asset Prices

A Unified Framework via Stochastic Calculus and Hurst Exponent Analysis

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Outline

- Bessel processes: definition and key properties
- Fractional Brownian motion and the Hurst exponent
- Fractional GBM as an asset price model
- Long memory in financial time series
- Connection between Bessel processes and fBm
- Pricing under fractional dynamics
- Numerical results

Definition. The Bessel process of dimension $\delta > 0$ is the solution to

$$dR_t = \frac{\delta - 1}{2R_t}\, dt + dW_t, \quad R_0 = r_0 > 0$$

- For $\delta \geq 2$: process never hits zero
- For $0 < \delta < 2$: zero is accessible (reflecting boundary)
- $\delta = 3$: radial part of 3D Brownian motion

Squared Bessel process $\rho_t = R_t^2$:

$$d\rho_t = \delta\, dt + 2\sqrt{\rho_t}\, dW_t$$

This is the CIR process with zero mean-reversion.

Definition. A fractional Brownian motion B_t^H with Hurst exponent $H \in (0,1)$ is a centred Gaussian process with covariance

$$\mathbb{E}[B_s^H B_t^H] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right)$$

H	**Memory**	**Increments**
$H < \frac{1}{2}$	Short memory	Anti-persistent
$H = \frac{1}{2}$	No memory	Standard BM
$H > \frac{1}{2}$	Long memory	Persistent

- **Not a semimartingale** for $H \neq \frac{1}{2}$ — Itô calculus does not apply directly

Fractional GBM

Model. The fractional GBM asset price:

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{1}{2}\sigma^2 t^{2H}\right)$$

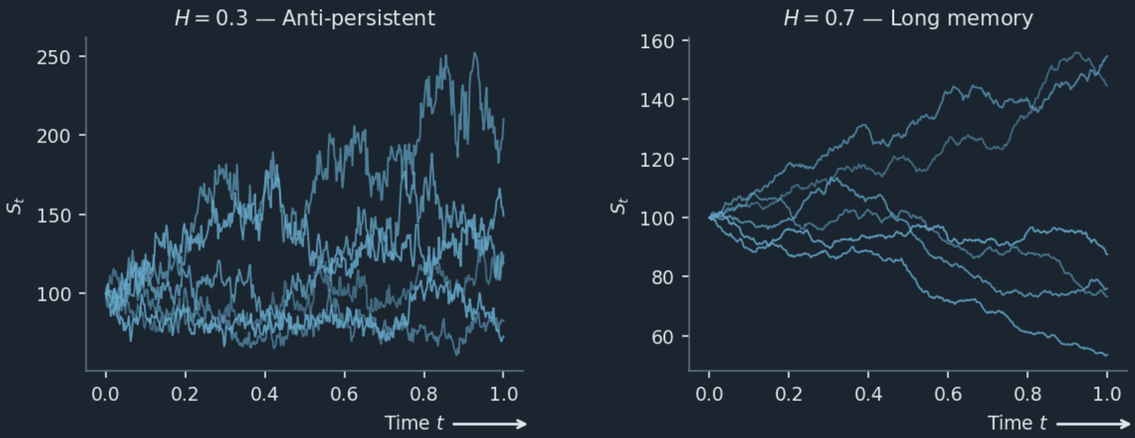
Properties:

- Captures long-range dependence in log-returns
- Volatility of $\log S_t$ scales as σt^H
- Reduces to standard GBM when $H = \frac{1}{2}$

Hurst estimation from realized data:

$$\hat{H} = \frac{\log \mathbb{E}[|B_{t+\tau}^H - B_t^H|]}{\log \tau}$$

Fractional GBM paths for different Hurst exponents



Long Memory in Asset Prices

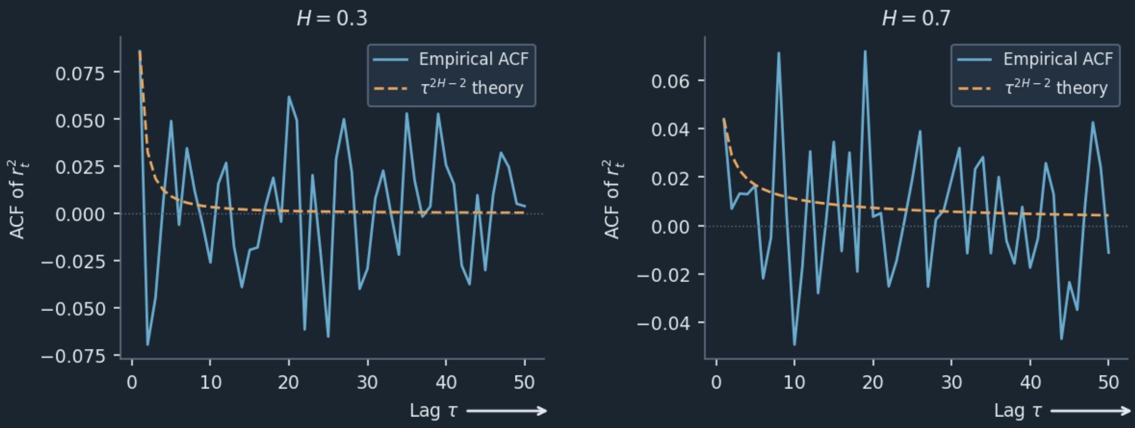
Empirical evidence:

- Volatility autocorrelations decay as τ^{2H-2} — hyperbolic, not exponential
- Realized variance exhibits Hurst $H \approx 0.7$ across major equity indices

Why Bessel processes appear:

- The modulus of fBm satisfies a Bessel-type SDE in the rough regime $H < \frac{1}{2}$
- Zero-crossing structure of fBm linked to $\delta = 2 - 2H$ Bessel dimension

Autocorrelation of squared returns vs. theoretical τ^{2H-2} decay



Connection: Bessel and fBm

Key result. For $H \in (0, \frac{1}{2})$, the local behaviour of $|B_t^H|$ near zero is governed by a Bessel process of dimension

$$\delta = 2(1 - H)$$

- Rough fBm $(H \to 0) \Rightarrow \delta \to 2$: borderline recurrence
- Standard BM $(H = \frac{1}{2}) \Rightarrow \delta = 1$: standard reflected BM

Implication for finance: volatility processes with rough paths $(H < \frac{1}{2})$ inherit the boundary behaviour of low-dimensional Bessel processes — explaining near-zero volatility clustering.

Challenge. fBm is not a semimartingale $\Rightarrow$ standard risk-neutral pricing breaks down.
Resolution (Wick–Itô calculus):

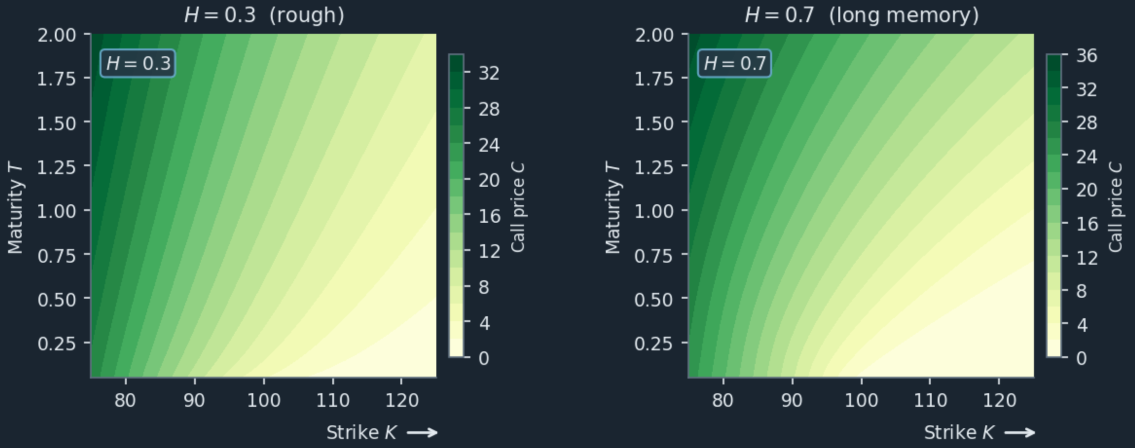
$$dS_t = \mu S_t\, dt + \sigma S_t\, d^{\diamond} B_t^H$$

where $d^{\diamond}$ denotes the Wick product integral.
European call price:

$$C(S_0, K, T) = S_0 \Phi(d_1^H) - K e^{-rT} \Phi(d_2^H)$$

with $d_{1,2}^H = \dfrac{\log(S_0/K) + (r \pm \frac{1}{2}\sigma^2 T^{2H})}{\sigma T^H}$

Fractional Black-Scholes call price surface for $H \in \{0.3, 0.5, 0.7\}$



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