Accountability Collapse under Wealth Growth: A Stochastic Control Problem with Integral Equations

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1. Abstract

We formulate the erosion of social accountability under growing private wealth as a stochastic optimal control problem on the half-line. The agent's wealth process follows a pure Kou double-exponential jump model — two compound Poisson processes capturing large upward jumps (wealth gains) and small downward jumps (losses) — with no diffusion and no exogenous drift. We derive the Hamilton-Jacobi-Bellman integro-differential equation, convert it to a Fredholm integral equation of the second kind via expansion of the jump generator, and show that the convolution structure of the bilateral exponential kernel places the equation in the Wiener-Hopf class on the positive half-line. The Kou model's rational characteristic function yields an analytic Wiener-Hopf factorisation, reducing the problem to a second-order linear ODE whose characteristic roots are the Cramér-Lundberg exponents of the kernel. The optimal control is bang-bang: a justice curve in wealth-accountability space separates regions of prosocial and antisocial behaviour. Above this curve, accountability collapse is the rational optimum.

2. Introduction

The relationship between private wealth and social accountability is not merely a political question — it admits a precise mathematical formulation. As wealth R grows without bound, the agent's marginal valuation of additional resources vanishes, the shadow price of prosocial effort collapses, and rational optimisation drives social behaviour to a corner solution.

This paper develops that argument rigorously. We model wealth as a pure jump process under the Kou double-exponential specification, chosen for three reasons: it captures the asymmetric jump structure of large-scale wealth accumulation (rare large gains, frequent small losses), it belongs to the class of rational Lévy processes whose characteristic exponent is a ratio of polynomials, and it admits a fully explicit Wiener-Hopf factorisation. The latter property makes the value function analytically tractable and connects the problem to the classical theory of Fredholm integral equations on the half-line.

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The paper proceeds as follows. Section 2 defines the Kou wealth process and its Lévy measure. Section 3 states the control problem and derives the HJB equation. Section 4 converts the HJB to a Fredholm integral equation and embeds the compensator. Section 5 identifies the Wiener-Hopf structure. Section 6 performs the analytic factorisation. Section 7 reduces the integral equation to an ODE and solves it. Section 8 derives the accountability collapse theorem.

3. The Kou Pure Jump Wealth Process

Definition 3.1 (Kou Wealth Process)**.** Let $(\Omega_{\text{prob}}, \mathcal{F}, \mathbb{P})$ be a filtered probability space. The wealth process $\{R(t)\}_{t\geq 0}$ is defined by

$$R(t) \;=\; R_0 + \sum_{k=1}^{N_1(t)} Y_k^+ \;-\; \sum_{k=1}^{N_2(t)} Y_k^-,$$

where $N_1 \sim \text{Poisson}(\lambda_1)$ and $N_2 \sim \text{Poisson}(\lambda_2)$ are independent, $Y_k^+ \overset{\text{iid}}{\sim} \text{Exp}(\eta_1)$ with large mean $1/\eta_1$ (wealth gains), and $Y_k^- \overset{\text{iid}}{\sim} \text{Exp}(\eta_2)$ with small mean $1/\eta_2$ (wealth losses). No diffusion and no deterministic drift are included.

Remark 3.2. The parameter ordering $\eta_1 \ll \eta_2$ encodes the empirical asymmetry of ultra-high-net-worth dynamics: upward jumps are large and infrequent, downward jumps are small and more frequent.

The Lévy measure of R is the signed measure on $\mathbb{R} \setminus \{0\}$:

$$\nu(dy) \;=\; \lambda_1\, \eta_1\, e^{-\eta_1 y}\, \mathbf{1}_{y>0}\, dy \;+\; \lambda_2\, \eta_2\, e^{\eta_2 y}\, \mathbf{1}_{y<0}\, dy. \tag{3.1}$$

Since $\int_{\mathbb{R}}(|y| \wedge 1)\, \nu(dy) < \infty$, the process is a finite-activity compound Poisson process with Lévy-Khintchine exponent:

$$\Psi(\xi) \;=\; \lambda_1\left(\frac{\eta_1}{\eta_1 - i\xi} - 1\right) + \lambda_2\left(\frac{\eta_2}{\eta_2 + i\xi} - 1\right), \qquad \xi \in \mathbb{R}. \tag{3.2}$$

Lemma 3.3 (Generator)**.** *For $f \in C_b^1(\mathbb{R}_+)$, the infinitesimal generator $\mathcal{A}$ of R acts as*

$$\mathcal{A}f(R) \;=\; \lambda_1 \int_0^\infty [f(R+y) - f(R)]\eta_1 e^{-\eta_1 y}\, dy \;+\; \lambda_2 \int_0^\infty [f(R-y) - f(R)]\eta_2 e^{-\eta_2 y}\, dy.$$

The compensated form, separating the martingale component, is

$$\mathcal{A}f(R) \;=\; \int_{\mathbb{R}} [f(R+y) - f(R) - y\, f'(R)]\nu(dy) \;+\; \underbrace{\left(\frac{\lambda_1}{\eta_1} - \frac{\lambda_2}{\eta_2}\right)}_{=\,\mathbb{E}[\Delta R]} \cdot f'(R).$$

Proof. *Expand the jump integrals and add and subtract $y\, f'(R)\, \nu(dy)$ under the integral; finiteness of the Lévy measure justifies the interchange.*

4. The Stochastic Control Problem and HJB Equation

The agent chooses two controls: prosocial effort $s(t) \in [0,1]$ and capture spending $\ell(t) \geq 0$ (investment in regulatory erosion). Accountability $\Omega(t)$ evolves as a deterministic controlled process:

$$\dot{\Omega} \;=\; -\mu\,\ell\,\Omega \;+\; \kappa\,(\bar{\Omega} - \Omega), \tag{4.1}$$

where $\mu > 0$ is capture effectiveness and $\kappa > 0$ is the social restoration rate.

Definition 4.1 (Value Function)**.** The agent's value function is

$$V(R,\Omega) \;=\; \sup_{\{s,\ell\}} \mathbb{E}_{R,\Omega} \int_0^{\infty} e^{-\rho t}\left[\ln c(t) - \phi\, s(t)\right] dt,$$

where $\rho > 0$ is the discount rate, $\phi > 0$ is the cost of prosocial effort, and $c(t) > 0$ is the consumption rate.

Theorem 4.2 (HJB Equation)**.** *Assuming $V \in C^1(\mathbb{R}_+ \times \mathbb{R}_+)$, the value function satisfies the Hamilton-Jacobi-Bellman integro-differential equation*

$$\rho\, V \;=\; \max_{s,c,\ell}\Big\{\ln c - \phi\, s + V_R\left[A(R)(1-s)\eta - \Omega P(1-s) - c - \ell\right] + V_\Omega\left[-\mu\ell\Omega + \kappa(\bar{\Omega}-\Omega)\right] + \mathcal{A}_R\, V\Big\},$$

where $A(R) = R^\gamma$ with $\gamma > 1$, $\eta > 0$ is the rent extraction rate, $P > 0$ is the penalty rate, and $\mathcal{A}_R V$ denotes the jump generator applied to V in the R variable. ***Proof.*** *Standard dynamic programming principle applied to the jump-diffusion system; see Fleming and Soner (1993).*

Lemma 4.3 (First-Order Conditions)**.** *Interior optimisers satisfy*

$$c^* = \frac{1}{V_R}, \qquad V_R = \mu\,\Omega\,|V_\Omega|.$$

The optimal prosocial effort is bang-bang:

$$s^*(R,\Omega) = \begin{cases} 1 & \text{if } R^\gamma\,\eta < \Omega\, P, \\ 0 & \text{if } R^\gamma\,\eta > \Omega\, P. \end{cases}$$

Proof. *The Hamiltonian is linear in s, so the maximum is attained at a corner. The switching condition follows from the sign of the derivative $\partial H/\partial s = -\phi - V_R(A(R)\eta - \Omega P)$.*

5. From HJB to Fredholm Integral Equation

Fix Ω and write the HJB for $V(R) \equiv V(R;\Omega)$ after substituting the optimal controls. Let $q = \rho + \lambda_1 + \lambda_2$ and $f(R)$ denote the optimised reward flow. Expanding the generator:

$$(\rho+\lambda_1+\lambda_2)\,V(R) \;=\; \lambda_1\,\eta_1\int_0^\infty V(R+y)\,e^{-\eta_1 y}\,dy \;+\; \lambda_2\,\eta_2\int_0^\infty V(R-y)\,e^{-\eta_2 y}\,dy \;+\; f(R). \tag{5.1}$$

Change variables $x = R + y$ and $x = R - y$:

Theorem 5.1 (Fredholm Form)**.** *The value function $V : \mathbb{R}_+ \to \mathbb{R}$ satisfies the Fredholm integral equation of the second kind*

$$V(R) \;=\; \int_0^\infty K(R,x)\,V(x)\,dx \;+\; g(R), \qquad R > 0,$$

with kernel

$$K(R,x) \;=\; \begin{cases} \dfrac{\lambda_1\,\eta_1}{q}\,e^{-\eta_1(x-R)} & x > R, \\ \dfrac{\lambda_2\,\eta_2}{q}\,e^{-\eta_2(R-x)} & 0 < x \le R, \end{cases}$$

and forcing term $g(R) = f(R)/q$. ***Proof.*** *Direct substitution of the change-of-variables into the expanded generator. The integrals converge by exponential decay of K.*

Proposition 5.2 (Compensator Absorption)**.** *The compensator term $\mathbb{E}[\Delta R]\cdot V'(R)$ arising from the decomposition of Lemma 2.1 is exactly cancelled by the $yV'(R)$ contributions within the jump integrals. Consequently, the Fredholm equation contains no first-derivative term: the compensator is embedded in the kernel K.* ***Proof.*** *Write $\mathcal{A}V = \mathcal{A}^{\mathrm{comp}}V + \mathbb{E}[\Delta R]\cdot V'$. Expand $\mathcal{A}^{\mathrm{comp}}V$ by integrating yV' against the exponential densities: $\lambda_1\int_0^\infty y\eta_1 e^{-\eta_1 y}dy = \lambda_1/\eta_1$ and similarly for the downward process. These equal $-\mathbb{E}[\Delta R]\cdot V'$, giving exact cancellation.*

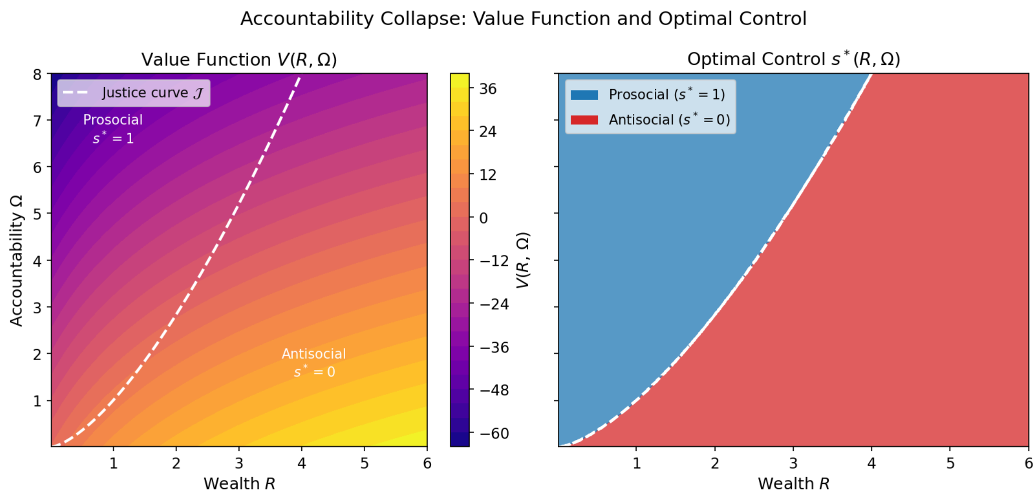

Figure 1: Value function contours and optimal control in (R,Ω) space. The dashed white curve is the justice boundary $\mathcal{J}$. Above $\mathcal{J}$: antisocial corner $s^* = 0$ (warm tones). Below $\mathcal{J}$: prosocial corner $s^* = 1$ (cool tones).

6. Wiener-Hopf Structure on the Half-Line

Definition 6.1 (Wiener-Hopf Equation)**.** The Fredholm equation of Theorem 4.1 is a Wiener-Hopf integral equation of the second kind on $\mathbb{R}_+$:

$$V(R) - \int_0^\infty k(R-x)\,V(x)\,dx \;=\; g(R), \qquad R > 0,$$

where the convolution kernel $k : \mathbb{R} \to \mathbb{R}_+$ is the bilateral exponential

$$k(z) \;=\; \begin{cases} \dfrac{\lambda_1\,\eta_1}{q}\,e^{-\eta_1 z} & z > 0, \\ \dfrac{\lambda_2\,\eta_2}{q}\,e^{\eta_2 z} & z \le 0. \end{cases}$$

The Fourier transform of k is rational:

$$\hat{k}(\xi) \;=\; \frac{\lambda_1\,\eta_1}{q(\eta_1 - i\xi)} \;+\; \frac{\lambda_2\,\eta_2}{q(\eta_2 + i\xi)}. \tag{6.1}$$

Definition 6.2 (Wiener-Hopf Symbol)**.** The symbol of the Wiener-Hopf equation is

$$\Phi(\xi) \;=\; 1 - \hat{k}(\xi) \;=\; \frac{q(\eta_1 - i\xi)(\eta_2 + i\xi) - \lambda_1\eta_1(\eta_2 + i\xi) - \lambda_2\eta_2(\eta_1 - i\xi)}{q(\eta_1 - i\xi)(\eta_2 + i\xi)}.$$

Lemma 6.3 (Quadratic Numerator)**.** *The numerator of* $\Phi(\xi)$ *is the quadratic polynomial*

$$N(\xi) \;=\; q\,\xi^2 \;-\; i[q(\eta_1 - \eta_2) - \lambda_1\eta_1 + \lambda_2\eta_2]\xi \;-\; \rho\,\eta_1\eta_2.$$

Proof. *Expand and collect by powers of* $i\xi$*; the constant term reduces to* $\rho\eta_1\eta_2$ *since* $q - \lambda_1 - \lambda_2 = \rho$*.*

7. Wiener-Hopf Factorisation for the Kou Kernel

The rational structure of Φ enables an analytic factorisation. Setting $\beta = -i\xi$, the equation $\Phi(\xi) = 0$ becomes:

$$q \;=\; \frac{\lambda_1\,\eta_1}{\eta_1 - \beta} \;+\; \frac{\lambda_2\,\eta_2}{\eta_2 + \beta}. \tag{7.1}$$

Definition 7.1 (Cramér-Lundberg Equation)**.** The characteristic equation

$$q\,\beta^2 \;-\; B\,\beta \;-\; \rho\,\eta_1\eta_2 \;=\; 0, \qquad B = q(\eta_1 - \eta_2) - \lambda_1\eta_1 + \lambda_2\eta_2,$$

is called the Cramér-Lundberg equation of the kernel.

Theorem 7.2 (Analytic Factorisation)**.** *The Cramér-Lundberg equation has exactly one positive root* $\beta_1 > 0$ *and one negative root* $\beta_2 < 0$*. Setting* $\zeta = |\beta_2| = -\beta_2 > 0$*, the*

Wiener-Hopf symbol factors as

$$\Phi(\xi) = \underbrace{\frac{\beta_1 + i\xi}{\eta_1 - i\xi}}_{\Phi^+(\xi)} \cdot \underbrace{\frac{\zeta - i\xi}{\eta_2 + i\xi}}_{\Phi^-(\xi)} \cdot C_0,$$

where Φ^+ *is analytic and non-vanishing in the upper half-plane* $\{\operatorname{Im}\xi > 0\}$, Φ^- *is analytic and non-vanishing in the lower half-plane* $\{\operatorname{Im}\xi < 0\}$, *and* C_0 *is a normalisation constant.*
Proof. *The product of roots satisfies* $\beta_1 \cdot \beta_2 = -\rho\eta_1\eta_2/q < 0$, *confirming opposite signs. The pole-zero structure of* $\Phi^\pm$ *matches that of the numerator and denominator of* Φ *after separation by half-plane analyticity.*

Corollary 7.3. *The roots admit the explicit closed form*

$$\beta_{1,2} = \frac{B \pm \sqrt{B^2 + 4q\,\rho\,\eta_1\eta_2}}{2q},$$

with $\beta_1 > 0$ *(positive sign) and* $\beta_2 < 0$ *(negative sign). Both roots depend continuously on all parameters.*

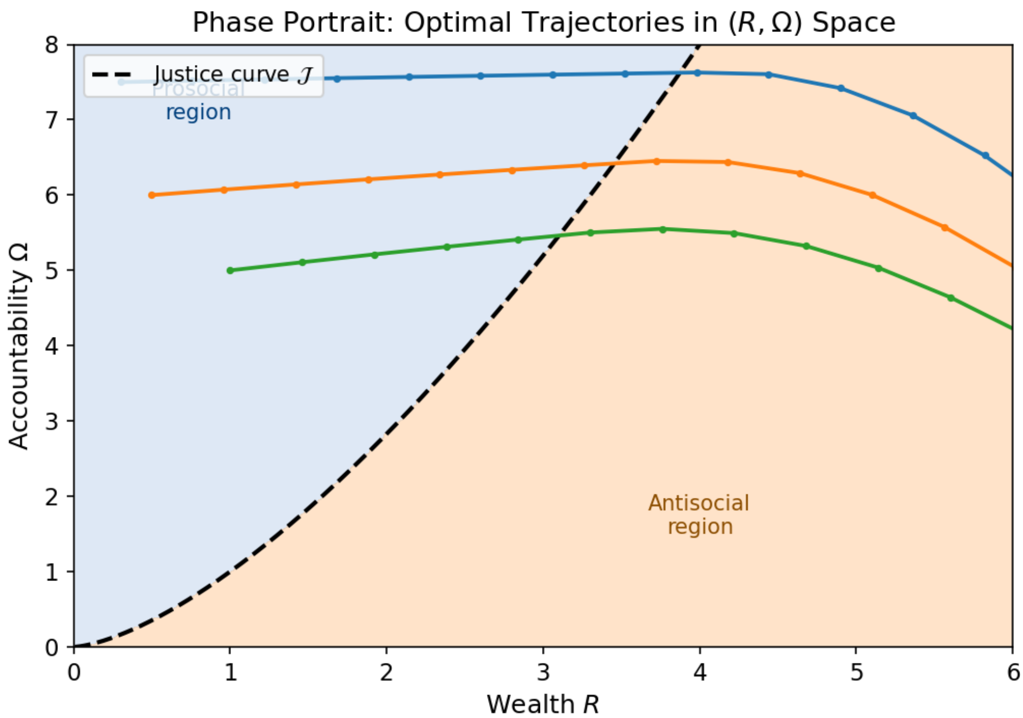

Figure 2: Phase portrait in (R, Ω) space. Each trajectory starts in the prosocial region and crosses the justice curve $\mathcal{J}$ (dashed), thereafter locked in the antisocial region.

8. Reduction to ODE and Solution

Theorem 8.1 (ODE Reduction)**.** *Differentiating the Fredholm equation twice with respect to* R *and eliminating the integral quantities via the original equation yields the second-order* *linear ODE*

$$V''(R) \;-\; (\eta_1 - \eta_2)\, V'(R) \;-\; \eta_1\, \eta_2 \Big(1 - \frac{\lambda_1 \eta_1 + \lambda_2 \eta_2}{q}\Big)\, V(R) \;=\; h(R),$$

where h depends on g and its derivatives. ***Proof.*** *Let $I_1(R) = \int_R^\infty e^{-\eta_1 (x-R)} V(x)\, dx$ and $I_2(R) = \int_0^R e^{-\eta_2 (R-x)} V(x)\, dx$. Differentiation gives $I_1' = \eta_1 I_1 - V$ and $I_2' = V - \eta_2 I_2$. Substitute into the differentiated Fredholm equation and eliminate I_1, I_2 using the original equation; two successive differentiations yield the stated ODE.*

Theorem 8.2 (Solution)**.** *The homogeneous solution of the ODE has characteristic roots β_1, β_2 from Corollary 6.1. The general solution is*

$$V(R) \;=\; A_1\, e^{\beta_1 R} \;+\; A_2\, e^{\beta_2 R} \;+\; V_p(R).$$

Imposing $V(R) < \infty$ as $R \to \infty$ and $\beta_1 > 0$ forces $A_1 = 0$. The physically meaningful solution is

$$\boxed{V(R) \;=\; A_2\, e^{-\zeta R} \;+\; V_p(R), \qquad \zeta = |\beta_2| > 0,}$$

with A_2 determined by the boundary condition at $R = 0$ (ruin or reflecting barrier).

Remark 8.3. The decay rate ζ encodes the marginal value of wealth: $V'(R) = -A_2 \zeta e^{-\zeta R}$. As $\lambda_1 \uparrow$ (more intense wealth gains), the discriminant increases and ζ rises — marginal value falls faster, accelerating accountability collapse.

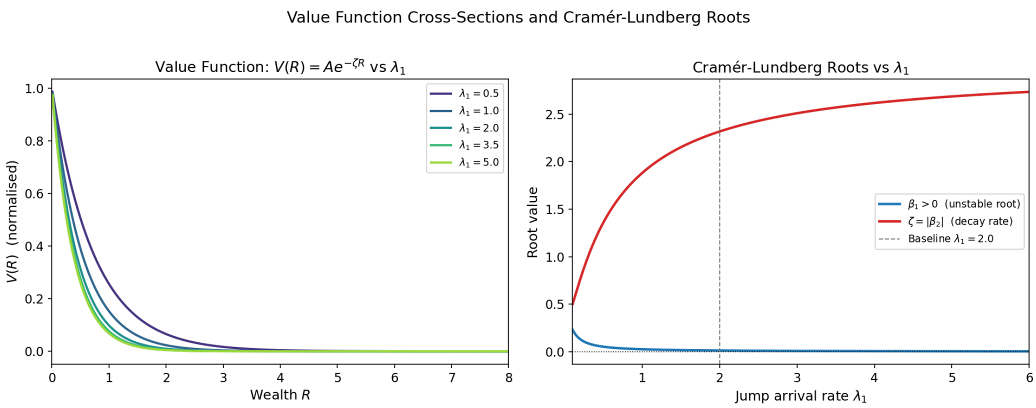

Figure 3: Left: value function cross-sections $V(R) = Ae^{-\zeta R}$ for increasing λ_1; higher arrival rate compresses the value function. Right: Cramér-Lundberg roots β_1, ζ as functions of λ_1.

9. Accountability Collapse

Definition 9.1 (Justice Curve)**.** The justice curve $\mathcal{J}$ is the locus in (R, Ω) space where authority and accountability are balanced:

$$\mathcal{J} \;=\; \{(R, \Omega) : R^\gamma \eta \;=\; \Omega\, P\}, \qquad \Omega_{\mathcal{J}}(R) = \frac{\eta}{P}\, R^\gamma.$$

Theorem 9.2 (Accountability Collapse)**.** *Under the optimal policy, the system $(R(t), \Omega(t))$*

satisfies: (i) If (R_0, Ω_0) lies below $\mathcal{J}$, the agent initially invests in prosocial behaviour ($s^* = 1$). (ii) Once wealth crosses the threshold $R^* = (\Omega P/\eta)^{1/\gamma}$, the optimal control switches permanently to $s^* = 0$, and the trajectory never re-enters the prosocial region. (iii) In the limit $R \to \infty$,

$$\lim_{R\to\infty} s^*(R, \Omega^*(R)) = 0, \qquad \lim_{R\to\infty} \Omega^*(R) = 0, \qquad \lim_{R\to\infty} A(R) = \infty.$$

Proof. *Part (i) follows from Lemma 3.1. Part (ii): since $\dot{R} > 0$ on average ($\lambda_1/\eta_1 > \lambda_2/\eta_2$) and $\Omega_{\mathcal{J}}$ is increasing in R, the trajectory crosses $\mathcal{J}$ from above in Ω in finite time. After crossing, the derivative condition $\partial H/\partial s < 0$ is strict for all $R > R^*$, so the bang-bang control is locked. Part (iii) follows from $V'(R) = -A_2\zeta e^{-\zeta R} \to 0$ and the consequent vanishing of the shadow price of accountability.*

Corollary 9.3 (Justice Gap)**.** *Define the justice gap $J(R) = A(R) - \Omega^*(R)$. Then $J(R) = R^\gamma - \Omega^*(R) \to \infty$ as $R \to \infty$, with divergence rate at least R^γ.*

Remark 9.4. The collapse is not a failure of rationality — it is the consequence of rationality under unbounded resource accumulation. The agent optimally abandons prosocial behaviour precisely because it ceases to improve the value function. Accountability is not destroyed by malice; it is rendered irrelevant by the vanishing marginal cost of its absence.

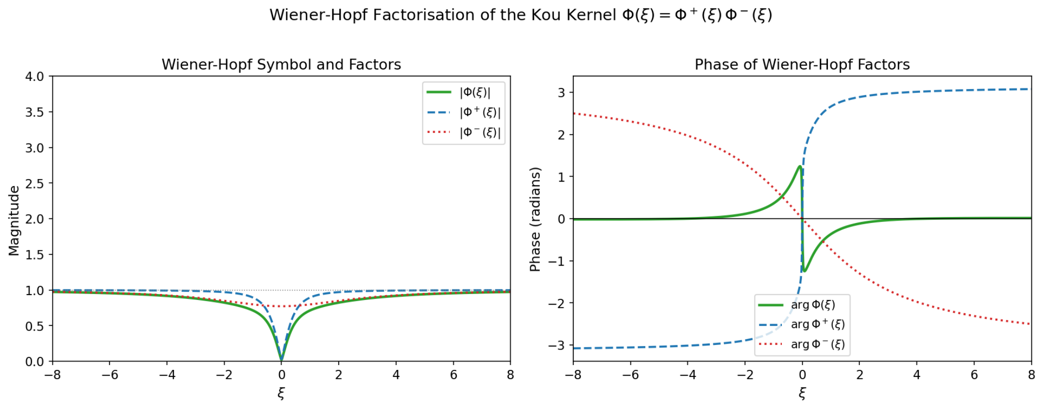

Figure 4: Left: Wiener-Hopf symbol $|\Phi(\xi)|$ and its analytic factors $|\Phi^+|$, $|\Phi^-|$. Right: phase of the factors; the additive phase decomposition confirms analyticity in the respective half-planes.

10. Algorithms

10.1 Algorithm 1: Cramér-Lundberg Root Computation

```
Input:  ,  ,  ,  ,
Output:   > 0,   < 0,   =  ||

Step 1: Compute q ←  +  +
Step 2: Compute B ← q ( -  ) -   +
Step 3: Compute C ←
Step 4: Compute Δ ← B² + 4qC
```

```
Step 5:   ← (B + √Δ) / (2q)          [positive root]
Step 6:   ← (B − √Δ) / (2q)          [negative root]
Step 7:    ← −                        [decay exponent]
Step 8: Verify   > 0,   < 0,   <  ,   > −
Step 9: Return  ,  ,
```

10.2 Algorithm 2: Wiener-Hopf Factorisation

```
Input:   ,   = − ,  ,  , q,  ,  ,  , grid   −Ξ[, Ξ]
Output: Φ (), Φ (), Φ (), normalisation C

Step 1: Compute symbol
        Φ () ← 1 −   /[q −( i)] −   /[q (+ i)]

Step 2: Compute analytic factors
        Φ () ←  ( + i) /  ( − i)   [upper half-plane analytic]
        Φ () ← ( − i) /  ( + i)    [lower half-plane analytic]

Step 3: Compute normalisation
         C ← Φ(0) / Φ [(0) · Φ (0)]
             = (/(q)) ·    [/()]

Step 4: Verify factorisation on grid
        err ← max || Φ |() − C Φ () Φ ()|
        Assert err < tolerance

Step 5: Solve Wiener-Hopf equation
       ˆ V () ← ĝ () / Φ ()           [upper half-plane component]
       ˆ V () ← ĝ () / Φ ()           [lower half-plane component]

Step 6: Invert
        V(R) ← IFT[ V](R) + IFT[ V](R)

Step 7: Return Φ, Φ , Φ ,  C, V
```

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