SINH-ACCELERATION: EFFICIENT EVALUATION OF PROBABILITY DISTRIBUTIONS, OPTION PRICING, AND MONTE-CARLO SIMULATIONS

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Abstract. Characteristic functions of several popular classes of distributions and processes admit analytic continuation into unions of strips and open coni around $\mathbb{R} \subset \mathbb{C}$. The Fourier transform techniques reduces calculation of probability distributions and option prices to evaluation of integrals whose integrands are analytic in domains enjoying these properties. In the paper, we suggest to use changes of variables of the form $\xi = \sqrt{-1}\omega_1 + b\sinh(\sqrt{-1}\omega + y)$ and the simplified trapezoid rule to evaluate the integrals accurately and fast. We formulate the general scheme, and apply the scheme for calculation probability distributions and pricing European options in Lévy models, the Heston model, the CIR model, and a subordinated NTS model. We outline applications to fast and accurate calibration procedures and Monte Carlo simulations.

Key words: sinh-regular Lévy processes, sinh-regular distributions, sinh-acceleration, Heston model, KoBoL, CGMY, CIR, CIR subordinator, Monte-Carlo simulations

1. Introduction

In the paper, we formulate general conditions on integrals, which arise in the Laplace and Fourier inversion, the Wiener-Hopf factorization, calculation of probability distributions, and pricing options and other derivative securities, which make it possible to evaluate these integrals very accurately and fast. We start with the explanation of the main idea of the suggested methodology in the case of one-dimensional integrals of the form

$$I = \int_{\operatorname{Im}\xi=\omega_0} e^{-ix\xi} g(\xi) d\xi, \tag{1.1}$$

where $i = \sqrt{-1}$, $x \in \mathbb{R}$, the line of integration $\{\operatorname{Im}\xi = \omega_0\}$ is in the domain of analyticity of g, and $g(\xi)$ decays sufficiently fast as $\xi \to \infty$ remaining in the strip sandwiched between the lines $\mathbb{R}$ and $\{\operatorname{Im}\xi = \omega_0\}$. In probability, the simplest integrals of this kind appear when the characteristic function $g(\xi) = \mathbb{E}[e^{i\xi Y}]$ of a random variable Y is well-defined not only on $\mathbb{R}$ but on the line $\{\operatorname{Im}\xi = \omega_0\}$ as well. Then the RHS of (1.1) is the probability distribution function (pdf) $p_Y(x)$ of Y evaluated at x (and multiplied by 2π).

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The first step of our methodology is the following change the variable in (1.1):

$$\xi = \chi_{\omega_1,\omega;b}(y) = i\omega_1 + b\sinh(i\omega + y), \tag{1.2}$$

where $\omega_0, \omega_1 \in \mathbb{R}, \omega \in [-\pi/2, \pi/2]$ and $b > 0$ are related as follows: $\omega_0 = \omega_1 + b\sin(\omega)$. We call the change of variables (1.2) the *sinh-acceleration* (in the case of multiple integrals, we make an appropriate sinh-accelerations w.r.t. to each argument). In the y- coordinate, we integrate over the real line. The change of variables can be justified if the integrand $f(y) = e^{-ix\chi_{\omega_1,\omega;b}(y)}g(\chi_{\omega_1,\omega;b}(y))\chi'_{\omega_1,\omega;b}(y)$ admits analytic continuation to a sufficiently wide strip $S_{(d_-,d_+)} = \{y \in \mathbb{C} \mid \operatorname{Im} y \in (d_-, d_+)\}$ and decays sufficiently fast as $y \to \infty$ remaining in the strip. In more detail, the Cauchy integral theorem allows us to deform the line of integration $\{\operatorname{Im}\xi = \omega_0\}$ into the contour $\mathcal{L}_{\omega_1,\omega;b} := \chi_{\omega_1,\omega;b}(\mathbb{R})$. In the integral over $\mathcal{L}_{\omega_1,\omega;b}$, we make the change of variables (1.2).

The sinh-acceleration is possible iff the integrand admits analytic continuation to the union of a strip S around the line of integration $\{\operatorname{Im}\xi = \omega_0\}$ and an appropriate conus, and vanishes as $\xi \to \infty$ remaining in the conus. In some important cases, analytic continuation to a wider region in an appropriate Riemann surface is possible, and then the speed of the method improves.

We failed to invent a short name for a class of functions enjoying these properties: whereas the name "functions analytic in a strip (and decaying at infinity)" is not exceedingly clumsy, the name "functions analytic in a union of a strip and conus" does seem clumsy. We suggest the name *sinh-regular functions*. We use the same adjective *sinh-regular* for distributions and processes that lead to integrals of sinh-regular functions. For the same process, in different problems, the sinh-acceleration with different sets of parameters ω_1, ω, b needs to be used, therefore, we will formulate general conditions on the characteristic function of the process in terms of the strip (in multi-factor models, tube domain) and conus of analyticity, and list several wide classes of sinh-regular processes and distributions.

The second step of our methodology is quite standard: the discretization of the integral using the infinite trapezoid rule. If the integrand is analytic in a strip $S_{(\omega_0-d,\omega_0+d)}$ around the line of integration and decays sufficiently fast as $\xi \to \infty$ remaining in the strip, the discretization error of the simplified trapezoid rule decays exponentially as a function of the reciprocal to the mesh size ζ. Hence, a small error tolerance can be satisfied quite easily. Next, the infinite sum must be truncated; the resulting formula is called the simplified trapezoid rule:

$$I = \zeta \sum_{|j|\le N} f(j\zeta).$$

As it is common in the literature, one can apply the simplifying trapezoid rule to the initial integral. However, in many cases of interest, $g(\xi)$ decays slowly as $\xi \to \infty$ remaining in the strip, hence, the truncation error decays slowly as the number of terms of the simplified trapezoid rule increases. For instance, even a moderately accurate evaluation of the probability distribution of a Lévy process may require dozens of million of terms and more. The sinh-acceleration exponentially increases the rate of decay of the integrand, and the number N of terms sufficient to satisfy a given error significantly decreases. In many cases, $N < 10$ suffice to satisfy the error tolerance $\epsilon = 10^{-7}$; typically, less than 50 terms suffice, and in essentially all cases of interest, N of the order of 100-150 suffices to satisfy the error tolerance 10^{-12}.

A similar trick with the fractional-parabolic changes of the variables of the form

$$\xi = \chi^{\pm}_{\omega;\sigma;\alpha}(\eta) = i\omega \pm i\sigma(1 \mp i\eta)^{\alpha}, \tag{1.3}$$

where $\omega \in \mathbb{R}, \sigma > 0, \alpha > 1$, was systematically used in a series of papers [7, 28, 6, 32, 13, 5, 30, 8, 31, 20, 21]. In the working paper [29], it was suggested to use the sinh-acceleration $\eta = \sinh(ay)/a$ with integration over the real line after the fractional-parabolic change of variables has been made. In the present paper, we use the sinh-acceleration only, in the more general form (1.2). Depending on the sign of ω, the new contour of integration $\mathcal{L}_{\omega_1,\omega,b}$ is deformed either upward or downward, and the deformed contours enjoy properties similar to the properties of the contours in [7, 28, 6, 32, 13, 5, 30, 8, 31, 20, 21, 29]. The number of terms of the simplified trapezoid rule is approximately the same as in [29] (and smaller than in [7, 28, 6, 32, 13, 5, 30, 8, 31, 20, 21], where thousands of terms are needed in some cases) but the number of elementary operations needed to calculate individual terms decreases. The general scheme is simpler than the one in [29].

The rest of the paper is organized as follows. In Section 2, we explain the sinh-acceleration techniques in applications to evaluation of probability distribution functions of wide classes of Lévy processes and infinitely divisible distributions, which we call sinh-regular processes and distributions. The class contains almost all popular classes of Lévy processes used in finance, conditional probability distributions in the Heston model, more general stochastic volatility models, affine and quadratic interest rate models, models with Wishart dynamics, Barndorf-Nielsen and Shephard model, 3/2 model,.... As a basic numerical example, we consider the probability distributions in the NTS model [3]. In Section 3, 4 and 5, we consider pricing European options in sinh-regular Lévy models and the Heston model, the CIR model, and the subordinated NTS model, the subordinator being the aggregated square root process. Section 7 summarizes the results of the paper and outlines natural extensions. Technical results and tables are relegated to Sections A and Section B, respectively.

2. SINH-regular Lévy processes and infinitely divisible distributions

2.1. **Definition.** In [10], two almost equivalent definitions of a wide family of *Regular Lévy Processes of exponential type* (RLPE) are given: one in terms of the Lévy density (exponential decay at infinity), the other one in terms of the characteristic exponent (analytic in a strip around the real axis for processes on $\mathbb{R}$, and, for processes on $\mathbb{R}^n$, in a tube domain $\mathbb{R}^n + iU$, where $U \subset \mathbb{R}^n$ is an open set containing 0). The class of RLPEs contains all classes of processes (model classes) popular in quantitative finance. The class of tempered stable Lévy process as defined in [41] is a subclass of RPLEs. In [7, 6, 32], it was noticed that the characteristic exponents of processes of the model classes admit analytic continuation to much wider regions of the complex plane and appropriate Riemann surfaces, and enjoy several properties useful for the construction of new efficient methods for pricing contingent claims. In this paper, we relax the general conditions of the definition of *strongly regular Lévy processes of exponential type* (sRLPE) introduced in [6]. Additional conditions formulated in [6, 32] were needed for the construction of more efficient methods when the Wiener-Hopf factorization and the fractional-parabolic change of variables were used. When the sinh-acceleration is used instead, the advantage of these additional conditions is marginal.

Let $\lambda_- < 0 < \lambda_+$ and $-\pi/2 \le \gamma^- < \gamma^+ \le \pi/2$ and either $\gamma^- \le 0 < \gamma^+$ or $\gamma^- < 0 \le \gamma^+$. We define the conus $\mathcal{C}_{\gamma^-,\gamma^+} = \{e^{i\varphi}\rho \mid \rho \ge 0, \varphi \in (\gamma^-,\gamma^+) \cup (\pi-\gamma^+, \pi-\gamma^-)\}$, and the set

$$U(\lambda_-,\lambda_+;\gamma^-,\gamma^+) = i(\lambda_-,\lambda_+) + \mathcal{C}_{\gamma^-,\gamma^+} := \{ia + z \mid \lambda_- < a < \lambda_+, z \in \mathcal{C}_{\gamma^-,\gamma^+}\}. \tag{2.1}$$

As in [6], we represent the characteristic exponent in the form

$$\psi(\xi) = -i\mu\xi + \psi^0(\xi), \tag{2.2}$$

and impose conditions on ψ^0.

We need coni $\mathcal{C} = \mathcal{C}_{\gamma^-,\gamma^+}$ of several kind:

(1) to describe a domain $U = i(\lambda_-,\lambda_+) + \mathcal{C}_{\gamma^-,\gamma^+}$ of analyticity of the characteristic exponent:
(2) to introduce a subset $U^u = i(\lambda_-,\lambda_+) + \mathcal{C}^u$ where $|\psi^0(\xi)|$ admits a useful upper bound;
(3) to introduce a subset $U^l = i(\lambda_-,\lambda_+) + \mathcal{C}^l$ where $\operatorname{Re}\psi^0(\xi)$ admits a useful lower bound.

In many cases, the coni are around the real axis. However, if $\psi^0(\xi) = o(|\xi|)$ as $\xi \to \infty$ in the domain of analyticity, and $\mu \ne 0$, then, to calculate the pdf not at the peak and price options that are not at the money, we will have to choose a domain $U' = i(\lambda_-,\lambda_+) + \mathcal{C}'$, where the conus $\mathcal{C}'$ is either in the upper half-plane or low half-plane.[1]

Definition 2.1. *We say that X is a SINH-regular Lévy process (on $\mathbb{R}$) of type $((\mu_-,\mu_+);\mathcal{C};\mathcal{C}_+)$ and order $\nu \in (0,2]$ iff the following conditions are satisfied:*

(i) $\mu_- < 0 < \mu_+$;
(ii) $\mathcal{C}_+ \subset \mathcal{C} \subset \mathbb{C}$ are open coni adjacent to or containing the real axis;
(iii) ψ, the characteristic exponent of X, admits analytic continuation to $i(\mu_-,\mu_+) + \mathcal{C}$;
(iv) for any $\mu'_- \in (\mu_-, 0)$, $\mu'_+ \in (0,\mu_+)$ and an open sub-cone $\mathcal{C}^u \subset \mathcal{C}$ adjacent to or containing the real axis, there exist $C > 0$ such that

$$|\psi^0(\xi)| \le C(1+|\xi|)^\nu, \quad \forall\ \xi \in i[\mu'_-,\mu'_+] + \mathcal{C}^u; \tag{2.3}$$

(v) for any closed sub-cone $\mathcal{C}^l_+ \subset \mathcal{C}_+$ and any $[\mu'_-,\mu'_+] \subset (\mu_-,\mu_+)$, there exist $c, C > 0$ such that

$$\operatorname{Re}\psi^0(\xi) \ge c|\xi|^\nu - C, \quad \forall\ \xi \in i[\mu'_-,\mu'_+] + \mathcal{C}^l_+. \tag{2.4}$$

We say that a distribution is a SINH-regular infinitely divisible distribution of type $((\mu_-,\mu_+);\mathcal{C};\mathcal{C}_+)$ and order ν iff it is the distribution of X_1, where X is a SINH-regular Lévy process of type $((\mu_-,\mu_+);\mathcal{C};\mathcal{C}_+)$ and order ν.

Definition 2.2. *We say that X is an elliptic SINH-regular Lévy process of type $((\mu_-,\mu_+);\mathcal{C};\mathcal{C}_+)$ and order ν if conditions (i)-(iii) and the following two conditions hold*

(iv)' for any $\xi_0 \in \mathcal{C} \cap \{\xi \mid |\xi| = 1\}$, as $\rho \to +\infty$,

$$\psi^0(\rho\xi_0) \sim c_\infty(\arg\xi_0)\rho^\nu, \tag{2.5}$$

where $c_\infty(\arg\xi_0) := c_\infty(\psi^0, \arg\xi_0)$ is continuous;

(v)' for any $\xi_0 \in \overline{\mathcal{C}^l_+} \cap \{\xi \mid |\xi| = 1\}$, $\operatorname{Re} c_\infty(\arg\xi_0) > 0$.

We say that a distribution is an elliptic SINH-regular infinitely divisible distribution iff it is the distribution of X_1, where X is an elliptic SINH-regular Lévy process.

[1] Additional conditions on sets of the form (2.1) are needed when the Wiener-Hopf factors are calculated; these conditions depend on the spectral parameter.

Remark 2.1. a) The properties needed for efficient calculations are formulated in the language of Complex Analysis, and cannot be naturally formulated in the probabilistic language. Indeed, such simple processes as the Brownian motion with an embedded compound Poisson process with the Lévy density $\mathbf{1}_{[a,b]}$, where $a < 0 < b$, are RLPEs and their characteristic exponents are analytic in the complex plane but, on any conus, $\operatorname{Re}\psi^0(\xi)$ is not semi-bounded, and the crucial property (2.4) fails. However, if either $a = 0, b > 0$ or $a < 0, b = 0$, then such a process is an elliptic SINH-regular process of type $((\mu_-, \mu_+); \mathcal{C}; \mathcal{C}_+)$, where $\mu_- < 0 < \mu_+$ are arbitrary, and $\mathcal{C}, \mathcal{C}_+$ are coni in the lower half plane if $a < 0, b = 0$, and upper half plane if $a = 0, b > 0$.

b) The properties formalized in Definitions 2.1 and 2.2 for characteristic exponents hold for wide classes of the symbols of pseudo-differential operators (PDO) which have no relation to probability, and the methods based on these properties can be applied to develop efficient numerical methods for various boundary problems for such PDOs.

Remark 2.2. If a subordinator Y and process X are elliptic SINH-regular Lévy processes, then $\{X_{Y_t}\}$ is an elliptic SINH-regular Lévy process. See Section 5 for an example.

2.2. **Examples and some generalizations.**

(1) Essentially all Lévy processes used in quantitative finance are elliptic SINH-regular Lévy processes: Brownian motion (BM), Merton model [40], NIG (normal inverse Gaussian model) [2], hyperbolic processes [14], double-exponential jump-diffusion model [33, 34, 22, 23, 24], its generalization: hyper-exponential jump-diffusion model, introduced in [26, 33] and studied in detail in [26, 27], the majority of processes of the β-class [25]; the generalized Koponen's family [9] and its subclass KoBoL [10]. A subclass of KoBoL (known as the CGMY model - see [11]) is given by the characteristic exponent

$$\psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu], \tag{2.6}$$

where $\nu \in (0,2), \nu \neq 1$ (in the case $\nu = 1$, the analytical expression is different: see [9, 10]). Thus, KoBoL is SINH-regular of type $((\lambda_-, \lambda_+); \mathcal{C}, \mathcal{C}_+)$ of order ν, where $\mathcal{C} = \mathcal{C}_{-\gamma,\gamma}$, $\gamma \geq \pi/2$, and $\mathcal{C}_+ = \mathcal{C}_{-\gamma',\gamma'}$, where $\gamma' = \pi/(2\nu)$ (see (3) below for the meaning of $\gamma' > \pi/2$). BM, DEJD and HEJD are of order $\nu = 2$, and NIG is of order $\nu = 1$.

The characteristic exponents of NTS processes constructed in [3] are given by

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 + (\xi + i\beta)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2}], \tag{2.7}$$

where $\nu \in (0,2)$, $\delta > 0$, $|\beta| < \alpha$. This is a process of type $((\alpha + \beta, \alpha - \beta); \mathcal{C}, \mathcal{C}_+)$ of order ν, where $\mathcal{C}$ and $\mathcal{C}_+$ are the same as for KoBoL of the same order.

(2) In order to consider Variance Gamma processes (VG) [37], Definitions 2.1-2.2 must be generalized replacing the function $\rho \mapsto \rho^\nu$ with a strictly increasing function $w : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $w(+\infty) = +\infty$. We say: X is an (elliptic) SINH-regular Lévy process of type $(S, \mathcal{C}, \mathcal{C}_+, w)$. For Variance Gamma processes, $w(\rho) = \ln(1 + \rho)$.

(3) For KoBoL, VG and NTS, ψ^0 admits analytic continuation into an appropriate Riemann surface $\mathcal{R}$, and $\mathcal{C}$ can be defined as an appropriate subset of $\mathcal{R}$. Formally, in (1), $\gamma > \pi/2$ is admissible with the understanding that seemingly overlapping parts of $\mathcal{C}_{-\gamma,\gamma}$ lie on different sheets of $\mathcal{R}$. See [7, 6, 32], where advantages of $\mathcal{C}_+ \subset \mathcal{R}$ were utilized to increase the speed. The same extension is very useful when the SINH-acceleration is applied to calculate the Wiener-Hopf factors, and less so for pricing European options.

(4) The asymptotic coefficient $c_\infty(\arg \xi_0)$ is

(i) if X is BM, DEJD and HEJD, $c_\infty(\varphi) = (\sigma^2/2)e^{i2\varphi}$, hence, $\mathcal{C}_+ = \mathcal{C}_{-\pi/4,\pi/4}$;
(ii) if X is given by (2.6), then, for $\varphi \in [-\pi/(2\nu), \pi/(2\nu)]$,

$$c_\infty(\varphi) = -2c\Gamma(-\nu)\cos(\pi\nu/2)e^{i\nu\varphi}, \tag{2.8}$$

hence, $\mathcal{C}_+ = \mathcal{C}_{-\pi/(2\nu),\pi/(2\nu)}$;
(iii) if X is given by (2.7), then, for $\varphi \in [-\pi/(2\nu), \pi/(2\nu)]$,

$$c_\infty(i\varphi) = \delta e^{i\nu\varphi}, \tag{2.9}$$

hence, $\mathcal{C}_+ = \mathcal{C}_{-\pi/(2\nu),\pi/(2\nu)}$.

(5) In [9], we constructed more general classes of Lévy processes, with the characteristic exponents of the form

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}], \tag{2.10}$$

where $c_\pm \geq 0$, $c_+ + c_- > 0$, $\lambda_- < 0 < \lambda_+, \nu_\pm \in (0,2), \nu_\pm \neq 1$, with modifications in the case $\nu_+ = 1$ and/or $\nu_- = 1$. For these processes, the domains of analyticity and bounds are more involved. In particular, in general, the coni are not symmetric w.r.t. the real axis.
(6) In examples above, $\operatorname{Re}\psi^0(\xi) \to +\infty$ as $\xi \to \infty$ in a conus around the real axis, due to special forms of the Lévy densities, hence, characteristic functions. If $\psi^0(\xi)$ contains terms of the form $Ce^{ic\xi}$, where $c \in \mathbb{R}$, then
(i) if $c < 0$, then $\operatorname{Re}\psi^0(\xi)$ is not semibounded (from below) as $\xi \to \infty$ in any conus $\mathcal{C}$ in the upper half-plane;
(ii) if $c > 0$, then $\operatorname{Re}\psi^0(\xi)$ is not semibounded (from below) as $\xi \to \infty$ in any conus in the lower half-plane.

The simplest example is the BM with the embedded jumps, the Lévy density being $\mathbf{1}_{[-a,b]}$. If $a < 0 < b$, then $\operatorname{Re}\psi^0(\xi)$ is not semibounded as $\xi \to \infty$ in any conus $\mathcal{C}$.
(7) If X is the BM with embedded negative jumps only, and the jump density decays exponentially at infinity, then X is an elliptic SINH-regular process of order 2 and type $((0,+\infty);\mathcal{C},\mathcal{C}_{0,\pi/4})$, where $\mathcal{C}$ is the upper half-plane. If X is the BM motion with embedded positive jumps only, then X is an elliptic SINH-regular of order 2 and type $((-\infty,0);\mathcal{C},\mathcal{C}_{-\pi/4,0})$, where $\mathcal{C}$ is the lower half-plane.
(8) In Example (7), one may any add a positive (resp., negative) jump component as in KoBoL or exponential jump-diffusions. One can also replace the BM with one-sided SINH-regular processes of type $((\mu_-,\mu_+),\mathcal{C},\mathcal{C}_+)$, where $\mu_- < 0 < \mu_+$ and $\mathcal{C}_+, \mathcal{C}$ are coni around the real axis. In both cases, the type of the resulting process X will be characterized by smaller domains of analyticity than in the case of the BM with embedded one-sided jump components.
(9) Conditional distributions in affine stochastic volatility models and affine and quadratic interest rate models are sinh-regular.

2.3. **Calculation of probability distributions.** The pdf of X_t equals

$$p_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix'\xi - t\psi^0(\xi)}d\xi, \tag{2.11}$$

where $x' = x - \mu t$. Denote $g(\xi) = e^{-t\psi^0(\xi)}$. The change of variables (1.2) can be justified if the integrand $f(y) = e^{-ix'\chi_{\omega_1,\omega;b}(y)}g(\chi_{\omega_1,\omega;b}(y))\chi'_{\omega_1,\omega;b}(y)$ admits analytic continuation to a strip $S_{(-d,d)} = \{y \in \mathbb{C} \mid \operatorname{Im} y \in (-d,d)\}$ around the real line and decays sufficiently fast as

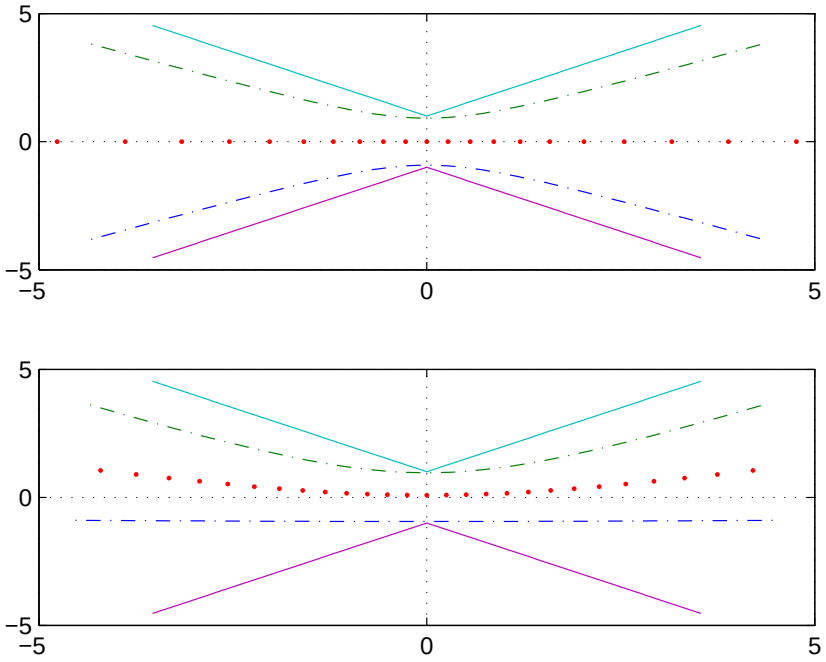

FIGURE 1. Solid lines: boundaries of the domain of analyticity $S_{(-1,1)}+\mathcal{C}_{-\pi/4,\pi/4}$ in ξ-coordinate. Dots: points $\xi_j=\chi_{\omega_1,\omega;b}(y_j)=i\omega_1+b\sinh(i\omega+y_j)$ used in the simplified trapezoid rule. Dots-dashes: boundaries of the image $\chi_{\omega_1,\omega;b}(S_{(-d,d)})$ of the strip of analyticity $S_{(-d,d)}$. Upper panel: $\omega_1=\omega=0$, $d=\pi/4$, $b=1/\sin(\pi/4)$. Lower panel: $\omega_1=-1,\omega=d=\pi/8$, $b=2/\sin(\pi/8)$. For the calculations represented in the lower panel, only a smaller domain $S_{(-1,1)}+\mathcal{C}_{0,\pi/4}$ matters.

$y\to\infty$ remaining in the strip. In more detail, the Cauchy integral theorem allows us to deform the line of integration $\{\operatorname{Im}\xi=\omega_0\}$ into the contour $\mathcal{L}_{\omega_1,\omega;b}:=\chi_{\omega_1,\omega;b}(\mathbb{R})$. In the integral over $\mathcal{L}_{\omega_1,\omega;b}$, we make the change of variables (1.2).

The choice of the parameters of the sinh-acceleration depends on the type of the process, its order, and x'. It is convenient to consider separately the following cases:

(1) $\mathcal{C}_+=\mathcal{C}_{\gamma^-,\gamma^+}$, where $\gamma^-<0<\gamma^+$, $\nu\in(1,2]$ or $\nu\in(0,1]$ and $x'=0$: as $\xi\to\infty$, the leading term of the asymptotics of $-ix'\xi-t\psi^0(\xi)$ is the same as the one of $-t\psi^0(\xi)$, hence, one can choose the parameters of the sinh-acceleration without taking x' into account. The line of integration may remain flat or it can be deformed either upward or downward. The conus of analyticity used to derive the error bound can be around the real axis (see the upper panel in Fig.1 for illustration), which allows one to use the mesh of a larger size than in the other cases. Naturally, if $\nu>1$ is close to 1 and/or x' is large in the absolute value, then it is safer to take the sign of x' into account, and deform the contour as in the case $\nu\in(0,1)$.

(2) $\mathcal{C}_+ = \mathcal{C}_{\gamma^-,\gamma^+}$, where $\gamma^- = 0 < \gamma^+$, $\nu \in (0,1)$, $x' < 0$. As $\xi \to \infty$, the leading term of the asymptotics of $-ix'\xi - t\psi^0(\xi)$ is the same as the one of $-ix'\xi$, hence, the deformed contour and the conus of analyticity used to derive the error bound must be in the upper half-plane even if $\gamma^- < 0$ (see the lower panel in Fig.1 for illustration).
(3) $\mathcal{C}_+ = \mathcal{C}_{\gamma^-,\gamma^+}$, where $\gamma^- < 0 = \gamma^+$, $\nu \in (0,1)$, $x' > 0$. As $\xi \to \infty$, the leading term of the asymptotics of $-ix'\xi - t\psi^0(\xi)$ is the same as the one of $-ix'\xi$, hence, the deformed contour and the conus of analyticity used to derive the error bound must be in the lower half-plane even if $\gamma^+ > 0$.
(4) $\mathcal{C}_+ = \mathcal{C}_{\gamma^-,\gamma^+}$, where $\gamma^- < 0 < \gamma^+$, $\nu = 1$, $x' \neq 0$. It is optimal to deform the contour but the conus of analyticity used to derive the error bound can be around the real axis.
(5) $\mathcal{C}_+ = \mathcal{C}_{\gamma^-,\gamma^+}$, where either $\gamma^- = 0 < \gamma^+$ or $\gamma^- < 0 = \gamma^+$, and $\nu = 1$. The deformed contour and the conus of analyticity used to derive the error bound must be in the lower half-plane even if $\gamma^+ > 0$ and in the lower half-plane even if $\gamma^- < 0$.

In the next subsection, we assume that $\mathcal{C}_+ = \mathcal{C}_{\gamma^-,\gamma^+}$, where $\gamma^- < 0 < \gamma^+$. The reader can easily modify the constructions below for the cases $\gamma^- = 0 < \gamma^+$ and $\gamma^- < 0 = \gamma^+$.

2.3.1. *The case $\nu \in (1,2]$ and the case $\nu \in (0,1]$, $x' = 0$.* In these cases, for any $\gamma^{-,\prime} \in (\gamma^-, 0), \gamma^{+,\prime} \in (0, \gamma^+)$, there exists $c > 0$ such that

$$\operatorname{Re} c_\infty(\varphi) \geq c, \quad \gamma^{-,\prime} \leq \varphi \leq \gamma^{+,\prime}. \tag{2.12}$$

First, we choose $\omega \in \mathbb{R}$ and $d_0 > 0$ so that $\omega + d_0 \leq \gamma^+, \omega - d_0 \geq \gamma^-$. Since d_0 is the upper bound for the half-width of the strip of analyticity in the y-coordinate, we want to choose d_0 as large as possible. Hence, we set

$$\omega = (\gamma^+ + \gamma^-)/2, \ d_0 = (\gamma^+ - \gamma^-)/2; \tag{2.13}$$

then $\omega + d_0 = \gamma^+, \omega - d_0 = \gamma^-$. Next, we must ensure that the intersection of the imaginary axis and the image of $S_{(-d_0,d_0)}$ under $\chi_{\omega_1,\omega;b}$ is a subset of (μ_-, μ_+), which is equivalent to $\omega_1 + ba_+ \leq \mu_+, \omega_1 - b \geq a_-\mu_-$, where $a_- = \sin(\min\{\pi/2, -\gamma^-\}), a_+ = \sin(\min\{\pi/2, \gamma^+\})$. We define

$$\omega_1 = \frac{\mu_+ a_- + \mu_- a_+}{a_+ + a_-}, \ b_0 = \frac{\mu_+ - \mu_-}{a_+ + a_-}. \tag{2.14}$$

If $\gamma^- = -\gamma^+$ which is the case for KoBoL and the generalization of NIG, (2.14) simplify

$$\omega_1 = \frac{\mu_+ + \mu_-}{2}, \ b_0 = \frac{\mu_+ - \mu_-}{2\sin(\min\{\pi/2, \gamma^+\})}. \tag{2.15}$$

In (1.2), we choose $d < d_0$, $b < b_0$ close to d_0, b_0, respectively, e.g., $d = 0.95d_0, b = 0.95d_0$. Then the integrand

$$f(y) = (1/2\pi)\exp[-ix'\chi_{\omega_1,\omega;b}(y) - t\psi^0(\chi_{\omega_1,\omega;b}(y))]\chi'_{\omega_1,\omega;b}(y)$$

admits analytic continuation to the strip $S_{(-d,d)} = \{y \in \mathbb{C} \mid \operatorname{Im} y \in (-d,d)\}$ around the real line and decays sufficiently fast as $y \to \infty$ remaining in the strip so that

$$\lim_{A \to \pm\infty} \int_{-d}^{d} |f(ia + A)| da = 0,$$

and the Hardy norm

$$H(f,d)=\lim_{a\downarrow -d}\int_{\mathbb{R}}|f(ia+y)|dy+\lim_{a\uparrow d}\int_{\mathbb{R}}|f(ia+y)|dy \tag{2.16}$$

is finite. Fix $\zeta>0$ and construct the grid $\{y_j=j\zeta, j\in\mathbb{Z}\}$. The discretization error of the infinite trapezoid rule

$$p_t(x)=\zeta\sum_{j\in\mathbb{Z}}f(y_j) \tag{2.17}$$

admits an upper bound via $H(f,d)\exp[-2\pi d/\zeta]/(1-\exp[-2\pi d/\zeta])$ (see Theorem 3.2.1 in [42] and Appendix in [28] for a simple proof). In some cases, the Hardy norm of the integrand as a function on a maximal strip of analyticity is infinite. In such cases, in order to use the universal bound for the discretization error, one has to apply the bound to functions on a narrower strip of analyticity; this explains our choices $d<d_0$ and $b<b_0$. A fairly accurate approximate bound for $H(f,d)$ can be derived relatively easily but, as in the case of the fractional-parabolic deformations [28, 30], the following rude approximation works well if the initial strip of analyticity is not very narrow and, typically, leads to an overkill:

$$H(f,d)=C(|f(-id)|+|f(id)|), \tag{2.18}$$

where $C=10$. To satisfy a small error tolerance $\epsilon>0$, we choose $\zeta=2\pi d/(\ln(H(f,d)/\epsilon)\sim 2\pi d/E$, where $E=\ln(1/\epsilon)$. The choice of N, the number of terms of the simplified trapezoid rule, equivalently, of the truncation parameter $\Lambda=N\zeta$, is somewhat more involved. The truncation error of the simplified trapezoid rule

$$p_t(x)=\zeta\sum_{|j|\le N}f(y_j) \tag{2.19}$$

can be approximated by the truncation error of the integral

$$Err_{tr}=2\int_{\Lambda}^{+\infty}|f(y)|dy. \tag{2.20}$$

If Λ is large, then, on $[\Lambda,+\infty)$, we can use an approximation

$$\chi_{\omega_1,\omega;b}(y)\sim(b/2)e^y e^{i\omega},\ \chi'_{\omega_1,\omega;b}(y)\sim(b/2)e^y e^{i\omega},$$

to derive an approximate upper bound

$$2\int_{\Lambda}^{+\infty}|f(y)|dy\le\frac{1}{\pi}\int_{\Lambda_1}^{+\infty}e^{\mathrm{Re}(-ix'e^{i\omega}\rho-t\psi(\rho e^{i\varphi})}d\rho,$$

where $\Lambda_1=(b/2)e^{\Lambda}$. The bound can be simplified (at the expense of some loss in accuracy) as

$$Err_{tr}(\Lambda_1)\le\frac{e^{tC_0}}{\pi}\int_{\Lambda_1}^{+\infty}e^{(x'\sin\omega)\rho-t\,\mathrm{Re}\,c_\infty(\omega)\rho^\nu}d\rho, \tag{2.21}$$

where $C_0=c\Gamma(-\nu)[\lambda_+^\nu+(-\lambda_-)^\nu]$ in the case of KoBoL, and $C_0=\delta(\alpha^2-\beta^2)^{\nu/2}$ in the case of NTS processes. If $c_\infty(\omega)=c_\infty(0)e^{i\omega\nu}$ as in the cases of KoBoL and NTS, then (2.21) can be written as

$$Err_{tr}(\Lambda_1)\le\frac{e^{tC_0}}{\pi}\int_{\Lambda_1}^{+\infty}e^{(x'\sin\omega)\rho-tc_\infty(0)\cos(\omega\nu)\rho^\nu}d\rho. \tag{2.22}$$

Given the error tolerance $\epsilon > 0$, one can find an approximation to Λ_1 satisfying $Err_{tr}(\Lambda_1) < \epsilon$ quite easily (see [7, 28, 30]), and then define

$$\Lambda = \ln(2\Lambda_1/b),\ N = \mathrm{ceil}(\Lambda/\zeta). \tag{2.23}$$

2.4. **The case $\nu \in (0,1)$, $x' < 0$.** Since $-ix'\xi - t\psi^0(\xi) \sim -ix'\xi$ as $\xi \to \infty$, we use the same construction as above replacing γ^- and γ^+ with $\gamma_0^- = 0$ and $\gamma_0^+ = \min\{\gamma^+, \pi\}$.

2.5. **The case $\nu \in (0,1)$, $x' > 0$.** We use the same construction as above replacing γ^- and γ^+ with $\gamma_0^- = \max\{\gamma^-, -\pi\}$ and $\gamma_0^+ = 0$.

2.6. **The case $\nu = 1$, $x' \neq 0$.** For simplicity, consider the case $c_\infty(\varphi) = c_\infty(0)e^{i\varphi}$, where $c_\infty(0) > 0$ is independent of φ. As $\rho \to +\infty$,

$$-ix'\rho e^{i\varphi} - t\psi^0(\rho e^{i\varphi}) = (-ix' - tc_\infty(0))e^{i\varphi} = -\sqrt{(x')^2 + (tc_\infty(0))^2}e^{i(\varphi+\varphi_0)},$$

where $\varphi_0 = \arctan(x'/(tc_\infty))$. Hence, we use the same constructions as above with

(i) $\gamma^- = -\pi/2 - \varphi_0$, $\gamma^+ = \pi/2$, if $x' < 0$ (hence, $\varphi_0 < 0$);
(ii) $\gamma^- = -\pi/2, \gamma^+ = \pi/2 - \varphi_0$, if $x' > 0$ (hence, $\varphi_0 > 0$).

The bound for the truncation error (2.21) can be made explicit

$$Err_{tr}(\Lambda_1) \le \frac{e^{tC_0}}{\pi(x'\sin\omega + tc_\infty\cos\omega)} e^{-(-x'\sin\omega + tc_\infty(0)\cos\omega)\Lambda_1}. \tag{2.24}$$

Given $\epsilon > 0$, we find

$$\Lambda_1 = \frac{\ln(1/\epsilon) + tC_0 - \ln(\pi(-x'\sin\omega + tc_\infty\cos\omega))}{x'\sin\omega + tc_\infty\cos\omega}, \tag{2.25}$$

and then apply (2.23).

2.7. **Complexity of the scheme based on the sinh-acceleration.** As $\epsilon \downarrow 0$, $\Lambda \sim \ln E$, where $E = E(\epsilon) = \ln(1/\epsilon)$, and $\zeta \sim E/(2\pi d)$, where $d < (\gamma^+ - \gamma^-)/2$ is fixed. Hence, the complexity of the scheme is of the order of $A(d)E\ln E$, where $A(d)$ can be arbitrarily close to $1/(\pi(\gamma^+ - \gamma^-))$ if d is chosen sufficiently close to $(\gamma^+ - \gamma^-)/2$. Note that if the integrand admits the analytic continuation to an appropriate Riemann surface, then γ^+ and/or $-\gamma^-$ can be larger than $\pi/2$. In particular, then, for NTS and KoBoL of order $\nu \in (0,1)$,

(i) if $x' = 0$, then $\gamma^+ = -\gamma^- = \pi/(2\nu)$,
(ii) if $x' < 0$, then $\gamma^+ = \min\{\pi, \pi/(2\nu)\}$, $\gamma^- = 0$;
(iii) if $x' > 0$, then $\gamma^+ = 0$, $\gamma^- = -\min\{\pi, \pi/(2\nu)\}$.

This implies that, rather counter-intuitively, the (asymptotic) complexity of the scheme decreases with ν, whereas for the flat iFT and the scheme based on the fractional-parabolic deformations, the complexity of the scheme increases as ν decreases.

Remark 2.3. The approximate bound for the complexity of the scheme derived above implicitly assumes that the strip is neither too wide nor too narrow, hence, b is neither too large nor too small. If the width of the strip becomes too large or too small, the Hardy norm may become too large. Hence, the approximations $\ln(2\Lambda_1/b) \sim \ln\Lambda_1$ and $\ln(H/\zeta) \sim \ln(1/\epsilon)$ which we used to access the complexity of the scheme may become not very accurate. The problem of a too wide strip can be fixed using a moderately wide strip instead of a very wide one.

If the strip $S_{(\mu_-,\mu_+)}$ is too narrow, but the Hardy norm does not tend to infinity as the strip shrinks (the case of the NTS model and KoBoL models), then the sinh-change of variables implies the rescaling which reduces the calculations to the case of a strip $S_{(-d,d)}$, where $d = k_d d_0$, $d_0 = (\gamma^+ - \gamma^-)/2$, $k_d = 0.9 - 0.95$. Hence, a more accurate approximation of the truncation parameter is $\ln \Lambda_1 + \ln(1/(\mu_+ - \mu_-))$ instead of $\ln \Lambda_1$. Even if the initial strip is extremely narrow, say, of the width 10^{-8}, the recommended truncation parameter increases by less than 30, and the number of terms needed to satisfy even vary small error tolerance remains quite moderate. In the case of the fractional-parabolic change of variables of order α (typically, $\alpha \in (1, 2)$), the truncation parameter increases by a factor of the order of $(\mu_+ - \mu_-)^{-1/\alpha}$, which can be large if $\mu_+ - \mu_-$, the width of the strip, is very small (see Section A.1 for the choice of the parameters of the fractional-parabolic method and analysis of its complexity). However, it is important that the angle $\gamma^- - \gamma^+$ between the rays $e^{i\gamma^-}\mathbb{R}_+$ and $e^{i\gamma^+}\mathbb{R}_+$ be not too small.

2.8. **Numerical example.** The calculations in the paper were performed in MATLAB© 8.0.0 (R2012b), on a PC with characteristics Intel (R) Core (TM) i7 CPU M 640, 2.80GHz, 8MB, under the Genuine Windows 7 Professional operating system.

In Section B.1, Tables 1 and 2, we show the pdf of X_t with the characteristic exponent (2.7) calculated using the sinh-acceleration, fractional parabolic change of variables and the standard inverse Fourier transform method (flat iFT). The parameters of the process are $\mu = 0$, $\alpha = 10, \beta = 0$ for $t = 0.004$; $\delta = m_2 \lambda^{\nu-2}$, where $m_2 = \psi''(0) = 0.1$ is the second instantaneous moment. In Table 1, ν varies, and the pdf is calculated at the peak. In Table 2, $\nu = 0.3$ is fixed, and x varies.

For the flat iFT, we use the accurate prescriptions for the choice of the mesh size ζ derived in [7], for the error tolerance $\epsilon = 10^{-7}$. By trial and error, we find that, due to the oscillation of terms in the infinite trapezoid rule, it is possible to increase the size of the mesh: ζ/k_ζ where $k_\zeta = 0.6$. Hence, we may use a smaller number of terms in the simplified trapezoid rule to satisfy a given error tolerance for the truncation error. Nevertheless, as the results shown in Tables 1 and 2 demonstrate, the flat iFT may require extremely large number of terms. At the same time, the sinh-acceleration allows one to satisfy a small error tolerance with several dozen of terms; fractional-parabolic method is less efficient than the sinh-acceleration.

It is easily see that unless ν is not small, equivalently, the process is close to the BM, it is essentially impossible to calculate the pdf at the peak sufficiently accurately and fast which is needed for an efficient MLE. A similar problem arises when the cumulative pdf is calculated and applied for simulation purposes.

3. Meromorphic SINH-regular distributions on $\mathbb{R}$

3.1. **Cumulative pdf in SINH-regular models.** In this case, we need to evaluate the integral of the same type but with an additional factor $-1/(i\xi)$ under the integral sign:

$$\mathbb{P}[X_t < x] = \frac{1}{2\pi} \int_{\operatorname{Im} \xi = \omega_0} \frac{e^{-ix'\xi - t\psi^0(\xi)}}{-i\xi} d\xi, \tag{3.1}$$

where $\omega_0 \in (0, \mu_+)$. Since there is a pole at 0, we can apply the same scheme as in Section 2.3 replacing μ_- with 0; the truncation parameter will be somewhat smaller due to the additional

decaying factor $1/(i\xi)$. To be more specific, instead of (2.21), we have the error bound

$$Err_{tr}(\Lambda_1) \le \frac{e^{tC_0}}{\pi}\int_{\Lambda_1}^{+\infty} e^{-(x'\sin\omega)\rho - t\,\mathrm{Re}\,c_\infty(\omega)\rho^\nu}\rho^{-1}d\rho. \tag{3.2}$$

If $x' > 0$ or μ_+ is small and $-\mu_- > \mu_+$, it is advantageous to move the line of integration in (3.1) down, and, on crossing the simple pole, apply the residue theorem:

$$\mathbb{P}[X_t < x] = 1 + \frac{1}{2\pi}\int_{\mathrm{Im}\,\xi=\omega_0'} \frac{e^{-ix'\xi - t\psi^0(\xi)}}{-i\xi}d\xi, \tag{3.3}$$

where $\omega_0' \in (\mu_-, 0)$. The integral on the RHS of (3.3) is calculated as in Section 2.3, with $\mu_+ = 0$.

3.2. **Puts and calls in SINH-regular models.** Let r be the riskless rate, τ the time to maturity, K the strike, and S the spot. Set $x' = \ln(S/K) + \mu\tau$. Assuming that $\mu_- < -1$, the price of the call option is given by

$$V_{\mathrm{call}}(\tau, S) = -\frac{Ke^{-r\tau}}{2\pi}\int_{\mathrm{Im}\,\xi=\omega_0} \frac{e^{ix'\xi - \tau\psi^0(\xi)}}{\xi(\xi+i)}d\xi, \tag{3.4}$$

where $\omega_0 \in (\mu_-, -1)$. The put price is given by the same integral but with $\omega_0 \in (0, \mu_+)$, and the price of the covered call by the same integral but with $\omega_0 \in (-1, 0)$.

For the call, we use the same scheme as above with μ_+ replaced with -1, for the put, μ_- is replaced by 0, and and for the covered call, we use $\mu_- = -1, \mu_+ = 0$. If $x' = 0$, we use the $\gamma^- < 0 < \gamma^+$ from the definition of the sinh-regular process to define $\omega = (\gamma^+ + \gamma^-)/2$ and use the conus $\mathcal{C}_{\gamma^-,\gamma^+}$ to derive the recommendation for the choice of ζ and N; if $x' > 0$, we replace γ^- with 0 so that the wings of the deformed contour point upward and the factor $e^{ix'\xi}$ decays as $\xi \to \infty$ in the conus $\mathcal{C}_{0,\gamma^+}$ used to derive the recommendations for the choice of ζ and N, and set $\omega = \gamma^+/2$; if $x' < 0$, we replace γ^+ with 0 so that the wings of the deformed contour point downward and the factor $e^{ix'\xi}$ decays as $\xi \to \infty$ in the conus $\mathcal{C}_{\gamma^-,0}$ used to derive the recommendations for the choice of ζ and N, and set $\omega = \gamma^-/2$. Note that in all cases, $A(\omega) := -x'\sin\omega \ge 0, B(\omega) = \tau\,\mathrm{Re}\,c_\infty(\omega) > 0$.

The bound for the truncation error is

$$Err_{tr}(\Lambda_1) \le \frac{e^{\tau C_0}}{\pi}\int_{\Lambda_1}^{+\infty} e^{(x'\sin\omega)\rho - \tau\,\mathrm{Re}\,c_\infty(\omega)\rho^\nu}\rho^{-2}d\rho. \tag{3.5}$$

We can use a more stringent bound

$$Err_{tr}(\Lambda_1) \le \frac{e^{\tau C_0}}{\pi\Lambda_1} e^{(x'\sin\omega)\Lambda_1 - \tau\,\mathrm{Re}\,c_\infty(\omega)\Lambda_1^\nu}. \tag{3.6}$$

Given the error tolerance $\epsilon > 0$, we have the equation for Λ_1:

$$F(\Lambda_1) := A(\omega)\Lambda_1 + B(\omega)\Lambda_1^\nu + \ln\Lambda_1 - C = 0, \tag{3.7}$$

where $C = \tau C_0 - \ln(\pi\epsilon)$. The equation can be solved easily and fast since it is unnecessary to achieve a high precision.

3.3. **Pricing European puts and calls in the Heston model.** Consider the Heston model [19] with constant riskless and dividend rates r and δ on stock (or exchange rate) S_t. To be more specific, we assume that, under an EMM $\mathbb{Q}$ chosen for pricing, S_t and the stock volatility v_t follow the system of stochastic differential equations (SDE)

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{v_t}d\hat{W}_{1,t}, \tag{3.8}$$

$$dv_t = \kappa(m-v_t)dt + \sigma_0\sqrt{v_t}dW_{2,t}, \tag{3.9}$$

where $\hat{W}_{1,t}, W_{2,t}$ are components of the Brownian motion in 2D with unit variances and correlation coefficient ρ. Starting with [19], prices of European options in the Heston model have been calculated using the Fourier transform technique (the first instance of using this standard technique in finance). For an overview of different realizations of the pricing formula, see [28]. Below, we will use the realization derived in [28]. Let $V_{\rm put}(t,S_t,v_t)$ and $V_{\rm call}(t,S_t,v_t)$ be the put and call options on S_t with strike K and maturity date T, at time $t<T$.

Theorem 3.1 ([28])**.** *Let $\tau = T-t$ be the time to maturity and let $\lambda_-(\tau) < -1 < 0 < \lambda_+(\tau)$ be reals such that $\mathbb{E}^{\mathbb{Q}}[S_T^{\lambda_\pm(\tau)} \mid S_t, v_t] < \infty$. Then, for any $\omega_0 \in (0, \lambda_+(\tau))$,*

$$V_{\rm put}(t,S_t,v_t) = -\frac{Ke^{-r\tau}}{2\pi}\int_{\mathrm{Im}\,\xi=\omega_0} \frac{e^{i\xi z_t + (v_t B_0(\tau,\xi)+C_0(\tau,\xi))/\sigma_0^2}}{\xi(\xi+i)}d\xi, \tag{3.10}$$

and for any $\omega_0 \in (\lambda_-(\tau), -1)$,

$$V_{\rm call}(t,S_t,v_t) = -\frac{Ke^{-r\tau}}{2\pi}\int_{\mathrm{Im}\,\xi=\omega_0} \frac{e^{i\xi z_t + (v_t B_0(\tau,\xi)+C_0(\tau,\xi))/\sigma_0^2}}{\xi(\xi+i)}d\xi, \tag{3.11}$$

where $z_t = \log(S_t/K) - (\rho/\sigma_0)v_t + \mu_0\tau$, $\mu_0 = r-\delta-\kappa m\rho/\sigma_0$,

$$B_0(\tau,\xi) = (\kappa - R(\xi))\frac{1-D_1(\xi)e^{-\tau R(\xi)}}{1-D(\xi)e^{-\tau R(\xi)}} \tag{3.12}$$

$$C_0(\tau,\xi) = \kappa m\left((\kappa-R(\xi))\tau - 2\ln\frac{1-D(\xi)e^{-\tau R(\xi)}}{1-D(\xi)}\right), \tag{3.13}$$

$$R(\xi) = \sqrt{\kappa^2 + (\sigma_0^2 - 2\rho\kappa)i\xi + \sigma_0^2(1-\rho^2)\xi^2} \tag{3.14}$$

$$D(\xi) = \frac{\rho\sigma_0 i\xi - \kappa + R(\xi)}{\rho\sigma_0 i\xi - \kappa - R(\xi)} \tag{3.15}$$

$$D_1(\xi) = D(\xi)\frac{\kappa+R(\xi)}{\kappa-R(\xi)} \tag{3.16}$$

Remark 3.1. a) Let $\{\mathrm{Im}\,\xi \in (\lambda_-(\tau), \lambda_+(\tau))\}$ be the maximal strip of analyticity of the characteristic function. Introduce a quadratic polynomial $P(\beta) = \kappa^2 - (\sigma_0^2 - 2\rho\kappa)\beta - \sigma_0^2(1-\rho^2)\beta^2$, and denote by $\lambda_-^0 < 0 < \lambda_+^0$ its roots. It is proved in [28] that $\lambda_-(\tau) \le \lambda_-^0$ and $\lambda_+^0 \le \lambda_+(\tau)$, and a procedure for the calculation of $\lambda_-(\tau)$ and $\lambda_+(\tau)$ is derived. As numerical examples in [28] indicate, typically, λ_-^0 and λ_+^0 are rather close to $\lambda_-(\tau)$ and $\lambda_+(\tau)$, respectively, hence, there is no sizable advantage in using a rather complicated procedure for the calculation of $\lambda_-(\tau)$ and $\lambda_+(\tau)$. In the numerical procedure of the present paper, we will use λ_-^0 and λ_+^0.

b) As it is proved in [35], the (conditional) characteristic function admits the analytic continuation to the complex plane with the cuts $i(-\infty,\lambda_-(\tau)]$ and $i[\lambda_+(\tau),+\infty)$, hence, to the complex plane with the cuts $i(-\infty,\lambda_-^0]$ and $i[\lambda_+^0,+\infty)$.

Choose the strip of analyticity $S_{(\lambda_-^0,-1)}$, $S_{(-1,0)}$, or $S_{(0,\lambda_+^0)}$, and move the line of integration into the strip; use the residue theorem when a pole (or the two poles) of the integrand is (are) crossed. Unless a strip is too narrow or wide (see Remark 2.3), one should choose a curve in the upper half-plane if $z_t>0$, and in the lower half-plane if $z_t<0$. Otherwise, the universal recommendation for the choice of the truncation parameter becomes inaccurate: one must add the length of the part of the deformed contour which is in the "incorrect" half-plane where the factor $e^{iz_t\xi}$ is large in the absolute value.

Let $S_{(\mu_-,\mu_+)}$ be the chosen strip. It follows from Remark 3.1 that the conditional distribution of the price is sinh-regular of order $\nu=1$ and type $((\mu_-,\mu_+);\mathcal{C}_{-\pi/2,\pi/2},\mathcal{C}_{-\gamma^-,\gamma^+})$, where $\gamma^\pm$ are defined as for elliptic Lévy processes of order $\nu=1$ with $x'=z_t$ and

$$c_\infty(\varphi)=\frac{1}{\sigma_0^2}\lim_{\rho\to+\infty}\rho^{-1}(v_tB_0(\tau,\rho e^{i\varphi})+C_0(\tau,e^{i\varphi})).$$

To find $c_\infty(\varphi)$, we calculate the asymptotics of $R(\xi),D(\xi),B_0(\tau,\xi)$ and $C_0(\tau,\xi)$ as $\xi\to\infty$ remaining in the right-half plane:

$$\begin{aligned}
R(\xi) &= \sigma_0(1-\rho^2)^{1/2}\xi\left(1+\frac{\sigma_0^2-2\rho\kappa}{\sigma_0^2(1-\rho^2)}i\xi^{-1}+O(\xi^{-2})\right)^{1/2} \qquad (3.17)\\
&= \sigma_0(1-\rho^2)^{1/2}\xi+i\frac{\sigma_0^2-2\rho\kappa}{2\sigma_0^2(1-\rho^2)}+O(\xi^{-1})\\
1-D(\xi) &= \frac{-2R(\xi)}{\rho\sigma_0 i\xi-\kappa-R(\xi)}=\frac{-2}{\rho\sigma_0 i\xi/R(\xi)-1+O(\xi^{-1})} \qquad (3.18)\\
&= \frac{2}{1-i\rho/(1-\rho^2)^{1/2}}+O(\xi^{-1})\\
v_tB_0(\tau,\xi)+C_0(\tau,\xi) &= (v_t+\kappa m\tau)(\kappa-R(\xi))+2\kappa m\ln(1-D(\xi))+O(\xi^{-1}) \qquad (3.19)\\
&= -(v_t+\kappa m\tau)\left(\sigma_0(1-\rho^2)^{1/2}\xi-\kappa+i\frac{\sigma_0^2-2\rho\kappa}{2\sigma_0(1-\rho^2)^{1/2}}\right)\\
&\quad +2\kappa m\ln(1-D(\xi))+O(\xi^{-1})\\
e^{2\kappa m\ln(1-D(\xi)} &= \left(\frac{2}{1-i\rho/(1-\rho^2)^{1/2}}\right)^{2\kappa m}. \qquad (3.20)
\end{aligned}$$

It follows that, as $\xi\to\infty$ in the right half-plane, the integrand on the RHS of (3.11) has the following asymptotics:

$$C_\infty\frac{e^{iz_t\xi-c_\infty(0)\xi}}{\xi^2}(1+O(\xi^{-1})), \qquad (3.21)$$

where

$$C_\infty = \frac{Ke^{-r\tau}}{2\pi}\left(\frac{2}{i\rho/(1-\rho^2)^{1/2}-1}\right)^{2\kappa m} \cdot \exp\left[(v_t+\kappa m\tau)\left(\kappa - i\frac{\sigma_0^2-2\rho\kappa}{2\sigma_0(1-\rho^2)^{1/2}}\right)\right], \tag{3.22}$$

$$c_\infty(0) = (v_t+\kappa m\tau)\sigma_0(1-\rho^2)^{1/2}. \tag{3.23}$$

Note that

$$|C_\infty| = \frac{Ke^{-r\tau}}{2\pi}e^{(v_t+\kappa m\tau)\kappa}\left(4(1-\rho^2)\right)^{\kappa m}. \tag{3.24}$$

Set $\varphi_0 = -\arctan(z_t/c_\infty(0))$, and

(i) $\gamma^- = -\pi/2-\varphi_0$, $\gamma^+ = \pi/2$, if $z_t > 0$ (hence, $\varphi_0 < 0$),
(ii) $\gamma^- = -\pi/2, \gamma^+ = \pi/2-\varphi_0$, if $z_t < 0$ (hence, $\varphi_0 > 0$).

Thus, $\gamma^- \in [-\pi/2, 0), \gamma^+ \in (0, \pi/2]$. We define ω and d_0 by (2.13), then $\omega + d_0 = \gamma^+, \omega - d_0 = \gamma^-$. Next, we must ensure that the intersection of the imaginary axis and the image of $S_{(-d_0,d_0)}$ under $\chi_{\omega_1,\omega;b}$ is a subset of $i(\mu_-,\mu_+)$, which is equivalent to $\omega_1 + ba_+ \le \mu_+, \omega_1 - b \ge a_-\mu_-$, where $a_- = -\sin\gamma^-, a_+ = \sin\gamma^+$. Hence, we define ω_1 and b_0 by (2.14). We choose $d < d_0$, $b < b_0$ close to d_0, b_0, respectively, e.g., $d = 0.95d_0, b = 0.95d_0$, and, for the given error tolerance, set $\zeta = 2\pi d/(\ln(H(f,d)/\epsilon) \sim 2\pi d/E$, where $E = \ln(1/\epsilon)$.

The approximate bound for the truncation error is

$$Err_{tr}(\Lambda_1) \le 2|C_\infty|e^{-(z_t\sin\omega + c_\infty(0)\cos\omega)\Lambda_1}/\Lambda_1. \tag{3.25}$$

Given $\epsilon > 0$, we find a moderately accurate approximation to the solution of the equation

$$(z_t\sin\omega + c_\infty(0)\cos\omega)\Lambda_1 + \ln\Lambda_1 - \ln(|C_\infty|/\epsilon) = 0, \tag{3.26}$$

and then calculate $\Lambda = \ln(2\Lambda_1/b),\ N = \mathrm{ceil}(\Lambda/\zeta)$.

3.4. **Complexity of the scheme.** As $\epsilon \downarrow 0$, $\Lambda \sim \ln E$, where $E = E(\epsilon) = \ln(1/\epsilon)$, and $\zeta \sim E/(2\pi d)$, where $d < (\gamma^+ - \gamma^-)/2$ is fixed. Hence, the complexity of the scheme is of the order of $A(d)E\ln E$, where $A(d) = 1/(2\pi d) < 2/\pi^2$ if d is chosen sufficiently close to $(\gamma^+ - \gamma^-)/2 > \pi/4$.

Remark 3.2. For the calibration of the model, calculations need to be done many times. The trick explained in Section A.2 allows one to use $\mu_- = \lambda_-^0, \mu_+ = \lambda_+^0$, hence, a larger ζ and smaller grid, which decreases the CPU time.

3.5. **Numerical results.** In Section B.2, Tables 3-8, we produce results for European put in the Heston model, and compare the performance of the sinh-acceleration with the fractional-parabolic change of variables. We adjust the recommended ζ and $\Lambda = N\zeta$ dividing ζ by k_ζ and multiplying Λ by k_Λ. We show ζ, N and the resulting errors for each choice of ϵ, k_ζ and k_Λ. The errors (rounded) are calculated with respect to the benchmark prices (rounded). The latter are obtained using several sets of the parameters of the numerical scheme; the results differed by less than E-13. Time: the CPU time in msc of calculations for each point, the average over 1000 runs.

In Table 9, we compare the performance of the sinh-acceleration method with the Lewis-Lipton and Carr-Madan realizations of the flat iFT method. In all cases, the standard prescriptions ($\zeta = 0.125$, $N = 4096$) imply negligible truncation errors, hence, the errors shown are, essentially, the discretization errors.

4. Options on bond in the CIR model

4.1. **Characteristic function.** In the CIR model, the state space is $\mathbb{R}_+$, the dynamics of the short rate is given by

$$dr_t = \kappa(\theta - r)dt + \sigma\sqrt{r_t}dW_t, \tag{4.1}$$

where $\kappa, \theta, \sigma > 0$, and dW_t is the increment of the standard Wiener process. For $t < T$ and $r > 0$, the characteristic function

$$W(t, T; r, \xi) = \mathbb{E}_t^{\mathbb{Q},r}\left[\exp\left(-\int_t^T r_s ds\right) e^{i\xi r_T}\right],\ \xi \in \mathbb{R},$$

is of the form

$$W(t, T; r, \xi) = \exp[B(\tau, \xi)r + C(\tau, \xi)], \tag{4.2}$$

where $\tau = T - t$, and B, C can be found solving the system of Riccati equations associated with the model. We reproduce the well-known solution in Section A.3, in the form convenient for application of the sinh-acceleration method. We will use the representation

$$B(\tau, \xi) = \frac{B_+B_- + i\xi B_{+,n}(\tau)}{B_{++}(\tau) - i\xi}, \tag{4.3}$$

$$C(\tau, \xi) = \kappa\theta\left[B_-\tau + \frac{2}{\sigma^2}\ln\frac{B_+ - B_-}{1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}}} - \frac{2}{\sigma^2}\ln(B_{++}(\tau) - i\xi)\right], \tag{4.4}$$

where $B_\pm = (\kappa \pm \sqrt{\kappa^2 + 2\sigma^2})/\sigma^2$,

$$B_{+,n}(\tau) = \frac{B_+e^{-\tau\sqrt{\kappa^2+2\sigma^2}} - B_-}{1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}}}, \tag{4.5}$$

$$B_{++}(\tau) = \frac{B_+ - B_-e^{-\tau\sqrt{\kappa^2+2\sigma^2}}}{1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}}}. \tag{4.6}$$

Lemma 4.1. *a) The characteristic function is analytic in $\mathbb{C} \setminus i(-\infty, -B_{++}(\tau)]$.*
b) As $\xi \to \infty$ remaining in $\mathbb{C} \setminus i(-\infty, -B_{++}(\tau)]$,

$$\left|e^{B(\tau,\xi)r + C(\tau,\xi)}\right| \sim C_\infty(\tau, r)|\xi|^{-2\kappa\theta/\sigma^2}, \tag{4.7}$$

where

$$C_\infty(\tau, r) = \left(\frac{B_+ - B_-}{1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}}}\right)^{2\kappa\theta/\sigma^2} \exp\left[r\frac{B_+ - B_-e^{\tau\sqrt{\kappa^2+2\sigma^2}}}{1 - e^{\tau\sqrt{\kappa^2+2\sigma^2}}} + \kappa\theta B_-\tau\right]. \tag{4.8}$$

Proof. a) It follows from (4.3) and (4.4) that $B(\tau, \xi)$ is meromorphic in $\mathbb{C}$ with the only simple pole at the root of the equation

$$i\xi - B_- - (i\xi - B_+)e^{\tau\sqrt{\kappa^2+2\sigma^2}} = 0,$$

which is $-B_{++}(\tau)$, and $C(\tau,\xi)$ is analytic in $\mathbb{C}\setminus i(-\infty,B_{++}(\tau)]$.

b) As $\xi\to\infty$, $\Lambda(\xi)\to 1$, hence, (4.7)-(4.8) follow from (4.3) and (4.8). □

4.2. **The bond price.** We let $\xi=0$ in the formula for the characteristic function

$$P(T,r)=\exp[B(T,0)r+C(T,0)].$$

Since $\Lambda(0)=B_+/B_-$, we have

$$\begin{aligned} B(T,0) &= \frac{B_+-B_-(B_+/B_-)e^{T\sqrt{\kappa^2+2\sigma^2}}}{1-(B_+/B_-)e^{T\sqrt{\kappa^2+2\sigma^2}}}=B_-\frac{e^{T\sqrt{\kappa^2+2\sigma^2}}-1}{e^{T\sqrt{\kappa^2+2\sigma^2}}-B_-/B_+}\\ &= B_-\frac{1-e^{-T\sqrt{\kappa^2+2\sigma^2}}}{1-(B_-/B_+)e^{-T\sqrt{\kappa^2+2\sigma^2}}},\\ C(T,0) &= \kappa\theta\left[B_-T+2\sigma^{-2}\ln\frac{1-B_-/B_+}{1-(B_-/B_+)e^{-T\sqrt{\kappa^2+2\sigma^2}}}\right]. \end{aligned}$$

4.3. **Call and put options.** Consider now the call option with the maturity date τ and strike $K<e^{C(T,0)}$, on the bond maturing at $T+\tau$. Set $z_{T,K}=(C(T,0)-\ln K)/B(T,0)$. Since $B(T,0)<0$, the Fourier transform of option's payoff

$$G(\xi) = \int_{\mathbb{R}} e^{-ir\xi}(e^{B(T,0)r+C(T,0)}-K)_+dr=\frac{KB(T,0)e^{iz_{T,K}\xi}}{\xi(\xi+iB(T,0))},$$

is well-defined in the half-plane $\{\operatorname{Im}\xi>-B(T,0)\}$, and admits meromorphic continuation to the complex plane with two simple poles at 0 and $-iB(T,0)$. The characteristic function $\exp[B(\tau,\xi)r+C(\tau,\xi)]$ of $r_\tau|r$ admits analytic continuation to the complex plane with the cut $i(-\infty,-B_{++}(\tau)]$, and decays as $|\xi|^{-2\kappa\theta/\sigma^2}$ as $\xi\to\infty$ in the complex plane with the cut.

Hence, the price of the call option on the bond can be calculated as

$$V_{\text{call}}(\tau,r)=\frac{KB(T,0)}{2\pi}\int_{\operatorname{Im}\xi=\omega_0}\frac{e^{iz_{T,K}\xi+B(\tau,\xi)r+C(\tau,\xi)}}{\xi(\xi+iB(T,0))}d\xi, \tag{4.9}$$

where $\omega_0>-B(T,0)$ is arbitrary. The integrand is meromorphic in $\mathbb{C}\setminus i(-\infty,-B_{++}(\tau)]$, with two simple poles at 0 and $-iB(T,0)$, hence, we have 3 strips of the analyticity $S_{(-B_{++},0)}$, $S_{(0,-B(T,0))}$ and $S_{(-B(T,0),+\infty)}$ of the integrand. We can move the line of integration into any strip or, as we outlined in the case of the Heston model, get rid of both simple poles at 0 and $-iB(T,0)$ and reduce to the integrand analytic in the complex plane with the cut $i(-\infty,-B_{++}(\tau)]$. On crossing one or two poles, we apply the residue theorem. When both poles are crossed, we obtain the price of the put. Thus,

$$V_{\text{call}}(\tau,r)=V_{\text{put}}(\tau,r)+e^{z_{T,K}B(T,0)}P(T+\tau,r)-KP(\tau,r), \tag{4.10}$$

which can be used to double-check the accuracy of calculations. Indeed, if $V_{\text{call}}(\tau,r)$ and $V_{\text{put}}(\tau,r)$ are calculated directly, with no pole crossed, then a random agreement between the two completely differently sums, with the difference of the order of 10^{12}, say, has a negligible probability unless the errors of both results are of the same order of magnitude.

Let $S_{(\mu_-,\mu_+)}$ be the chosen strip. According to the general scheme, the choice of the parameters of the sinh-acceleration depends on the sign of $z_{T,K}$. If $z_{T,K}=0$, we can apply the sinh-acceleration with $\gamma^-=-\pi/2,\gamma^+=\pi/2$; if $z_{T,K}>0$ with $\gamma^-=0,\gamma^+=\pi/2$; and if $z_{T,K}<0$,

with $\gamma^- = -\pi/2, \gamma^+ = 0$. If $z_{T,K} \geq 0$, then, for any $r \geq 0$, $\exp[B(T,0)r + C(T,0)] \leq K$, hence, the price of the call option is 0, and it is unnecessary to apply a numerical method to recover this zero. However, to test the accuracy of the method, we applied our general recommendations to this case as well, and, in numerical examples, verified that the call option price calculated numerically is of the order of the error tolerance used to choose the parameters of the scheme.

We define ω and d_0 by (2.13), and ω_1 and b_0 by (2.14). We choose $d < d_0$, $b < b_0$ close to d_0, b_0, respectively, e.g., $d = 0.95d_0, b = 0.95d_0$. Then, after the change of variables (1.2), the integrand in the pricing formula, denote it $f(y)$, admits analytic continuation to the strip $S_{(-d,d)}$ and decays sufficiently fast as $y \to \infty$ remaining in the strip. The Hardy norm (2.16) is finite and can be approximated well by (2.18). To satisfy a small error tolerance $\epsilon > 0$, we choose $\zeta = 2\pi d/(\ln(H(f,d)/\epsilon) \sim 2\pi d/E$, where $E = \ln(1/\epsilon)$.

The truncation error of the simplified trapezoid rule can be approximated by the truncation error of the integral (2.20). The rate of decay of the integrand is the smallest if $z_{T,K} = 0$. In this case, the integrand decays as

$$\frac{K(-B(T,0))C_\infty(\tau,r)}{2\pi}|\xi|^{-2-2\kappa\theta/\sigma^2}$$

(see (4.7)-(4.8)), therefore, given the error tolerance ϵ, we can find $\Lambda_1 = be^\Lambda/2$ from

$$\frac{K(-B(T,0))C_\infty(\tau,r)}{\pi}\int_{\Lambda_1} y^{-2-2\kappa\theta/\sigma^2}dy = \epsilon,$$

which gives $\Lambda_1 = \epsilon_1^{-1/(1+2\kappa\theta/\sigma^2)}$, where $\epsilon_1 = \epsilon(1+2\kappa\theta/\sigma^2)/(K(-B(T,0))C_\infty(\tau,r))$. Thus,

$$\Lambda = \ln(2/b) + (1+2\kappa\theta/\sigma^2)^{-1}\ln(1/\epsilon_1),\ N = \mathrm{ceil}(\Lambda/\zeta).$$

If $z_{T,K} \neq 0$, then $|\omega| = \pi/4$, and $\mathrm{Re}(iz_{T,K}\xi) \sim -c_\infty(T,K)|\xi|$ along the contour $\xi = \chi_{\omega_1,\omega,b}(\mathbb{R})$, where $c_\infty(T,K) = |z_{T,K}\sin\omega|$. Hence, we need to find $\Lambda_1 = be^\Lambda/2$ satisfying

$$\frac{K(-B(T,0))C_\infty(\tau,r)}{\pi}\int_{\Lambda_1} e^{-c_\infty(T,K)y}y^{-2-2\kappa\theta/\sigma^2}dy \leq \epsilon.$$

We solve a stronger equation

$$e^{-c_\infty(T,K)\Lambda_1}\Lambda_1^{-1-2\kappa\theta/\sigma^2} = \epsilon_1,$$

equivalently,

$$F(\Lambda_1) := c_\infty(T,K)\Lambda_1 + (1+2\kappa\theta/\sigma^2)\ln\Lambda_1 - \ln(1/\epsilon_1) = 0,$$

as follows: $\Lambda_1 = (1/c_\infty(T,K))\ln(1/\epsilon_1)$,

$$\Lambda_1 := \Lambda_1 - \frac{1+2\kappa\theta/\sigma^2}{c_\infty(T,K)}\ln\Lambda_1.$$

Finally, we calculate $\Lambda = \ln(2\Lambda_1/b)$, $N = \mathrm{ceil}(\Lambda/\zeta)$.

Remark 4.1. (1) If $z_{T,K} \neq 0$ (and not too small in absolute value), then the rate of decay is, essentially, as in the case of the call option in regular SINH-models of order $\nu \in (0,1)$ and non-zero x'. However, if $z_{T,K}$ is zero or close to zero, then the number of terms

needed to satisfy a given error tolerance can be larger - and very large if the sinh-acceleration is not used. Even the fractional-parabolic deformation requires 10 times more terms (for some parameters, even more) to achieve the same accuracy.

(2) Formally, one should use the widest strip $S_{(-B(T,0),+\infty)}$ and choose an arbitrary large $\omega_0 > -B(T,0)$. However, if ω_0 is large, then the simplified general recommendations for the choice of ζ and, especially, Λ can become too inaccurate. Indeed, if $z_{T,K} < 0$, then the wings of the curve $\mathcal{L}_{\omega_1,\omega;b} := \chi_{\omega_1,\omega;b}(\mathbb{R})$ point downward, and we can truncated the sum in the infinite trapezoid rule where the integrand becomes sufficiently small. However, if $\omega_0 > 0$ is not small, the significant number of points $\xi_j = \chi_{\omega_1,\omega;b}(y_j) = i\omega_1 + b\sinh(i\omega + y_j)$ used in the simplified trapezoid rule are in the upper half-plane but the simplified recommendation implicitly presumes that all the points are in the low half-plane.

Thus, if $\omega_0 > 0$ and $z_{T,K} < 0$, we need to modify the prescription above by adding to Λ the half-length Λ_0 of the intersection of the curve $\mathcal{L}_{\omega_1,\omega;b}$ with the upper half-plane. To find Λ_0, we solve the equality $\omega_1 + b\,\mathrm{Im}\sinh(i\omega + y) = 0$, equivalently, $e^y - e^{-y} + 2\omega_1/(b\sin\omega) = 0$. Thus,

$$\Lambda_0 = -\omega_1/(b\sin\omega) + \sqrt{(\omega_1/b\sin\omega)^2 + 1)},$$

and $\Lambda = \ln(2\Lambda_1/b) + \Lambda_0$, $N = \mathrm{ceil}(\Lambda/\zeta)$. This increases the number of terms. Hence, it is advisable to choose $\omega_0 \in (-B_{++}, 0)$ unless B_{++} is very small.

4.4. **Numerical examples.** A numerical example in Section B.3 (Table 10) shows that the sinh-acceleration is significantly faster than the fractional-parabolic method (the number of terms is 10-30 times fewer and the CPU time in Matlab realization is about 5 times smaller); the flat iFT can satisfy the error tolerance 10 mln times larger only when the number of terms is of the order of 10^5, and the CPU time is 100 times larger.

5. Subordination

We consider the following example. Let y_t be the square root process with the dynamics

$$dy_t = \kappa(\theta - y_t)dt + \lambda\sqrt{y_t}dW_t, \tag{5.1}$$

where $\kappa > 0, \lambda > 0, \theta > 0$ and dW_t is the increment of the standard Wiener process. A popular subordinator $Y_t = \int_0^t y_s ds$ conditioned on y_0 has the characteristic function

$$\begin{aligned} \Phi_{\mathrm{CIR}}(t, y_0; \eta) &:= \mathbb{E}\left[e^{i\xi Y_t} \mid Y_0 = y_0\right] \\ &= \frac{\exp(\kappa^2\theta t/\lambda^2)\exp(2y_0 i\eta/(\kappa + u(\eta)\coth(u(\eta)t/2)))}{[\cosh(u(\eta)t/2) + \kappa\sinh(u(\eta)t/2)/u(\eta)]^{2\kappa\theta/\lambda^2}}, \end{aligned} \tag{5.2}$$

where $u(\eta) = \sqrt{\kappa^2 - 2\lambda^2 i\eta}$. The pdf of $Y_t|Y_0 = y_0$ is

$$p(y_0; \tau) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\tau\eta}\Phi_{\mathrm{CIR}}(t, y_0; \eta)d\eta. \tag{5.3}$$

Since $\tau > 0$, and $\Phi_{\mathrm{CIR}}(t, y_0; \eta)$ is uniformly bounded on the domain of analyticity, we must use a cone in the lower half plane $\mathrm{Im}\,\eta < 0$. Since $u(\eta)$ is analytic in the complex plane with the cut $i(-\infty, -\kappa^2/(2\lambda^2)]$, the simplest choice is $\gamma^- = -\pi/2, \gamma^+ = 0$, $\omega = -\pi/4$, $d_0 = \pi/4$.

Lemma 5.1. *a) For $\eta \in \mathbb{C} \setminus i(-\infty, -\kappa^2/(2\lambda^2)]$, $\operatorname{Re} u(\eta) > 0$.*
b) Fucntion $\mathbb{R} \ni \eta \mapsto \Phi_{\rm CIR}(t, y_0; \eta) \in \mathbb{C}$ admits analytic continuation to $\mathbb{C} \setminus i(-\infty, -\kappa^2/(2\lambda^2)]$.

Proof. a) is evident. b) We set $\gamma = 2\lambda\theta/\lambda^2$, $w = w(\eta) = e^{-u(\eta)t/2}$, and rewrite the denominator on the RHS of (5.2) as

$$\begin{aligned} & [\cosh(u(\eta)t/2) + \kappa \sinh(u(\eta)t/2)/u(\eta)]^{2\kappa\theta/\lambda^2} \\ = \ & \frac{e^{\gamma u(\eta)t/2}}{(2u(\eta))^\gamma}(1+w^2)^\gamma \left(u(\eta) + \kappa\frac{1-w^2}{1+w^2}\right)^\gamma. \end{aligned} \tag{5.4}$$

Since $\operatorname{Re} u(\eta) > 0$, $w_1 := w(\eta)^2 = e^{-u(\eta)t/2}$ belongs to the unit open disc $\mathcal{D}$, hence, the fraction on the RHS of (5.4) and the factor $(1+w^2)^\gamma$ are well-defined analytic functions on $\mathbb{C} \setminus i(-\infty, -\kappa^2/(2\lambda^2)]$. To prove that the same statement holds for the last factor, it suffices to show that $(1-w^2)/(1+w^2)$ is in the right half-plane if $w^2 \in \mathcal{D}$. Let $w^2 = a + ib$, where $a^2 + b^2 < 1$. Then

$$\operatorname{Re}\frac{1-w^2}{1+w^2} = \operatorname{Re}\frac{(1-a)-ib}{1+a+ib} = \operatorname{Re}\frac{((1-a)-ib)(1+a-ib)}{(1+a)^2+b^2} = \frac{1-a^2-b^2}{(1+a)^2+b^2} > 0.$$

□

The pricing formula for European options in the CIR-subordinated Lévy models changes as follows. Instead of the expectation $\mathbb{E}[e^{i\xi X_\tau} \mid X_0 = x] = e^{ix\xi - \tau\psi(\xi)}$, we have the expectation $\mathbb{E}[e^{i\xi X_{Y_\tau}} \mid X_0 = x, Y_0 = y_0] = e^{ix\xi}\Phi_{\rm CIR}(\tau, y_0; i\psi(\xi))$. Hence,

$$V_{\rm call}(S, K; y_0, \tau) = -\frac{Ke^{-r\tau}}{2\pi}\int_{\operatorname{Im}\xi=\omega_0} \frac{e^{ix\xi}\Phi_{\rm CIR}(\tau, y_0; i\psi(\xi))}{\xi(\xi+i)}d\xi, \tag{5.5}$$

where $x = \ln(S/K)$ and $\omega_0 < -1$ is such that, for all ξ on the line $\operatorname{Im}\xi = \omega_0$, $i\psi(\xi)$ is in the complex plane with the cut $i(-\infty, -\kappa^2/(2\lambda^2)]$, equivalently, $\psi(\xi)$ is in the complex plane with the cut $(-\infty, -\kappa^2/(2\lambda^2)]$. This implies that the CIR subordinator must satisfy the condition $\kappa^2/(2\lambda^2) > -\psi(-i)$.

Applying the sinh-acceleration, we need to choose the parameters of the scheme so that

(1) the image of the strip $S_{(-d_0,d_0)}$ under the composition $y \mapsto \psi(\chi_{\omega_1;\omega,b}(y))$ belongs to the complex plane with the cut $(-\infty, -\kappa^2/(2\lambda^2)]$ or to an appropriate Riemann surface;
(2) if $x > 0$, then $\chi_{\omega_1;\omega,b}(S_{(-d_0,d_0)})$ must be a subset of a half-plane of the form $\operatorname{Im}\xi > a$, where $a \in \mathbb{R}$ is a constant;
(3) if $x < 0$, then $\chi_{\omega_1;\omega,b}(S_{(-d_0,d_0)})$ must be a subset of a half-plane of the form $\operatorname{Im}\xi < a$, where $a \in \mathbb{R}$ is a constant.

Furthermore, in cases when $\Phi_{\rm CIR}(\tau, y_0; i\psi(\xi))$ admits analytic continuation to an appropriate Riemann surface, it is advisable to choose the parameters of the sinh-acceleration so that the deformed contour $\mathcal{L}_{\omega_1;\omega,b} = \chi_{\omega_1;\omega,b}(\mathbb{R})$ remains in the complex plane and additional operations in the program caused by appropriate phase shifts are not needed. If the deformed contour crosses a cut, the number of terms in the simplified trapezoid rule decreases somewhat but the number of elementary operations needed to evaluate individual terms increases. The total gain is small if any.

Let X be an elliptic sinh-regular process of type $((\lambda_-,\lambda_+);\mathcal{C},\mathcal{C}_+)$ and order $\nu\in(0,2]$, where and $\lambda_-<0<\lambda_+$; furthermore, as $\xi\to\infty$ in the conus $\mathcal{C}$,

$$\psi^0(\xi)\sim c_\infty e^{i\varphi\nu}|\xi|^\nu,\ \xi\to+\infty,$$

where $\varphi=\arg\xi$, and $c_\infty>0$.

First, we find a strip $S_{(\mu_-,\mu_+)}$ of analyticity of $\Phi_{\mathrm{CIR}}(t,y_0;i\psi(\xi))$. Here $\mu_-<0<\mu_+$ are such that $\kappa^2+2\lambda^2\psi(i\mu)>0$ for all $\mu\in(\mu_-,\mu_+)$. Since $\mu\mapsto\psi(i\mu)$ is convex on (λ_-,λ_+) and $\psi(0)=0$, we conclude that if $\kappa^2+2\lambda^2\psi(i(\lambda_+-0))\geq 0$, then $\mu_+=\lambda_+$, otherwise μ_+ is the only positive solution of the equation $\kappa^2+2\lambda^2\psi(i\mu)=0$. Similarly, if $\kappa^2+2\lambda^2\psi(i(\lambda_-+0))\geq 0$, then $\mu_-=\lambda_-$, otherwise μ_- is the only positive solution of the equation $\kappa^2+2\lambda^2\psi(i\mu)=0$.

Next, we need to find a conus of analyticity, and calculate the asymptotics of $\Phi_{\mathrm{CIR}}(t,y_0;i\psi(\xi))$ as $\xi\to\infty$ remaining in the conus. We consider two cases.

I. $\nu\in(1,2]$ or $\nu\in(0,1)$ and $\mu=0$. Then $\psi(\xi)=i\mu\xi+\psi^0(\xi)$ enjoys the same properties as ψ^0, and

$$\psi(\xi)\sim c_\infty e^{i\varphi\nu}|\xi|^\nu,\ \xi\to+\infty.$$

If $\nu=1$, the asymptotic coefficient and argument depend on μ:

$$\psi(\xi)\sim c_\infty(\mu)e^{i(\varphi+\gamma(\mu))}|\xi|,\ \xi\to+\infty.$$

Thus, there exist $-\pi\leq\gamma^-<0<\gamma^+\leq\pi$ such that for any $\varphi\in(\gamma^-,\gamma^+)$,

$$\psi(\rho e^{i\varphi})\sim c_\infty(\varphi)\rho^\nu,\ \rho\to+\infty,$$

where $c_\infty(\varphi)\notin(-\infty,0]$. Hence, for any $\varphi\in(\gamma^-,\gamma^+)$,

$$(\kappa^2+2\lambda^2\rho e^{i\varphi})^{1/2}\sim c_\infty(\varphi)^{1/2}\rho^{\nu/2},\ \rho\to+\infty,$$

where $\operatorname{Re}c_\infty(\varphi)^{1/2}>0$.

The argument above almost proves that $(\kappa^2+2\lambda^2\psi(\xi))^{1/2}$ admits analytic continuation to $i(\mu_-,\mu_+)+\mathcal{C}_{\gamma^-,\gamma^+}$. We say almost because the proof above demonstrates that, for $\xi\in i(\mu_-,\mu_+)+\mathcal{C}_{\gamma^-,\gamma^+}$, $\kappa^2+2\lambda^2\psi(\xi)\notin(-\infty,0]$ if ξ is in a certain neighborhood of 0 and a certain neighborhood of infinity. For NTS and KoBoL of order $\nu\in[1,2]$, one can demonstrate that the image of $i(\mu_-,\mu_+)+\mathcal{C}_{\gamma^-,\gamma^+}$ under the map $\xi\mapsto\kappa^2+2\lambda^2\psi(\xi)$ does not intersect $(-\infty,0]$; in general case, one should study the image on the case-by-case basis, and, if necessary, use $\mu_\pm$ closer to 0. Note that it suffices to ensure that the image does not intersect the essentially singular point 0, which is a much weaker condition. If the image intersects $(-\infty,0)$ but does not contain 0, the image is a subset of an appropriate Riemann surface, and a larger ζ can be chosen. However, it is advantageous to choose the parameters of the sinh-acceleration so that the image of the deformed contour under the map $\xi\mapsto\kappa^2+2\lambda^2\psi(\xi)$ is a subset of the complex plane, and there is no need to introduce phase shifts in the pricing formula, when the cut is crossed.

Once $i(\mu_-,\mu_+)+\mathcal{C}_{\gamma^-,\gamma^+}$ is found, we define the parameters of the sinh-acceleration and choose ζ for the given error tolerance using the general prescriptions. It remains to find the truncation parameterand N. If follows from (2.23) that as $\xi=\rho e^{i\varphi}\to\infty$ remaining in $\mathcal{C}_{\gamma^-,\gamma^+}$,

$$|\Phi_{\mathrm{CIR}}(t,y_0;i\psi(\xi))|\leq(1+o(1))\exp(\kappa^2\theta t/\lambda^2-B(\varphi)\rho^{\nu/2}).$$

where

$$B(\varphi)=\sqrt{2}\lambda\operatorname{Re}(c_\infty(\varphi\nu)^{1/2})(t/2)2\kappa\theta/\lambda^2=\sqrt{2}\operatorname{Re}(c_\infty(\varphi\nu)^{1/2})t\kappa\theta/\lambda.$$

Hence, we can find the truncation parameter $\Lambda = \ln(2\Lambda_1/b)$ solving approximately the equality

$$e^{-A(\omega,x)\Lambda_1 - B(\omega)\Lambda_1^{\nu/2}}\rho^{-1} = \epsilon_1,$$

where $A(\omega,x) = |x\sin\omega|$, $\epsilon_1 = \epsilon\pi\exp(r\tau - \kappa^2\theta t/\lambda^2)/K$, as follows.

If $\nu = 1$, then

$$\Lambda_1 := \ln(1/\epsilon_1)/(A(\omega)+B(\omega)),\ \Lambda_1 := \max\{2, \Lambda_1 - \ln\Lambda_1/(A(\omega)+B(\omega))\};$$

if $F := (A(\omega,x)+B(\omega))\Lambda_1 + \ln\Lambda_1 - \ln(1/\epsilon_1) < 0$, then

$$\Lambda_1 := \Lambda_1 - F/(A(\omega)+B(\omega)+1/\Lambda_1).$$

If $\nu \in (1,2]$ or $\nu \in (0,1)$ and $\mu = 0$, then we set

$$\Lambda_1 = (\ln(1/\epsilon_1)/B(\omega)^{2/\nu},\ \Lambda_1 = \left(\frac{\max\{2, \ln(1/\epsilon_1) - A(\omega)\Lambda_1 - \ln\Lambda_1\}}{B(\omega)}\right)^{2/\nu};$$

if $F := A(\omega,x)\Lambda + B(\omega)\Lambda_1^{\nu/2} + \ln\Lambda_1 - \ln(1/\epsilon_1) < 0$, then $\Lambda_1 := \Lambda_1 - F/DF$, where

$$DF := A(\omega,x) + (\nu/2)B(\omega)\Lambda^{\nu/2-1} + 1/\Lambda_1.$$

Note that if $B(\omega)$ is very small and $|x|$ is not very small, then it is safer to use the same rule as in the case $\nu = 1$; this leads to a moderate overkill.

II. $\nu \in (0,1)$ and $\mu \neq 0$. If $x > 0$, we take arbitrary $0 < \gamma^- < \gamma^+ < \pi/2$; as $\xi \to \infty$ remaining in $\mathcal{C}_{\gamma^-,\gamma^+}$, $\kappa^2 - 2\lambda^2 i\psi(\xi) \sim \kappa^2 - 2\lambda^2\mu\xi$, hence, if $\mu > 0$, then

$$u(i\psi(\rho e^{i\varphi})) \sim e^{i((\varphi-\pi/2)}\rho^{1/2},\ \rho \to +\infty,$$

and if $\mu < 0$, then

$$u(i\psi(\rho e^{i\varphi})) \sim e^{i(\varphi/2)}\rho^{1/2},\ \rho \to +\infty.$$

If $\varphi \in (0,\pi)$, $\cos(\varphi - \pi/2) > 0$ and $\cos(\varphi/2) > 0$. Hence, if $\nu \in (0,1)$ and $\mu \neq 0$, the rate of the decay of the integrand is as in the case $\nu = 1$ but the asymptotic coefficient is different. We leave the details to the reader.

The results of a numerical experiment can be found in Table 11.

6. Quantiles of KoBoL and Monte-Carlo simulations

6.1. **One-factor KoBoL.** We consider efficient evaluation of quantiles, that is, solution of the equation $F(x) = A$, where $A \in (0,1)$ and F is the cumulative distribution function; once an efficient procedure for quantile evaluation is available, the procedure can be used for the Monte-Carlo simulations.

Monte-Carlo simulation remains to this day the most universal method of pricing path-dependent options on financial assets. In the case of Lévy-driven models, a basic building block of any Monte-Carlo method is the simulation of an increment of the underlying Lévy process. In some situations — for instance, the Variance Gamma model [38, 37, 36] — the process can be expressed in terms of simpler ones using time subordination, and hence its increments can be simulated (almost) exactly. However, in other cases no exact simulation algorithm is known. Madan and Yor [39] proposed an algorithm for simulating KoBoL increments based on representing the process as Brownian motion subordinated by a stable Lévy process, truncating the small jumps of the subordinator and replacing them with their average. However, as extensive numerical examples in [4] demonstrate, an efficient implementation of the standard

approach described below is 10-100 faster; the variation which we introduce in this Section, is much faster and more accurate than the realization in [4].

If Z is any random variable with continuous distribution, one can simulate Z sampling a uniformly distributed random variable U on $(0,1)$ and calculate $F^{-1}(U)$, where F denotes the cumulative distribution function of Z. When an explicit formula for F^{-1} is not available, it becomes important to be able to calculate its values very quickly and sufficiently accurately. A straightforward approach that was used with a limited success for simulation of stable Lévy processes (the tails of the distributions decay too slowly, hence, the Monte-Carlo simulations are moderately efficient only if the index of the process is close to 2, and the distribution does not differ much from the normal distribution, with the exception of far parts of the tails, which can be safely disregarded in this case) is as follows. One tabulates the values of F on a sufficiently long and fine grid of points $x_0, x_1, \dots, x_M$ on the real line and approximates F^{-1} using linear interpolation. This method is very attractive from the practical viewpoint, because the values $F(x_i)$ only have to be calculated once, and afterward the computational cost of each simulation of Z is extremely low: one has to draw a sample A of U, find j satisfying $F(x_j) \le A < F(x_{j+1})$ (which requires about $\log_2(M)$ comparisons) and perform 4–5 arithmetic operations required for linear interpolation:

$$x = x_j + (x_{j+1} - x_j)(A - F_j)/(F_{j+1} - F_j). \tag{6.1}$$

If $A < F_1$, one assigns $x = x_1$, and if $A > F_M$, then one assigns $x = x_M$. In many important examples, an explicit formula is known for the characteristic function of the random variable Z. In such a case, the calculation of the values $F(x_i)$ reduces to computing certain inverse Fourier transforms (see Glasserman and Liu [16, 18]). In the case of Lévy processes with exponentially decaying tails, the problem of a slow decay is less serious than in the case of stable Lévy processes unless the exponential rate is too small but the peak of the probability distribution remains very high if the order of KoBoL is close to 0.

In the application to the Monte-Carlo simulations, this method has 3 sources of errors:

(1) truncation error;
(2) errors of linear interpolation;
(3) errors of evaluation of F_k.

A popular approach (see, e.g., [16, 17, 18, 12, 1, 15]) is to use either the fast Fourier transform (FFT) or fast Hilbert transform (fast HT), which allows one to calculate the values F_k at all points of interest faster than point-by-point, especially if the number of points is large. This approach faces the following fundamental difficulties:

(a) if the time step is small, which is necessary for accurate simulation of barrier options with continuous monitoring, then, in a neighborhood of $x' = 0$, the $F'(x) = p(x)$ is very large, hence, in order that the linear interpolation (6.1) be sufficiently accurate, the size of the mesh $\Delta x = x_{j+1} - x_j$ must be very small; in fact, even in relatively nice cases of $\nu \in (0.5, 1)$, $\Delta x = 10^{-5}$ can lead to interpolation errors greater than 10^{-8} (see an example below); for ν closer to 0, much smaller Δx can be insufficient;
(b) if one of the steepness parameters $\lambda_\pm$ is small in the absolute value, e.g., $\lambda_+ > 0$ is small, then x_1 must be negative and very large in absolute value to ensure that the truncation error in the neighborhood of $-\infty$ is sufficiently small. In view of difficulty (a), the total number of points can be measured in millions and dozens of million;

(c) as examples in Section 2 demonstrate, accurate evaluation of $F(x_k)$ for x_k large in the absolute value can be either too time consuming or virtually impossible if the flat FT is used; the same difficulties arise is HT is used.

The sinh-acceleration allows us to calculate $F_k = F(x_k)$ very accurately and fast; for x_k large in absolute value, especially fast. The fractional-parabolic method used in [4] to calculate F_k is faster and more accurate than FFT and HT-based methods but, after the fractional-parabolic change of variables the number of terms N in the simplified trapezoid rule depends on x_k much stronger than after the sinh-acceleration. When the latter is applied, given the error tolerance, the parameters of the sinh-acceleration procedure bar the number of terms N can be chosen the same for all x'. Furthermore, N decreases as x' increases in the absolute value, there is no need to assign $x = x_1$ if $a < F_1$ and $x = x_M$ if $a > F_M$. Instead, we calculate $x = F^{-1}(a)$ using the Newton method, and 2-3 iterations suffice to satisfy the error tolerance 10^{-12} and less if the initial approximation (e.g., x_1 or x_M) is not small in absolute value.

To apply the Newton method

$$x_{n+1} = x_n - (F(x_n) - A)/F'(x_n), \tag{6.2}$$

one has to evaluate the pdf $p_n = F'(x_n)$ as well but the sinh-acceleration method allows one to calculate both $F(x_n)$ and $F'(x_n)$ using the same parameters of the numerical scheme; moreover, only one step of calculations is different: in the case of $F(x_n)$, we have an additional factor $1/(-i\xi)$. We can simultaneously calculate $F''(x_n)$ inserting the factor $-i\xi$ instead of $1/(-i\xi)$; this allows us to use the second order approximation

$$F(x) = F(x_n) + (x - x_n)F'(x_n) + \frac{(x - x_n)^2}{2}F''(x_n)$$

to solve $F(x) - A = 0$ on a sufficiently small interval $[x_{n-1}, x_n]$:

$$x = x_n - \frac{2(F(x_n) - A)}{F'(x_n) + \sqrt{F'(x_n)^2 - 2(F(x_n) - A)F''(x_n)}}. \tag{6.3}$$

In general, we can calculate $F(x_n), F'(x_n), F''(x_n)$ for x_n over a wide interval using the same parameters of the scheme. This implies that the bulk of the CPU time is spent on calculations of the parameters of the scheme, and the arrays $\xi_k = i\omega_1 + b\sinh(i\omega + y_k)$ and $Exp_k := \exp[-t\psi(i\omega_1 + b\sinh(i\omega + y_k))]\cosh(i\omega + y_k)$, $k = 1, 2, \dots, N$. The last step is the fast and straightforward evaluation of the quantities

$$F(x_n) = \frac{ib\zeta}{\pi}\sum_{j=0}^{N}(1 - \delta_{j0}/2)\,\mathrm{Re}\,\frac{\exp[-ix'_n\xi_j]Exp_j}{\xi_j} \tag{6.4}$$

$$F'(x_n) = \frac{b\zeta}{\pi}\sum_{j=0}^{N}(1 - \delta_{j0}/2)\,\mathrm{Re}\exp[-ix'_n\xi_j]Exp_j \tag{6.5}$$

$$F''(x_n) = \frac{ib\zeta}{\pi}\sum_{j=0}^{N}(1 - \delta_{j0}/2)\,\mathrm{Re}\exp[-ix'_n\xi_j]\xi_j Exp_j. \tag{6.6}$$

For the application of the Newton method for $a < F_1$ and $a > F_M$, at each step of the iteration procedure, we can use the arrays ξ and Exp calculated for $N = N(x_1)$ and $N = N(x_M)$, respectively.

The quadratic approximation (6.3) allows us to use much sparser grids than the linear approximations (6.1) and (6.2), for x_n not small in absolute value especially.

The next trick allows us to decrease the number of points smaller still. Instead of the equation $F(x) = A$, we solve the equation $f(x) = a$, where $f(x) = \ln F(x)$ and $a = \ln A$. Since f is more regular than F, the same approximations work better, and $f'_k = f'(x_k) = F'_k/F_k$, $f''_k = (F''_k F_k - (F'_k)^2)/F_k$ are easy to calculate.

Example 6.1. Consider KoBoL of order $\nu = 0.7$ with $c_+ = c_+ = 0.6$, $\lambda_+ = 5$, $\lambda_- = -10$, $\mu = 0$; $t = 0.001$. The second instantaneous moment $m_2 = 0.093440429$ (rounded) is not small, and time step $t = 0.001$ is not exceedingly small. The order $\nu = 0.7$ is not small as well; in the empirical literature, one find numerous examples of ν close to 0. The steepness parameter λ_+ is not too small as well (one can find examples of $\lambda_+ < 1$). Nevertheless, as several examples for quantiles demonstrate,

1. accurate Monte-Carlo simulations using FFT or HT will require grids with the size of the mesh 10^{-5} or less;
2. if the truncation is made at the level $F_1 = 10^{-8}$, then $x_1 = -1.6707581397416$ (the result is found using the Newton method with the initial approximation -1; three iterations were needed to satisfy the error tolerance 10^{-12}.) Hence, FFT or HT method would require the uniformly spaced grid of the length of the order of 160k, and the errors of truncation and evaluation of F_k would be non-negligible;
3. with the exception of a very small neighborhood of 0, the quadratic approximation applied to f requires much sparser grid than other approximations.

In Table 12 (see Appendix B.4), we list the errors of several approximations, for several values of A and several widths h of the interval (x_{j-1}, x_j) containing $f^{-1}(a) = F^{-1}(A)$.

Labels for approximations used:

- L: linear interpolation (6.1);
- N: Newton approximation (6.2);
- LL: linear interpolation applied to $f = \ln F$;
- LN: Newton approximation applied to $f = \ln F$;
- QT: quadratic approximation (6.3) applied to $f = \ln F$.

From Table 12, it is clearly seen that QT allows one to use much sparser grid $x_1 < x_2 < \cdots < x_M$; the grid must be non-uniformly spaced. At the points of the grid, $f = \ln F$ and its first and second derivatives must be precalculated, which can be done very fast using the sinh-acceleration. For evaluation of $x = f^{-1}(a)$ for $a < f_1$ and $a > f_M$, we use the Newton method and two precalculated arrays of a small size which represent functions in the dual space. No truncation is needed.

Outline of the algorithm.

I. In a neighborhood of $x' = 0$, e.g., in the interval $[F^{-1}(0.3), F^{-1}(0.7)]$, the steps $h_j = h_{j+1} - h_j$ should be of the order 10^{-5} if t is small. For larger t, larger steps are admissible. E.g., for $t = 1$, h_j of the order of 0.001 can be admissible.

II. As $|x_j|$ increases, h_j can be made larger. As the rule of thumb, for the left tail, we would recommend $h_j = -0.01 f_{j+1}/f'_{j+1}$ for points below $x_{low} := F^{-1}(0.3)$; the points x_j below x_{low} are calculated in the same cycle as the values f_j, f'_j, f''_j. For the right tail, the recommendations are by symmetry.

III. The grid is truncated at $F^{-1}[0.001]$ or $F^{-1}[0.0001]$, and at $F^{-1}[0.999]$ or $F^{-1}[0.9999]$. For all points of the grid, the values f_j, f'_j, f''_j should be calculated and stored.

IV. The parameters of the sinh-acceleration should be calculated for $x_0 = F^{-1}[0.001]$ (or $F^{-1}[0.0001]$), and arrays $\xi = \xi^-$ and Exp^- calculated and stored. (The sign minus indicates that the arrays will be used for calculations in the left tail).

V. The parameters of the sinh-acceleration should be calculated for $x_M = F^{-1}[0.999]$ (or $F^{-1}[0.9999]$), and arrays $\xi = \xi^+$ and Exp^+ calculated and stored. (The sign plus indicates that the arrays will be used for calculations in the right tail).

VI. If a realization $A \sim U[0,1]$ belongs to $[F^{-1}(0.001), 0]$ (resp., to $[0, F^{-1}(0.999)]$, then an interval $[x_{n-1}, x_n]$ s.t. $f_{n-1} < \ln A \le f_n$ (resp., an interval $[x_n, x_{n+1}]$ s.t. $f_n \le A < f_{n+1}$) should be found, and the quadratic approximation (6.3) applied.

VII. If $A < 0.001$, the Newton method is applied with x_0 as the initial approximation; the stored values are used to calculate $f(x_n)/f'(x_n)$ at each step of the Newton method.

VIII. If $A > 0.999$, the Newton method is applied with x_0 as the initial approximation; the stored values are used to calculate $f(x_n)/f'(x_n)$ at each step of the Newton method.

6.2. **Multi-factor KoBoL.** In [10, Section 9.1], we defined a general class of Lévy processes with exponentially decaying tails, and explicitly constructed several subclasses. In particular, the characteristic exponent of a multi-dimensional KoBoL of order $\nu \in (0,2), \nu \ne 1$, (see [10, Eq. (9.1)]) can be written in the form

$$\psi(\xi) = -i\langle \mu, \xi\rangle + c\Gamma(-\nu)\int_{S_{n-1}} [\lambda(\varphi)^\nu - (\lambda(\varphi) - i\langle \Sigma\xi, \varphi\rangle)^\nu]\,\Pi'(d\varphi), \tag{6.7}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in b^n_r, S_{n-1} is the unit sphere, $c > 0$, λ is a positive continuous function on S_{n-1}, Σ is a positive-definite matrix, and $\Pi'(d\varphi)$ is a measure on S_{n-1} of unit mass, which can be regarded as a probability distribution on S_{n-1}.

For simplicity, let $\mu = 0$. We suggest two natural methods of simulation of the multi-factor KoBoL with the characteristic exponent (6.7).

mKoBoL MC 1. Approximate the measure by an atomic measure. For each atom, design the simulation procedure of the corresponding 1-factor one-sided KoBoL. At each step of the general simulation procedure, simulate a point on the sphere (with the weight $\Pi'(\varphi_j)$), and, for φ_j run the simulation procedure for the corresponding one-dimensional KoBoL. To the current spot value x, add the realization of the one-factor KoBoL in the direction of φ.

mKoBoL MC 2. A slower but more accurate method. At each time step, simulate the distribution Π', and, for each realization, apply the full-scale MC method for one-factor KoBoL.

Variations: work with two-sided one-dimensional KoBoL writing

$$\begin{aligned}\psi(\xi) &= -i\langle \mu, \xi\rangle + \Gamma(-\nu)\int_{\mathbb{RP}^{n-1}} \Pi'(d\varphi)\\ &\quad\times \{c_+\left[(\lambda(\varphi))^\nu - (\lambda(\varphi)^\nu - i\langle \Sigma\xi, \varphi\rangle)^\nu\right] + c_-\left[(\lambda(-\varphi))^\nu - (\lambda(-\varphi)^\nu + i\langle \Sigma\xi, \varphi\rangle)^\nu\right]\},\end{aligned}$$

$c_\pm > 0$, where $\mathbb{RP}^{n-1}$ is the real projective space of dimension $n-1$ (informally, the unit sphere with opposite points identified).

7. Conclusion

In the paper, we developed a general methodology for fast and accurate evaluation of integrals of the form

$$I = \int_{\operatorname{Im}\xi=\omega_0} g(\xi)d\xi,$$

that appear in many problems in probability, mathematical finance, and other areas of applied mathematics, and formalized the properties of the integrands that can be calculated using this scheme. The methodology is applicable if an integrand $g(\xi)$ admits analytic continuation to a union of a strip around the line of integration and a conus that contains the strip, and decays sufficiently fast as $\xi \to \infty$ remaining in the union. The analyticity of the integrand in the strip and sufficiently fast decay at infinity allows one to exploit an important property of the infinite trapezoid rule, namely, exponential decay of the discretization error as function of $1/\zeta$, where $\zeta > 0$ is the mesh size. This property is well-known and widely used in the literature. In probability, the characteristic functions of various probability distributions related to diffusion processes and jump-diffusion processes with exponentially decaying densities of jumps are analytic in a strip around the real axis. Unfortunately, in many cases of interest such as the CIR model, VG model and KoBoL, the characteristic function decays slowly as $\xi \to \infty$, and millions of terms in the simplified trapezoid rule may be needed to satisfy even a moderate error tolerance.

However, if $g(\xi)$ admits analytic continuation to a conus and decays polynomially or exponentially as $\xi \to \infty$ remaining in the conus, then a change of the variable of the form $\xi = i\omega_1 + b\sinh(i\omega + y)$ in the integral is justified. After the change, the new integrand is analytic in a strip around the real axis and decays exponentially if the initial integrand decayed polynomially and as $\exp[-c\exp|y|]$, where $c > 0$, if the initial integral decayed exponentially. In the result, in many cases, $N < 10$ suffice to satisfy the error tolerance $\epsilon = 10^{-7}$; typically, less than 50 terms suffice, and in essentially all cases of interest, N of the order of 100-150 suffices to satisfy the error tolerance 10^{-12}.

We formalized the properties of the characteristic functions of processes and distributions that allow one to apply the sinh-acceleration, and illustrated the general scheme of the sinh-acceleration with several typical examples: pdf of Lévy processes; pricing of European options in Lévy models, Heston model, CIR model, and a subordinated Lévy model. The scheme admits straightforward modification to affine stochastic volatility models and interest rate models (it suffices to replace in [30] the fractional-parabolic change of variables $\xi = i\omega_1 \pm i\sigma(1 \mp i\eta)^\alpha$ with the sinh-acceleration, and take into account that the maximal conus of analyticity is, in the general case, narrower than in the case of the Heston model and CIR model); jumps can be included as in [30]. Note that if the fractional-parabolic change of variables is used, then the rate of decay of the integrand increases but the resulting number of terms remains too large in a number of important cases such as the evaluation of the probability distribution function at the peak (see [7, 30]), and pricing options in the interest models of the CIR-type.

An additional advantage of the sinh-acceleration as compared to the fractional-parabolic change of variables is that the width of the initial strip of analyticity is almost irrelevant in the former case as explained in Remark 2.3 whereas in the latter case, a narrow strip implies a very large number of terms, and makes it necessary to move the line of integration to a wider

strip [7, 28, 30]. However, the angle between the rays that define the conus of analyticity is important.

The general scheme of the sinh-acceleration consists of the following steps

I. Find $\gamma^- \le 0 < \gamma^+$ or $\gamma^- < 0 \le \gamma^+$ such that the integrand $g(\xi)$ is analytic in the cone $\mathcal{C}_{\gamma^-,\gamma^+}$, and, for any $\varphi \in (\gamma^-, \gamma^+)$, decays along the rays $e^{i\varphi}\cdot\mathbb{R}_+$ and $e^{i(\pi-\varphi)}\cdot\mathbb{R}_+$.

II. Set $\omega = (\gamma^+ + \gamma^-)/2$, $d_0 = (\gamma^+ - \gamma^-)/2$.

III. Find a strip $S_{(-\mu_-,\mu_+)}$ of analyticity of the integrand around the initial line $\operatorname{Im}\xi = \omega_0$ of integration.

IV. Set $a_- = \sin(\min\{\pi/2, -\gamma^-\}), a_+ = \sin(\min\{\pi/2, \gamma^+\})$, and

$$\omega_1 = \frac{\mu_+ a_- + \mu_- a_+}{a_+ + a_-},\ b_0 = \frac{\mu_+ - \mu_-}{a_+ + a_-}.$$

V. Choose $k_b = 0.8 - 0.95, k_d = 0.8 - 0.95$ and set $b = k_b b_0$, $d = k_d d_0$.

VI. Derive an upper bound for the Hardy norm H of $f(y) = g(i\omega_1 + b\sinh(i\omega+y))b\cos(i\omega+y)$ as an analytic function on $S_{(-d,d)}$. Typically, a simple approximation $H = 10(|f(id)| + |f(-id)|)$ works well.

VII. Given the error tolerance ϵ, choose the mesh size as $\zeta = 2\pi d/\ln(H/\epsilon)$.

VIII. Derive an approximate bound for $g(e^{i\omega}\rho)$ and $g(e^{i(\pi-\omega)}\rho)$ for ρ in a neighborhood of $+\infty$.

IX. Given the error tolerance ϵ, use the bound to find Λ_1 such that

$$\int_{\Lambda_1}^{+\infty} |g(e^{i\omega}\rho)|d\rho + \int_{\Lambda_1}^{+\infty} |g(e^{i(\pi-\omega)}\rho)|d\rho < \epsilon.$$

X. Set $\Lambda = \ln(2\Lambda_1/b)$, $N = \operatorname{ceil}(\Lambda/\zeta)$.

XI. Apply the simplified trapezoid rule

$$\int_{\operatorname{Im}\xi=\omega_0} g(\xi)d\xi \approx b\zeta \sum_{|j|\le N} g(i\omega_1 + b\sinh(i\omega + j\zeta))\cos(i\omega + j\zeta). \tag{7.1}$$

XII. If $\overline{g(\xi)} = g(-\xi),\ \forall\ \xi$, use the following faster version of (7.1)

$$\int_{\operatorname{Im}\xi=\omega_0} g(\xi)d\xi \approx 2b\zeta \sum_{0\le j\le N} (1-\delta_{j0}/2)\operatorname{Re}\left(g(i\omega_1 + b\sinh(i\omega + j\zeta))\cos(i\omega + j\zeta)\right), \tag{7.2}$$

where δ_{jk} is Kronecker's delta.

Note that, in addition to the theoretical bounds for the error of the sinh-acceleration, one can easily check the accuracy of the result choosing less optimal pair (γ^-, γ^+) so that the new ω is different from the old one and larger and finer grid than recommended, and recalculate the integral. The probability of a random agreement between the two results is negligible, hence, the absolute value of the difference is a good proxy for the error.

The sinh-acceleration change of variables can be applied to pricing European options in quadratic term structure models, models with Wishart dynamics (in both cases, the integrands decay very slowly, as in the CIR model, hence, accurate calculations using the popular FFT techniques are essentially impossible as demonstrated in [30] for affine models of $A_n(n)$ class), 3/2 models and, essentially, any model where the (conditional) characteristic function can be calculated, e.g., Barndorff-Nielsen and Shephard model, and subordinated models more general than the model considered in the paper. The methodology can be also applied to evaluation

of special functions [29], calculation of distributions of the infimum and supremum of Lévy processes, processes of Ornstein-Uhlenbeck type, and a number of other popular processes, calculation of the Wiener-Hopf factors, pricing of path-dependent options, Monte-Carlo simulations (in many cases, the CPU time decreases by a factor of 10-100 and more), in multi-factor models including. The efficiency of the calibration procedure of the Heston model in [20, 21] can also be improved. To apply the sinh-acceleration to regime-switching models, it suffices to use matrix operations instead of the scalar ones (and, naturally, study the region where the matrix functions and their reciprocals are analytic; formally, the scheme remains the same). Applications to stochastic covariance models are similar to applications to stochastic volatility models.

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Appendix A. Auxiliary results

A.1. **Complexity of the flat iFT scheme and the scheme based on the fractional-parabolic deformation of order α.** We consider KoBoL and NTS of order $\nu \in (0,2)$. In both cases, ψ^0 admits analytic continuation to the complex plane with the cuts $i(-\infty,\mu_-]$, $i[\mu_+,+\infty)$, where $\mu_- = \lambda_-, \mu_+ = \lambda_+$ in the case of KoBoL, and $\mu_- = -\alpha-\beta$, $\mu_+ = \alpha-\beta$ in the case of NTS. In the latter case, we represent ψ^0 in the form

$$\psi^0(\xi) = \delta[(\mu_+ + i\xi)^{\nu/2}(-\mu_- - i\xi)^{\nu/2} + \mu_+\mu_-].$$

If $x' \le 0$, we use the change of variables $\xi = i\mu_+ - i\sigma(1+i\eta)^\alpha$ (note that this α has no relation to the α in the definition of NTS processes), where $\sigma = (\mu_+ - \mu_-)/2^\alpha$, and the representations

$$(\mu_+ + i\xi)^{\nu/2} = \sigma^{\nu/2}\exp\left[\frac{\alpha\nu}{2}\ln(1+i\eta)\right] \tag{A.1}$$

$$(-\mu_- - i\xi)^{\nu/2} = \exp\left[\frac{\nu}{2}\ln[\mu_+ - \mu_- - \sigma\exp[\alpha\ln(1+i\eta)]]\right]. \tag{A.2}$$

The RHS of (A.1) is analytic in the strip $S_{(-1,1)}$, and the same holds for the RHS of (A.2) if $\alpha \le 4$. If $\alpha > 4$, we need to define analytic continuation of $\ln[\mu_+ - \mu_- - \sigma\exp[\alpha\ln(1+i\eta)]]$ on the RHS of (A.2) adding $i\pi$ (resp., $-i\pi$) as $\eta \mapsto \mu_+ - \mu_- - \sigma\exp[\alpha\ln(1+i\eta)]$ crosses $(-\infty,0)$ from above (resp., from below). Then, in the η-coordinate, the half-width of the maximal strip of analyticity is $d_0 = 1$. We take $d \in (0,1)$, and define the mesh $\zeta = 2\pi d/\ln(H(f,d)/\epsilon)$, where f is the integrand in the η-coordinate.

To find an approximation to the truncation parameter Λ, we note that

$$\psi^0(\xi(\eta)) \sim c_\infty(0)e^{i(\alpha-1)\nu\pi/2}\sigma^\nu|\eta|^{\alpha\nu},$$

as $\eta \to \infty$ remaining in the strip $S_{(-d,d)}$, therefore, given $\epsilon > 0$, we can find Λ_1 satisfying

$$\frac{e^{tC_0}}{\pi}\int_{\Lambda_1}^{+\infty} e^{-x'\cos(\alpha\pi/2)\rho - c^\infty\rho^\nu}d\rho = \epsilon, \tag{A.3}$$

where $c^\infty := c^\infty(tc_\infty(0),\alpha,\nu) = tc_\infty(0)\cos((\alpha-1)\nu\pi/2)$ (we use (2.22) with $\omega = (\alpha-1)\pi/2$), and then calculate $\Lambda = (\Lambda_1/\sigma)^{1/\alpha}$ and $N = \mathrm{ceil}(\Lambda/\zeta)$.

If $x' = 0$ or $\nu \ge 1$, the necessary and sufficient condition on $\alpha(\ge 1)$ is $\pi\nu(\alpha-1)/2 < \pi/2$, equivalently, $\alpha < 1 + 1/\nu$. If $x' < 0$ and $\nu < 1$, then the necessary and sufficient condition on α is $\pi\alpha/2 < 3\pi/2$, equivalently, $\alpha < 3$. However, if x' is small in absolute value, then it is advisable to ensure that not only $\mathrm{Re}(-ix'\xi(\eta))$ increases but $\mathrm{Re}\,\psi^0(\xi(\eta))$ does not tend to $-\infty$ as well. Hence, if $1 + 1/\nu \le 3$, we take $\alpha < 1 + 1/\nu$.

An approximate solution to (A.3) is found differently in cases

(i) $x' = 0$ and $\nu \in [1,2]$;
(ii) $x' = 0$ and $\nu \in (0,1)$;
(iii) $x' < 0$ and $\nu \in (1,2]$;
(iv) $x' < 0$ and $\nu \in (0,1)$;
(v) $x' < 0$ and $\nu = 1$.

Case (i). We change the variable $c^\infty\rho^\nu = y$, $\rho = (y/c^\infty)^{1/\nu}$

$$\frac{e^{tC_0}}{\nu\pi(c^\infty)^{1/\nu}}\int_{c^\infty\Lambda_1^\nu}^{+\infty} e^{-y}y^{1/\nu-1}dy = \epsilon. \tag{A.4}$$

Since $1/\nu \le 1$, we can use the following approximation:

$$\Lambda_2 = \ln(1/\epsilon) + tC_0 - \ln(\nu\pi(c^\infty)^{1/\nu}),\ \Lambda_1 = (\Lambda_2/c^\infty)^{1/\nu},$$

whose accuracy can be improved reassigning

$$\Lambda_2 := \ln(\Lambda_2^{1/\nu-1}/\epsilon) + tC_0 - \ln(\nu\pi(c^\infty)^{1/\nu}),\ \Lambda_1 = (\Lambda_2/c^\infty)^{1/\nu}.$$

The complexity of the scheme is of the order of $AE^{1+1/(\alpha\nu)}$, where

$$A = 1/(2\pi d(c^\infty)^{1/(\alpha\nu)}\sigma^{1/\alpha}) \to +\infty, \quad \text{as } \alpha \uparrow 1 + 1/\nu.$$

Case (ii). We integrate by parts in (A.3) and obtain an approximate equation

$$\frac{e^{tC_0}}{\pi} e^{-c^\infty \Lambda_1^\nu} \Lambda_1 = \epsilon,$$

equivalently,

$$c^\infty \Lambda_1^\nu - \ln \Lambda_1 - tC_0 - \ln(1/(\pi\epsilon) = 0. \tag{A.5}$$

We obtain an approximate solution of (A.5) in two steps: $\Lambda_1 = (\ln(1/(\pi\epsilon))/c^\infty)^{1/\nu}$, then reassign

$$\Lambda_1 := ((\ln(1/(\pi\epsilon)) + \ln \Lambda_1 + tC_0)/c^\infty)^{1/\nu}.$$

The complexity of the scheme is given by the same formula as in Case (i), with a different A. Note that the complexity increases as $\nu \downarrow 0$ and α fixed. However, we can choose $\alpha < 1 + 1/\nu$ arbitrary close to $1 + 1/\nu$. Hence, in both cases, the complexity of the scheme is of the order of $A(c)E^{(\nu+2)/(\nu+1)-c}$, where $c > 0$ can be made arbitrarily small choosing the parameters of the model appropriately; however, $A(c) \to +\infty$ as $c \downarrow 0$.

Case (iii). One can use the same prescriptions as in the case (i); the truncation parameter can be made smaller taking into account that $\mathrm{Re}(i\xi)$ increases fast (but slower than $\mathrm{Re}\,\psi^0(\xi)$).

(iv) We take $\alpha \le 1 + 1/\nu$, $\alpha < 3$, and define Λ_1 in two steps:

$$\Lambda_1 = (\ln(1/(\pi\epsilon))/(-x' \cos(\alpha\pi))),\ \Lambda_1 := ((\ln(1/(\pi\epsilon)) + tC_0 + c^\infty(\alpha,\nu)\Lambda_1^\nu)/(-x' \cos(\alpha\pi))).$$

The complexity of the scheme is of the order of $AE^{1+1/\alpha}$.

(v) We take $\alpha = 2$, find Λ_1 from (A.3), and set $\Lambda = (\Lambda_1/\sigma)^{1/2}$. The complexity of the scheme is of the order of $AE^{3/2}$.

A.2. **A modification of the general scheme for pricing European options in the Heston model.** We fix a moderately large λ (e.g., $\lambda = 10$) so that we do not expect that λ_+^0 and $-\lambda_-^0$ which will appear in the process of calibration turn out to be much larger than λ; in all cases, we will use $\mu_+ = \min\{\lambda_+^0, \lambda\}$, $\mu_- = \max\{\lambda_-^0, -\lambda\}$. For a fine and sufficiently long grid $\delta = \delta_j = \delta_0 j, \delta_0 > 0, j = 1, 2, \dots, N_\delta$, we precalculate the array

$$V(\delta_j, \lambda) = -\int_{\mathbb{R}} \frac{e^{-\delta((\lambda^2+\xi^2)^{1/2}-\lambda)}}{\xi(\xi+i)} d\xi. \tag{A.6}$$

In the process of calibration, for a set of parameters of the Heston model, we calculate $\Phi(\xi) := i\xi z_t + (v_t B_0(\tau,\xi) + C_0(\tau,\xi))/\sigma_0^2$ at $\xi = 0$ and $\xi = -i$. Set $a_0 = \Phi(0), a_1 = \Phi(-i)$, and find δ from the condition $a_1 - a_0 = -\delta((\lambda^2-1)^{1/2} - \lambda)$. Using interpolation, we calculate $V(\delta, \lambda)$

with the desired precision, define $\Phi_1(\xi) = a_0 - \delta((\lambda^2-1)^{1/2} - \lambda)$, and, in (3.11), replace $e^{\Phi(\xi)}$ with $e^{\Phi(\xi)} - e^{\Phi_1(\xi)}$:

$$V_{\text{call,aux}}(t, S_t, v_t) = -\frac{Ke^{-r\tau}}{2\pi} \int_{\text{Im}\,\xi=\omega} \frac{e^{\Phi(\xi)} - e^{\Phi_1(\xi)}}{\xi(\xi+i)} d\xi. \tag{A.7}$$

The integrand in (A.7) has no poles (whereas the integrand in (3.11) has two simple poles at 0 and $-i$), and we can evaluate the integral using $\mu_+ = \min\{\lambda_+^0, \lambda\}$, $\mu_- = \max\{\lambda_-^0, -\lambda\}$. The truncation parameter is chosen as the maximal of the truncation parameters defined by the procedure above for $\Phi(\xi)$ and $\Phi_1(\xi)$. After $V_{\text{call,aux}}(t, S_t, v_t)$ is calculated, we find

$$V_{\text{call}}(t, S_t, v_t) = V_{\text{call,aux}}(t, S_t, v_t) - \frac{Ke^{-r\tau}}{2\pi} V(\delta, \lambda).$$

A.3. **Characteristic function in the CIR model.** Function $W(t, T; r, \xi)$ is the unique bounded solution of the backward Kolmogorov equation

$$(\partial_t + \kappa(\theta - r)\partial_r + \frac{\sigma^2}{2} r \partial_r^2 - r) W(t, T; r, \xi) = 0, \tag{A.8}$$

subject to $W(T, T; r, \xi) = e^{ir\xi}$. Substituting (4.2) into (A.8), we obtain

$$-rB_\tau - C_\tau + \kappa(\theta - r)B + \frac{\sigma^2}{2} r B^2 - r = 0.$$

The system of Riccati equations for (B, C) is

$$B_\tau = \frac{\sigma^2}{2} B^2 - \kappa B - 1, \tag{A.9}$$

$$C_\tau = \kappa\theta B, \tag{A.10}$$

the initial condition being $B(0, \xi) = i\xi$, $C(0, \xi) = 0$. Let $B_\pm = (\kappa \pm \sqrt{\kappa^2 + 2\sigma^2})/\sigma^2$ be the roots of the quadratic equation $\frac{\sigma^2}{2} B^2 - \kappa B - 1 = 0$. We rewrite (A.9) in the form

$$\left[\frac{1}{B - B_+} - \frac{1}{B - B_-} \right] dB = \sqrt{\kappa^2 + 2\sigma^2} d\tau,$$

and integrate

$$\frac{B - B_+}{B - B_-} = \Lambda(\xi) e^{\tau\sqrt{\kappa^2+2\sigma^2}},$$

where $\Lambda(\xi) = \frac{i\xi - B_+}{i\xi - B_-}$. Thus,

$$\begin{aligned} B(\tau, \xi) &= \frac{B_+ - B_- \Lambda(\xi) e^{\tau\sqrt{\kappa^2+2\sigma^2}}}{1 - \Lambda(\xi) e^{\tau\sqrt{\kappa^2+2\sigma^2}}} \\ &= \frac{B_+(i\xi - B_-) - B_-(i\xi - B_+) e^{\tau\sqrt{\kappa^2+2\sigma^2}}}{i\xi - B_- - (i\xi - B_+) e^{\tau\sqrt{\kappa^2+2\sigma^2}}} \\ &= \frac{B_+(i\xi - B_-) e^{-\tau\sqrt{\kappa^2+2\sigma^2}} - B_-(i\xi - B_+)}{(i\xi - B_-) e^{-\tau\sqrt{\kappa^2+2\sigma^2}} - (i\xi - B_+)} \\ &= \frac{i\xi(B_+ e^{-\tau\sqrt{\kappa^2+2\sigma^2}} - B_-) - B_+(-B_-)(1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}})}{B_+ - B_- e^{-\tau\sqrt{\kappa^2+2\sigma^2}} - i\xi(1 - e^{-\tau\sqrt{\kappa^2+2\sigma^2}})}, \end{aligned} \tag{A.11}$$

and we obtain (4.3). Using (A.11), we write

$$C(\tau,\xi)=\kappa\theta\int_0^\tau\frac{B_+-B_-\Lambda(\xi)e^{\tau'\sqrt{\kappa^2+2\sigma^2}}}{1-\Lambda(\xi)e^{\tau'\sqrt{\kappa^2+2\sigma^2}}}d\tau',$$

and change the variable $\Lambda(\xi)e^{\tau'\sqrt{\kappa^2+2\sigma^2}}=y$, $d\tau'=dy/(y\sqrt{\kappa^2+2\sigma^2})$:

$$\begin{aligned}
C(\tau,\xi) &= \frac{\kappa\theta}{\sqrt{\kappa^2+2\sigma^2}}\int_{\Lambda(\xi)}^{\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}}\frac{B_+-B_-y}{y(1-y)}dy\\
&= \frac{\kappa\theta}{\sqrt{\kappa^2+2\sigma^2}}\int_{\Lambda(\xi)}^{\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}}\left[\frac{B_+}{y}-\frac{B_+-B_-}{y-1}\right]dy\\
&= \frac{\kappa\theta}{\sqrt{\kappa^2+2\sigma^2}}\left[B_+\ln\frac{\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}}{\Lambda(\xi)}-(B_+-B_-)\ln\frac{\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}-1}{\Lambda(\xi)-1}\right]\\
&= \frac{\kappa\theta}{\sqrt{\kappa^2+2\sigma^2}}\left[B_+\tau\sqrt{\kappa^2+2\sigma^2}-\frac{2\sqrt{\kappa^2+2\sigma^2}}{\sigma^2}\ln\frac{\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}-1}{\Lambda(\xi)-1}\right]\\
&= \kappa\theta\left[B_+\tau+2\sigma^{-2}\ln\frac{B_+-B_-}{(\Lambda(\xi)e^{\tau\sqrt{\kappa^2+2\sigma^2}}-1)(B_--i\xi)}\right]\\
&= \kappa\theta\left[B_-\tau+2\sigma^{-2}\ln\frac{B_+-B_-}{(\Lambda(\xi)-e^{-\tau\sqrt{\kappa^2+2\sigma^2}})(B_--i\xi)}\right]\\
&= \kappa\theta\left[B_-\tau+2\sigma^{-2}\ln\frac{B_+-B_-}{B_+-i\xi-e^{-\tau\sqrt{\kappa^2+2\sigma^2}}(B_--i\xi))}\right],
\end{aligned}$$

which gives (4.4).

Appendix B. Numerical examples

The calculations in the paper were performed in MATLAB© 8.0.0 (R2012b), on a PC with characteristics Intel (R) Core (TM) i7 CPU M 640, 2.80GHz, 8MB, under the Genuine Windows 7 Professional operating system.

B.1. **Tables I. Pdf of NTS.** The parameters of the process are $\mu=0$, $\alpha=10,\beta=0$ for $t=0.004$; $\delta=m_2\lambda^{\nu-2}$, where $m_2=\psi''(0)=0.1$ is the second instantaneous moment. In Table 1, ν varies, and the pdf is calculated at the peak. In Table 2, $\nu=0.3$ is fixed, and x varies.

The benchmark prices are obtained using the sinh-acceleration with different γ^+,γ^-, ζ and N; the results differ by less than E-15. For each ν, $x'=x-\mu t$ and the method of integration, the mesh size ζ and Λ are chosen using the universal prescriptions for the error tolerance ϵ. In some cases, latter are either inaccurate or lead to the overkill; then we show the results obtained with ζ/k_ζ and $k_\Lambda\Lambda$ instead of the prescribed ζ and Λ. Typically, approximate bounds for the Hardy norm are inaccurate for $\nu<1$ (ζ must be about 30% smaller) and lead to an overkill for $\nu>1$ (ζ can be about 5-10% larger). In some cases, Λ can be 5-10% smaller as well. The CPU time is in microseconds; the average is over 1 mln runs.

TABLE 1. Pdf of X_t at the peak at 0, rounded, and truncation errors of the calculation using sinh-acceleration and flat inverse Fourier transform. Dependence on the order ν.

ν	0.1	0.3	0.5	0.9	1.1	1.5	1.9
$p_t(0)$	1.64335E+11	27813.7583	1077.36380	111.103247	64.5381220	32.7368302	21.6193636
SINH							
	$\epsilon = 10^{-15}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	30	30	33	32	33	34	35
Error	0	0	0	0	0	0	0
Time	12.2	12.2	12.7	12.5	12.7	12.9	13.0
	$\epsilon = 10^{-15}$	$k_\zeta = 1$	$k_\Lambda = 0.95$				
N	29	29	31	31	31	32	33
Error	0	0	0	0	0	-1.2E-12	-9.9E-14
Time	12.0	12.0	12.4	12.4	12.3	12.5	12.7
	$\epsilon = 10^{-7}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	18	15	16	15	15	16	16
Error	-2.46E+04	-0.00019	-7.3E-06	2.6E-08	1.6E-08	1.3E-10	2.0E-11
Time	10.0	9.5	9.4	9.8	10.0	10.1	10.4
	$\epsilon = 10^{-7}$	$k_\zeta = 1.1$	$k_\Lambda = 1$				
N	19	17	17	17	17	18	18
Error	1.743E+03	2.7E-06	5.9E-07	2.3E-07	-9.6E-08	5.3E-09	-6.6E-09
Time	10.1	9.8	9.6	9.7	9.6	9.9	10.1
	$\epsilon = 10^{-4}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	13	10	10	9	9	10	10
Error	-5.1E+06	1.48	0.020	0.0013	0.0013	-0.00032	-8.3549E-05
Re.Err.	-3.1E-05	5.3E-05	1.9E-05	1.2E-05	2.0E-05	-9.8E-06	-3.9E-06
Time	9.0	8.5	8.3	8.3	8.5	8.4	8.4
Fract.	Parabolic						
	$\epsilon = 10^{-15}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	17851	10866	10244	1250	729	345	201
Error	-146.4	4.3E-09	-1.0E-11	0	0	0	0
Time	5066	2996	2759	617	365	186	109
	$\epsilon = 10^{-7}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	6512	3361	2921	460	268	130	78
Error	-113	8.0E-08	5.7E-10	-9.0E-11	-2.3E-10	-3.5E-10	-9.6E-11
Time	2010	1003	877	232	142	72.1	46.2
	$\epsilon = 10^{-4}$	$k_\zeta = 1$	$k_\Lambda = 1$				
N	3334	1558	1279	238	138	68	41
Error	-245	-0.0003	2.9E-05	1.3E-07	1.4E-06	1.7E-07	-6.0E-07
Time	1040	826	633	124	73.4	42.3	28.1
Errors	of flat IFT;	ζ is	fixed				
$N = 10^5$	-1.64E+11	-10795	-1.5E-04	1.1E-07	7.7E-08	3.1E-08	4.1E-09
$N = 10^6$	-1.64E+11	-1175	2.1E-07	1.1E-07	7.7E-08	3.1E-08	4.1E-09
$N = 10^7$	-1.64E+11	4.6	2.1E-07	1.1E-07	7.7E-08	3.1E-08	4.1E-09
$N = 2 \cdot 10^7$	-1.64E+11	0.28	2.1E-07	1.1E-07	7.7E-08	3.1E-08	4.1E-09

X: completely symmetric NTS Lévy process with $\lambda = 10$, $m_2 = \psi''(0) = 0.1$, $\delta = m_2\lambda^{\nu-2}$, $t = 0.004$, ν varies. Study of the efficiency of the universal recommendations for the parameter choice for the sinh-acceleration and fractional-parabolic transformation. For flat iFT, study of the dependence of the truncation error on ν and the number of terms N. Time: CPU time in microseconds.

B.2. **Tables II. The Heston model.** Table 3: $T = 0.004$, comparison of the sinh-acceleration with the fractional-parabolic method, for one strike.

Table 4: $T = 0.004$, calculation using the same set of parameters of SINH for all strikes. The errors and times for calculation of prices for different numbers of strikes.

Tables 5-8: the same as Table 4, for $T = 0.1, 1.0, 5, 15$.

Table 9: the comparison of the performance of the sinh-acceleration method with the Lewis-Lipton and Carr-Madan realizations of the flat iFT method. In all cases, the standard prescriptions ($\zeta = 0.125$, $N = 4096$) imply negligible truncation errors, hence, the errors shown are, essentially, the discretization errors.

TABLE 2. Left tail of pdf of X_t, rounded, and truncation errors of the calculation using the sinh-acceleration and flat inverse Fourier transform. Dependence on the distance from the peak.

x	-0.3	-0.25	-0.2	-0.15	-0.1	-0.05	-0.02	-0.01
$p_t(x)$	0.0029428	0.0059872	0.01277601	0.0294055	0.0777612	0.2894651	1.160531	2.93835839
SINH								
	$\epsilon = 10^{-15}$	$k_\zeta = 1$	$k_\Lambda = 1$					
N	19	20	22	24	26	31	37	42
Error	7.0E-17	6.9E-17	-3.0E-16	-1.0E-16	-4.0E-16	-1.0E-15	0	0
Time	16.5	15.5	16.2	16.9	17.6	19.3	21.5	23.3
	$\epsilon = 10^{-7}$	$k_\zeta = 1$	$k_\Lambda = 1$					
N	8	8	9	10	11	14	17	19
Error	1.5E-08	-1.2E-08	9.1E-09	2.1E-08	-6.1E-09	-4.2E-08	-4.6E-08	1.1E-07
Time	11.2	11.0	11.3	11.8	12.2	13.3	14.3	14.9
	$\epsilon = 10^{-4}$	$k_\zeta = 1$	$k_\Lambda = 1$					
N	4	4	5	5	6	8	10	12
Error	-1.4E-05	2.9E-05	-2.6E-05	3.1E-05	-1.9E-05	5.8E-05	1.1E-04	-1.6E-04
Time	9.5	9.3	9.7	9.8	10.2	11.0	11.7	12.3
Fract.	Parabolic							
	$\epsilon = 10^{-15}$	$k_\zeta = 1$	$k_\Lambda = 0.8$					
N	38	41	44	48	55	70	96	122
Error	5.0E-17	4.0E-17	-1.0E-16	0	1.9E-16	1.0E-15	0	0
Time	32.5	33.6	35.5	38.1	42.1	51.1	67.1	84.2
	$\epsilon = 10^{-7}$	$k_\zeta = 1$	$k_\Lambda = 0.8$					
N	14	15	16	17	20	25	35	44
Error	1.0E-10	5.7E-11	5.8E-11	-1.4E-11	8.9E-12	-4.6E-11	9.0E-09	3.2E-07
Time	17.6	18.1	18.8	19.2	21.2	24.0	30.0	35.7
	$\epsilon = 10^{-4}$	$k_\zeta = 1$	$k_\Lambda = 0.8$					
N	9	10	10	11	13	16	22	28
Error	5.5E-06	1.9E-06	9.0E-07	-5.0E-07	-9.4E-07	3.6E-07	-1.3E-07	6.5E-07
Time	14.2	14.9	15.0	15.6	16.7	18.5	22.4	26.2
	$\epsilon = 10^{-4}$	$k_\zeta = 0.95$	$k_\Lambda = 0.8$					
N	9	9	10	11	12	15	21	27
Error	1.7E-05	6.5E-06	1.2E-06	1.6E-06	2.4E-06	-3.5E-06	-8.5E-07	-3.5E-07
Time	14.5	14.3	15.1	15.6	16.2	18.1	21.9	26.9
Errors	of flat IFT;	ζ is	fixed					
$N = 10^5$	0.0057	-0.0056	0.0056	-0.0055	0.0054	-0.0054	-0.48	1.53
$N = 10^6$	0.0018	0.00040	-0.0018	-0.0045	-0.0070	-0.0088	-0.0094	-0.0094
$N = 10^7$	-1.3E-06	4.6E-06	7.1E-06	1.2E-06	-1.2E-05	-2.6E-05	-1.4E-05	-0.00015

X: completely symmetric NTS Lévy process of finite variation, with $\lambda = 10$, $m_2 = \psi''(0) = 0.1$, $\nu = 0.3$, $\delta = m_2\lambda^{\nu-2}$, $t = 0.004$, x varies. Study of the efficiency of the universal recommendations for the parameter choice for the sinh-acceleration and fractional-parabolic transformation. For flat iFT, study of the dependence of the truncation error on ν and the number of terms N. Time: CPU time in microseconds.

B.3. **Call option on the bond in CIR model (Table 10) and in CIR-subordinated NTS model (Table 11).**

B.4. **Examples of calculation of quantiles (Table 12).**

TABLE 3. Put in the Heston model. SINH acceleration vs Fractional Parabolic.

K	85	90	95	100	105	110	115
x'	0.205437159	0.1482787452	0.0942115239	0.0429182295	-0.0058719347	-0.0523919503	-0.0968437129
V_{put}	8.75606E-07	0.0004112657	0.046751956	1.0603962422	5.0125262734	9.991210204	14.9908003682
SINH							
	$\epsilon = 10^{-12}$	$k_\zeta = 1.8$	$k_\Lambda = 1.35$				
ζ	0.135219069	0.141969971	0.149051905	0.156224513	0.161505024	0.154669039	0.148348105
N	58	56	54	52	50	53	55
Error	-2.98E-12	3.95E-12	0	-4.00E-14	3.91E-14	0	9.95E-14
Time	0.207	0.203	0.199	0.195	0.184	0.193	0.188
	$\epsilon = 10^{-6}$	$k_\zeta = 1.8$	$k_\Lambda = 1.35$				
ζ	0.239504852	0.251634664	0.264309759	0.277086239	0.286405666	0.274054828	0.262614271
N	31	30	29	28	27	28	30
Error	-1.41E-07	-1.35E-07	-1.01E-07	-6.55E-10	5.52E-09	-6.43E-08	1.45E-07
Time	0.190	0.181	0.179	0.173	0.179	0.182	0.175
	$\epsilon = 10^{-2}$	$k_\zeta = 1.8$	$k_\Lambda = 1.35$				
ζ	0.450079392	0.473594213	0.497964069	0.522276556	0.539648211	0.515421352	0.492902392
N	14	14	13	13	13	13	14
Error	-2.16E-04	-6.09E-04	-1.63E-04	9.62E-06	-1.30E-04	-8.23E-04	-2.49E-03
Time	0.182	0.181	0.165	0.162	0.167	0.169	0.168
Fract. Para							
	$\epsilon = 10^{-12}$	$k_\zeta = 1$	$k_\Lambda = 1$				
ζ	0.1501040751	0.1498588017	0.149627309	0.1494081636	0.158551587	0.1583831405	0.1582217381
N	290	341	420	563	759	498	393
Error	-1.28E-13	2.80E-14	-2.80E-14	-1.01E-14	-7.02E-14	-2.90E-13	-2.01E-13
Time	0.459	0.413	0.480	0.601	0.792	0.544	0.460
	$\epsilon = 10^{-6}$	$k_\zeta = 0.85$	$k_\Lambda = 0.85$				
ζ	0.3006157713	0.299780534	0.2989939791	0.2982509424	0.3306004853	0.3299784849	0.3293834519
N	92	109	135	182	237	154	121
Error	-4.70E-07	-1.07E-06	3.05E-07	-1.91E-07	-1.53E-06	-1.55E-06	-1.5E-06
Time	0.349	0.224	0.241	0.280	0.352	0.261	0.240
	$\epsilon = 10^{-2}$	$k_\zeta = 0.85$	$k_\Lambda = 0.85$				
ζ	0.565	0.562	0.560	0.557	0.682	0.679	0.676
N	35	42	53	72	86	55	43
Error	1.2E-04	-4.3E-05	1.2E-04	-0.0057	2.0E-04	2.8E-05	-1.3E-04
Time	0.241	0.272	0.317	0.391	0.428	0.341	0.271

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 0.004$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Time: CPU time in microseconds.

Given the error tolerance ϵ, the parameters of the schemes are chosen for each point, using the universal prescriptions with the corrections factors $k_b = 0.8, k_d = 0.8$, k_ζ, k_Λ.

TABLE 4. Put in the Heston model, $T = 0.004$. Prices and errors of the SINH-acceleration.

K	85	90	95	100	105	110	115
x'	0.205437159	0.1482787452	0.0942115239	0.0429182295	-0.0058719347	-0.0523919503	-0.0968437129
V_{put}	8.75606E-07	0.0004112657	0.046751956	1.0603962422	5.0125262734	9.991210204	14.9908003682
$\epsilon = 10^{-12}$	4.26E-14	5.68E-14	-1.42E-14	-1.74E-12	6.01E-12	2.08E-11	2.26E-10
$\epsilon = 10^{-6}$	8.65E-10	7.82E-09	1.91E-08	-1.68E-07	4.23E-07	-6.85E-07	3.52E-06
$\epsilon = 10^{-2}$	-4.75E-03	1.06E-02	-2.25E-02	4.14E-02	-5.33E-02	6.09E-02	-6.89E-02

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 0.004$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Given the error tolerance, the parameters are chosen the same for strikes in the range $[85, 115]$, using the universal prescriptions with the corrections factors $k_b = 0.8, k_d = 0.8$, $k_\zeta = 1.85, k_\Lambda = 1.3$.

For $\epsilon = 10^{-12}$: $N = 89$, $\zeta = 0.081939329$, CPU time for 7 and 120 strikes: 0.323 and 2.79 msec., respectively.

For $\epsilon = 10^{-6}$: $N = 57$, $\zeta = 0.118334081$, CPU time for 7 and 120 strikes: 0.258 and 1.68 msec., respectively.

For $\epsilon = 10^{-2}$: $N = 30$, $\zeta = 0.181940202$, CPU time for 7 and 120 strikes: 0.221 and 1.40 msec., respectively.

TABLE 5. Put in the Heston model, $T = 0.1$. Prices and errors of the SINH-acceleration with the universal choice of the parameters.

K	85	90	95	100	105	110	115
x'	0.2085894213	0.1514310075	0.0973637862	0.0460704918	-0.0027196724	-0.049239688	-0.0936914506
$V_{\rm put}$	1.1764633175	1.8719759966	2.9150895284	4.5125209091	7.067104472	10.7962013124	15.2373482324
$\epsilon = 10^{-12}$	0	4.00E-14	-3.02E-14	-7.02E-14	0	9.95E-14	-3.00E-13
$\epsilon = 10^{-6}$	-1.44E-10	-3.26E-10	2.84E-09	1.31E-09	-5.60E-08	2.08E-07	-5.02E-07
$\epsilon = 10^{-2}$	-1.17E-08	-8.05E-08	2.96E-07	1.06E-06	-1.18E-06	-7.91E-06	-1.70E-05

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 0.1$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Given the error tolerance, the parameters are chosen the same for strikes in the range $[85, 115]$, using the universal prescriptions with the correction factors $k_b = 0.8, k_d = 0.8$, $k_\zeta = 1.85, k_\Lambda = 1.3$.

For $\epsilon = 10^{-12}$: $N = 94$, $\zeta = 0.080430727$, CPU time for 7 and 120 strikes: 0.321 and 2.59 msec., respectively.

For $\epsilon = 10^{-6}$: $N = 48$, $\zeta = 0.13343117$, CPU time for 7 and 120 strikes: 0.261 and 1.85 msec., respectively.

For $\epsilon = 10^{-2}$: $N = 30$, $\zeta = 0.181940202$, CPU time for 7 and 120 strikes: 0.222 and 1.37 msec., respectively.

TABLE 6. Put in the Heston model, $T = 1$. Prices and errors of the SINH-acceleration with the universal choice of the parameters.

K	85	90	95	100	105	110	115
x'	0.2381418803	0.1809834665	0.1269162452	0.0756229508	0.0268327867	-0.019687229	-0.0641389916
$V_{\rm put}$	4.7941827931	5.6161173264	6.646714606	8.0122168751	9.9462613433	12.730505446	16.3323981366
$\epsilon = 10^{-12}$	-1.96E-14	1.95E-14	9.95E-14	0	-6.04E-14	9.95E-14	1.00E-12
$\epsilon = 10^{-6}$	8.24E-10	1.41E-09	-1.36E-08	-5.52E-08	-5.51E-08	2.74E-07	1.06E-06
$\epsilon = 10^{-2}$	-1.31E-04	-1.11E-04	8.73E-04	-2.28E-03	3.50E-03	-2.14E-03	-3.56E-03

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 1$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Given the error tolerance, the parameters are chosen the same for strikes in the range $[85, 115]$, using the universal prescriptions with the correction factors $k_b = 0.8, k_d = 0.8$, $k_\zeta = 1.85, k_\Lambda = 1.3$.

For $\epsilon = 10^{-12}$: $N = 85$, $\zeta = 0.085671285$, CPU time for 7 and 120 strikes: 0.309 and 2.43 msec., respectively.

For $\epsilon = 10^{-6}$: $N = 49$, $\zeta = 0.130631744$, CPU time for 7 and 120 strikes: 0.252 and 1.75 msec., respectively.

For $\epsilon = 10^{-2}$: $N = 26$, $\zeta = 0.200931104$, CPU time for 7 and 120 strikes: 0.222 and 1.33 msec., respectively.

TABLE 7. Put in the Heston model, $T = 5$. Prices and errors of the SINH-acceleration with the universal choice of the parameters.

K	90	100	110	120	130	140	150
x'	0.3123277288	0.2069672131	0.1116570333	0.0246456563	-0.0553970514	-0.1295050235	-0.198497895
$V_{\rm put}$	8.9118170191	11.3017608315	14.4866039624	18.9062479333	24.8561314222	32.0308080039	39.9171298805
$\epsilon = 10^{-12}$	-6.04E-14	2.01E-13	-3.00E-13	-7.00E-13	-1.50E-12	-4.00E-12	2.20E-12
$\epsilon = 10^{-6}$	-2.70E-07	1.83E-09	4.67E-07	8.44E-07	2.73E-07	-2.09E-06	-2.78411E-06
$\epsilon = 10^{-2}$	5.00E-03	-8.27E-04	-9.73E-03	1.64E-02	-2.89E-03	-1.82E-02	7.70E-03

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 5$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Given the error tolerance, the parameters are chosen the same for strikes in the range $[90, 120]$, using the universal prescriptions with the correction factors $k_b = 0.8, k_d = 0.8$, $k_\zeta = 1.85, k_\Lambda = 1.3$.

For $\epsilon = 10^{-12}$: $N = 75$, $\zeta = 0.087187403$, CPU time for 7 and 120 strikes: 0.297 and 2.16 msec., respectively.

For $\epsilon = 10^{-6}$: $N = 43$, $\zeta = 0.13264446$, CPU time for 7 and 120 strikes: 0.246 and 1.71 msec., respectively.

For $\epsilon = 10^{-2}$: $N = 22$, $\zeta = 0.203311839$, CPU time for 7 and 120 strikes: 0.214 and 1.23 msec., respectively.

TABLE 8. Put in the Heston model, $T = 15$. Prices and errors of the SINH-acceleration with the universal choice of the parameters.

K	90	100	110	120	130	140	150
x'	0.6406883845	0.5353278689	0.440017689	0.3530063121	0.2729636044	0.1988556322	0.1298627607
V_{put}	12.4856557684	14.8462073848	17.4752559196	20.4094193312	23.6896491628	27.3577089222	31.4493345118
$\epsilon = 10^{-12}$	-3.00E-13	-1.40E-12	-6.50E-12	-4.90E-12	3.46E-11	-2.20E-11	-7.48E-11
$\epsilon = 10^{-6}$	-1.04E-06	4.53E-06	-9.60E-06	7.17E-06	1.37E-05	-3.16E-05	-6.82E-06
$\epsilon = 10^{-2}$	-0.0164	-0.053	-0.070	-0.051	0.00314	0.0753	0.139

Put option parameters: $r = 0.02$, $\delta = 0$, $T = 15$, $S = 100$.

Parameters of the Heston model: $v_0 = 0.18; \rho = -0.58, \sigma_0 = 2.44, \kappa = 0.30, m = 0.18$.

Given the error tolerance, the parameters are chosen the same for strikes in the range $[90, 120]$, using the universal prescriptions with the correction factors $k_b = 0.8, k_d = 0.8$, $k_\zeta = 1.85, k_\Lambda = 1.3$.

For $\epsilon = 10^{-12}$: $N = 58$, $\zeta = 0.095602143$, CPU time for 7 and 120 strikes: 0.273 and 1.93 msec., respectively.

For $\epsilon = 10^{-6}$: $N = 32$, $\zeta = 0.145718873$, CPU time for 7 and 120 strikes: 0.232 and 1.45 msec., respectively.

For $\epsilon = 10^{-2}$: $N = 15$, $\zeta = 0.224004098$, CPU time for 7 and 120 strikes: 0.205 and 1.12 msec., respectively.

TABLE 9. Put in the Heston model. Panel A: short and moderate maturities; panel B: long maturities. Errors (rounded) of the Lewis-Lipton choice of the line of integration $\omega = -0.5$ (LLT: simplified trapezoid rule, LLS: Simpson rule) and of Carr-Madan-Schoutens choice $\omega = -1.75$ (CMST: simplified trapezoid rule, CMSS: Simpson rule). In all cases, $\zeta = 0.125$, $N = 4096$, hence, the truncation errors are negligible, and the errors shown are, essentially, the discretization errors.

A							
K	85	90	95	100	105	110	115
$T = 0.004$							
x'	0.205437159	0.1482787452	0.0942115239	0.0429182295	-0.0058719347	-0.0523919503	-0.0968437129
LLT	-2.2504E-09	-2.310E-09	-2.372E-09	-2.433E-09	-2.4947E-09	-2.555E-09	-2.615E-09
LLS	2.14E-04	-1.90E-04	-0.0465	-1.060	-5.012	-9.991	-14.99
CMST	2.84E-14	5.68E-14	4.26E-14	-6.02E-14	-4.00E-14	-6.93E-14	0
CMSS	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07
$T = 0.1$							
x'	0.2085894213	0.1514310075	0.0973637862	0.0460704918	-0.0027196724	-0.049239688	-0.0936914506
LLT	-2.248E-09	-2.308E-09	-2.369E-09	-2.430E-09	-2.450E-09	-2.551E-09	-2.613E-09
LLS	2.15E-04	2.21E-04	2.26E-04	2.32E-04	2.38E-04	2.44E-04	2.50E-04
CMST	1.00E-13	9.99E-15	-7.99E-14	-3.02E-14	0	9.95E-14	-9.95E-14
CMSS	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07
$T = 1$							
x'	0.2381418803	0.1809834665	0.1269162452	0.0756229508	0.0268327867	-0.019687229	-0.0641389916
LLT	-2.229E-09	-2.290E-09	-2.348E-09	-2.408E-09	-2.468E-09	-2.527E-09	-2.587E-09
LLS	2.13E-04	2.19E-04	2.25E-04	2.30E-04	2.36	2.42	2.47
CMST	3.02E-14	1.71E-13	-1.09E-13	1.71E-13	0	9.95E-14	-9.95E-14
CMSS	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07	-2.17E-07
B							
K	90	100	110	120	130	140	150
$T = 5$							
x'	0.3123277288	0.2069672131	0.1116570333	0.0246456563	-0.0553970514	-0.1295050235	-0.198497895
LLT	-2.206E-09	-2.316E-09	-2.426E-09	-2.536E-09	-2.647E-09	-2.757E-09	-2.867E-09
LLS	2.107E-04	2.212E-04	2.317E-04	2.422E-04	2.527	2.631E-04	2.736
CMST	1.95E-04	1.77E-04	1.63E-04	1.50E-04	1.40E-04	1.31E-04	1.23E-04
CMSS	-0.0027	-0.0024	-0.0022	-0.0021	-0.0019	-0.0018	-0.0017
$T = 15$							
x'	0.6406883845	0.5353278689	0.440017689	0.3530063121	0.2729636044	0.1988556322	0.1298627607
LLT	-2.021E-09	-2.114E-09	-2.204E-09	-2.294E-09	-2.384E-09	-2.473E-09	-2.563E-009
LLS	1.929E-04	2.014E-04	2.099E-04	2.185E-04	2.270E-04	2.355	2.440
CMST	0.126	0.116	0.108	0.101	0.0947	0.0895	0.0848
CMSS	0.0471	0.0436	0.0407	0.0382	0.0360	0.0341	0.0324

TABLE 10. Prices of the call option on bond in CIR model, rounded. Errors, number of terms and CPU times (in msc) of different realizations of iFT.

K	97.50512024	97.6461914	97.78746667	97.92894634	98.0706307	98.21252005	98.35461469	98.49691491
z_{TK}	-0.02	-0.0175	-0.015	-0.0125	-0.01	-0.0075	-0.005	-0.0025
Call 1								
Price	0.876713465	0.756024612	0.636971345	0.519888515	0.40523729	0.293696753	0.186378527	0.08550053
ζ	0.110853	0.110844	0.110835	0.110826	0.110817	0.110808	0.110799	0.110791
N	43	44	45	46	48	50	53	58
Time	0.118	0.118	0.119	0.120	0.120	0.120	0.121	0.124
Call 2								
Err	1.54E-14	-5.11E-13	-5.46E-13	-5.93E-13	-6.53E-13	-6.83E-13	-7.36E-13	-7.61E-13
N	58	47	47	48	50	51	53	57
Time	0.123	0.123	0.124	0.123	0.124	0.125	0.126	0.128
FrPara	$\alpha = 2.8$							
Err	1.54E-14	-5.11E-13	-5.46E-13	-5.93E-13	-6.64E-13	-6.83E-13	-7.36E-13	-7.72E-13
N	759	795	838	891	961	1060	1217	1540
Time	0.619	0.617	0.618	0.619	0.620	0.618	0.619	0.628
Flat iFT	$N = 10^5$							
Err	-5.51E-06	-7.33E-06	-8.99E-06	-1.03E-05	-1.11E-05	-1.10E-05	-9.08E-06	5.29E-07
Time	11.7	11.4	11.4	11.4	11.5	11.4	11.4	11.4

Parameters of CIR model: $\kappa = 1.6; \theta = 0.01, \sigma = 0.5$.

Bond matures at $T + \tau = 3$, spot price 99.384925, implied $r_0 = 0.01$.

Call option matures at $\tau = 1$; strikes K and $z = \ln(C(2,0) - \ln K)/B(2,0)$ are shown in the table.

Call 1: call option prices calculated using the sinh-acceleration with the parameters chosen for a curve in the lower half-plane and put-call parity; errors of these prices is less than $2 * E - 14$. The mesh and truncation parameters used are 0.9 times larger and 0.95 smaller than the general prescription for the error tolerance $E - 13$ recommends.

Call 2: errors of option prices calculated using the sinh-acceleration with the parameters chosen for the curve that is above $-iB(T - \tau, 0)$ w.r.t. to prices Call 1; no put-call parity is needed. The mesh and truncation parameter are 0.9 times larger and 0.95 smaller than the general prescription for the error tolerance recommends.

FrPara: errors of option prices calculated using the fractional-parabolic method with the parameters chosen so that the errors are of the same order of magnitude as the ones for Call 1.

Flat iFT: errors of option prices calculated using the flat iFT with $N = 100,000$ terms.

Time: CPU time in msc, the average over 100,000 runs.

TABLE 11. Prices of the call option in the CIR-subordinated NTS model, rounded. Errors, meshes, and CPU times (in msc) of different realizations of iFT.

K	97.50512024	115.0273799	112.7496852	110.5170918	108.3287068	106.1836547	104.0810774	102.020134
$\ln(S_0/K)$	-0.14	-0.12	-0.1	-0.08	-0.06	-0.04	-0.02	0
V_{call}	0.000300627	0.000359782	0.000439703	0.000553604	0.00072976	0.001043745	0.001811214	0.047838815
SINH1	$\gamma^- = -\pi/3$	$\gamma^+ = 0$	$\epsilon = 10^{-15}$					
ζ	0.077345	0.0773358	0.0773263	0.0773167	0.0773070	0.0772970	0.0772870	0.0772767
N	79	81	83	86	90	95	103	172
Err	-2.81E-15	-1.11E-15	-1.62E-15	-4.52E-16	5.81E-15	-1.86E-15	-1.00E-15	6.80E-15
Time	0.221	0.235	0.243	0.244	0.247	0.241	0.251	0.245
SINH2	$\gamma^- = -\pi/2$	$\gamma^+ = 0$	$\epsilon = 10^{-15}$					
ζ	0.116423	0.1164011	0.1163788	0.1163560	0.1163329	0.1163093	0.1162853	0.116261
N	51	52	54	56	58	62	67	117
Err	1.20E-17	-1.80E-15	-1.45E-15	9.60E-16	-1.62E-15	-2.58E-15	-5.17E-15	1.15E-09
Time	0.195	0.197	0.202	0.199	0.207	0.214	0.219	0.261
SINH3	$\gamma^- = -\pi/2$	$\gamma^+ = 0$	$\epsilon = 10^{-7}$					
ζ	0.2251344	0.2250521	0.2249685	0.2248835	0.2247970	0.2247091	0.2246196	0.2245286
N	23	24	25	26	27	28	31	54
Err	-5.33E-07	-5.12E-07	-4.92E-07	-4.76E-07	-4.65E-07	-4.67E-07	-5.39E-07	-8.58E-06
Time	0.159	0.154	0.155	0.158	0.160	0.161	0.167	0.188
Flat iFT	$\omega_0 = -1.75$	$\zeta = 0.25$	$N = 16384$					
Err	5.02E-07	4.82E-07	4.63E-07	4.47E-07	4.36E-07	4.39E-07	5.11E-07	1.07E-05
Time	6.74	6.72	6.78	6.71	6.74	6.75	6.72	6.67

Parameters of CIR subordinator: $\kappa = 1.6; \theta = 0.01, \lambda = 0.25$, $y_0 = 0.02$.

Parameters of NTS model: $m_2 = 0.1$, $\nu = 1.6$, $\delta = 0.097$, $\alpha = 3$, $\beta = 0$, $\mu = 0$.

Call option: maturity $\tau = 0.004$, $r = 0.02$; spot $S_0 = 100$, strikes K and $\ln S_0/K$ are shown in the table.

Benchmark prices calculated using several sets of the parameters of the sinh-acceleration; errors less than $2E-15$ in the absolute value.

SINH1: ζ (rounded), N, errors and CPU time when using the general prescription with $\gamma^- = -\pi/3, \gamma^+ = 0$, for $\epsilon = 10^{-15}$.

SINH2: ζ (rounded), N, errors and CPU time when using the general prescription with $\gamma^- = -\pi/2, \gamma^+ = 0$, for $\epsilon = 10^{-15}$.

SINH3: ζ (rounded), N, errors and CPU time when using the general prescription with $\gamma^- = -\pi/2, \gamma^+ = 0$, for $\epsilon = 10^{-7}$.

Flat iFT: errors and CPU time of the calculation using the flat iFT with the standard prescription $\omega_0 = -1.75, \zeta = 0.25, N = 16384$.

Time: CPU time in msc, the average over 100,000 runs.

TABLE 12. Errors of approximations L,N,LL, LN and QT, for different fractiles $F^{-1}(A)$ and intervals of different length, containing $F^{-1}(A)$.

A	$2*10^{-9}$		e^{-16}		10^{-5}		0.001		0.3	
h	0.01	0.001	0.01	0.001	0.01	0.001	10^{-3}	10^{-4}	10^{-4}	10^{-5}
L	-7.3E-05	-6.8E-07	-7.7E-05	-7.9E-07	-9.3E-05	-9.4E-07	-2.4E-06	-2.5E-08	-6.4E-06	-5.0E-08
N	7.7E-05	1.2E-06	0.0001	8.8E-07	0.00015	1.4E-06	4.3E-06	2.0E-08	1.7E-06	9.5E-09
LL	-8.4E-07	-7.8E-09	-1.7E-06	-1.7E-08	-5.7E-06	-5.8E-08	-6.1E-07	-6.5E-09	-3.6E-06	-2.9E-08
LN	8.9E-07	1.4E-08	2.4E-06	2.0E-08	9.0E-06	8.6E-08	1.1E-06	5.0E-09	1.0E-06	5.4E-09
QT	-1.5E-09	-3.4E-12	-6.9E-09	-7.1E-12	-5.7E-08	-5.3E-11	-4.1E-09	-1.3E-12	-5.5E-08	-2.1E-11